Three-sets in a Union-Closed Family

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Abstract

A union closed (UC) family $\mathcal{A}$ is a finite family of sets such that the union of any two sets in $\mathcal{A}$ is also in $\mathcal{A}$. Peter Frankl conjectured in 1979, that for every union closed family $\mathcal{A}$, there exists some x contained in at least half the members of $\mathcal{A}$. In this paper, we show that if a UC family $\mathcal{A}$ fails the conjecture, then no element can appear in more than two of its 3-sets, and so the number of 3-sets in $\mathcal{A}$ can be no more than $2n/3$.

1 Introduction

The Frankl conjecture (or union closed sets conjecture) was stated by Peter Frankl in 1979: given a finite family of finite sets, closed under unions (a UC family), at least one element must be contained in at least half the members of the family. This apparently simple conjecture is remarkably resistant to proof, though many partial results have been found.

It is easy to see that a UC family which has any sets of cardinality one or two, satisfies the conjecture. In Section 2, we use a theorem of Bjorn Poonen [1], together with a technique described in [3] to show that if a UC family $\mathcal{A}$ with $\cup\mathcal{A} = \{1, 2, \ldots, n\}$ fails the conjecture, then $\mathcal{A}$ can contain no more than $2n/3$ sets of cardinality three.

The general outline of the proof is simple, though the details are rather complicated. We first prove that if a UC family $\mathcal{A}$ contains the 3-sets $\{1,2,3\}$, $\{1,4,5\}$, $\{1,6,7\}$, then one of the elements $1,2,3,4,5,6,7$ must appear in at least half the members of $\mathcal{A}$.

In [3], we proved that if $T = \{\{1,a,b\},\{1,c,d\},\{1,e,f\}\}$ and $|\{a,b,c,d,e,f\}| \leq 6$, then for every UC family containing T, one of the elements $1,a,b,c,d,e,f$ must appear in at least half the members of $\mathcal{A}$.

It follows that if any UC family contains three 3-sets with a common element, then one of the elements of those three 3-sets is contained in at least half the members of $\mathcal{A}$.

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From this, if a UC family $\mathcal{A}$ fails the conjecture, then no element can appear in more than two of its 3-sets; then the number of 3-sets in $\mathcal{A}$ can be no more than $2n/3$.

In Section 3, we prove a result which might lead to a proof that if a UC family $\mathcal{A}$ fails the conjecture, then the number of 3-sets in $\mathcal{A}$ can be no more than $n/2$. This proof could be completed by the successful verification of results like those of Section 2, for finitely many UC families involving 8, 9, or 10 elements.

For simplicity we use the notation e.g. 123 for $\{1,2,3\}$; and if T is a collection of sets, then $< T >$ represents the UC family generated by T. Throughout, $\mathcal{B}$ is the UC family generated by 123, 145, 167.

2 $\mathcal{B}$ is an FC family.

In this section, we prove that if a UC family $\mathcal{A}$ contains the three 3-sets $\{123, 145, 167\}$, then one of $1,2,3,4,5,6,7$ is contained in at least half the members of $\mathcal{A}$. The proof uses a theorem given by Bjorn Poonen in [1]. To state this theorem, we require some definitions.

Definition 2.1 *$N_i(\mathcal{A})$ is the number of members of $\mathcal{A}$ which contain the element i.*

Definition 2.2 *If $\mathcal{A}$ and $\mathcal{B}$ are UC families, then $\mathcal{A} \uplus \mathcal{B} = \{X \cup Y : X \in \mathcal{A}, Y \in \mathcal{B}\}$.*

Definition 2.3 *An FC family $\mathcal{B}$ is a UC family having the property that for every UC family $\mathcal{A}$ containing $\mathcal{B}$, it is true that one of the elements of $\cup\mathcal{B}$ is in at least half the members of $\mathcal{A}$.*

Theorem 2.4 *(Poonen) $\mathcal{B} \subseteq \mathcal{P}(n)$ is an FC family if and only if there exist non-negative real numbers $c_1, c_2, \cdots, c_n$ with sum 1 such that for every UC family $\mathcal{A} \in \mathcal{P}(n)$ satisfying $\mathcal{A} \uplus \mathcal{B} \subseteq \mathcal{A}$,*

$$\sum_{i=1}^{n} c_i N_i(\mathcal{A}) \geq |\mathcal{A}|/2.$$

In this section, we prove that the UC family

$$\mathcal{B} = \{123, 145, 167, 1234, 12345, 12367, 14567, 1234567\}$$

is an FC family. Using the method described in [3], we find that suitable numbers c_i are $c_1 = 1/4$, and $c_i = 1/8$ for $2 \leq i \leq 7$.

We first give some notation.

$S = \{1,2,3,4,5,6,7\}$, and $\mathcal{P}(S)$ is the power set of S.

For $X \neq \emptyset$ and $X \in \mathcal{P}(S)$, put $f(X) = \sum_{i \in X} c_i$ and $h(X) = f(X) - 1/2$. Put $f(\emptyset) = 0$, and $h(\emptyset) = -1/2$.

For a collection T of subsets, $h(T) = \sum_{X \in T} h(X)$.

For a UC family $\mathcal{A}$ in $\mathcal{P}(S)$, E_i is the collection of sets in $\mathcal{A}$ which have cardinality i.

Note that $\sum_{i=1}^{n} c_i N_i(\mathcal{A}) - |\mathcal{A}|/2 = h(\mathcal{A})$, so that in terms of the function h, the condition of Poonen's theorem is, that $h(\mathcal{A}) \geq 0$ for every $\mathcal{A} \subseteq \mathcal{P}(S)$ such that $\mathcal{A} \uplus \mathcal{B} \subseteq \mathcal{A}$.

Throughout, we consider only UC families $\mathcal{A} \subseteq \mathcal{P}(S)$ such that $\mathcal{A} \uplus \mathcal{B} \subseteq \mathcal{A}$.

Our procedure may be briefly described as follows: First, partition the subsets of $\mathcal{P}(S)$ in orbits of the automorphism group of $\mathcal{B}$. All the subsets in one orbit have the same h-value. Then we consider the sums $h(T)$ for each collection T of subsets of $\mathcal{A}$ having the same (plus or minus) h-value. Finally, we show that the sum of all these is non-negative.

The automorphism group of $\mathcal{B}$ consists of all permutations which permute the pairs 23, 45, 67 among themselves. This includes, for example, the transpositions (23), (45), (67), and "interlaced" cycles such as (2,4,3,5), or (2,4,6,3,5,7).

Given $\mathcal{A}$, each subset E_i is partitioned into sets E_{ij} by intersection with the orbits of the automorphism group. These sets, together with their h-values, are listed below, excluding 3-sets containing 1, and 4-sets which do not contain 1 (for such sets X, $h(X) = 0$, so they contribute nothing one way or the other, to $h(\mathcal{A})$).

$E_0 \subseteq \{\emptyset\}$; $h(E_0) = (-1/2)|E_0|$
$E_{11} \subseteq \{1\}$; $h(E_{11}) = (-1/4)|E_{11}|$
$E_{12} \subseteq \{2,3,4,5,6,7\}$; $h(E_{12}) = (-3/8)|E_{12}|$

$E_{21} \subseteq \{12,13,14,15,16,17\}$; $h(E_{21}) = (-1/8)|E_{21}|$
$E_{22} \subseteq \{23,45,67\}$; $h(E_{22}) = (-1/4)|E_{22}|$
$E_{23} \subseteq \{24,25,26,27,34,35,36,37,46,47,56,57\}$; $h(E_{23}) = (-1/4)|E_{23}|$

$E_{31} \subseteq \{abc | a,b,c \neq 1; ab \in E_{22}\}$; $h(E_{31}) = (-1/8)|E_{31}|$
$E_{32} \subseteq \{abc | a,b,c \neq 1; ab, ac, bc \notin E_{22}\}$; $h(E_{32}) = (-1/8)|E_{32}|$

$E_{41} \subseteq \{1abc | ab \in E_{22}\}$; $h(E_{41}) = (1/8)|E_{41}|$
$E_{42} \subseteq \{1abc | ab, ac, bc \notin E_{22}\}$; $h(E_{42}) = (1/8)|E_{42}|$

$E_{51} \subseteq \{1abcd | ab, cd \in E_{22}\}$; $h(E_{51}) = (1/4)|E_{51}|$
$E_{52} \subseteq \{1abcd | ab \in E_{22}, cd \notin E_{22}\}$; $h(E_{52}) = (1/4)|E_{52}|$
$E_{53} \subseteq \{S - \{1,i\}\}$; $h(E_{53}) = (1/8)|E_{53}|$

$E_{61} \subseteq \{S - \{i\}\}$ $(i > 1)$; $h(E_{61}) = (3/8)|E_{61}|$
$E_{62} \subseteq \{S - \{1\}\}$; $h(E_{62}) = (1/4)|E_{62}|$

$E_7 = \{1234567\}$; $h(E_7) = (1/2)|E_7| = 1/2$

In absolute value, there are just four h-values: $1/2, 1/4, 1/8, 3/8$, and we have four sums, listed below.

$$S_1 = (1/4)[|E_{62}| + |E_{51}| + |E_{52}| - |E_{11}| - |E_{22}| - |E_{23}|]$$
$$S_2 = (1/8)[|E_{41}| + |E_{42}| + |E_{53}| - |E_{21}| - |E_{31}| - |E_{32}|]$$
$$S_3 = (3/8)[|E_{61}| - |E_{12}|]$$
$$S_4 = (1/2)[|E_7| - |E_0|]$$

and we have $h(\mathcal{A}) = S_1 + S_2 + S_3 + S_4$.

Since we are assuming that $\mathcal{A} \uplus \mathcal{B} \subseteq \mathcal{A}$, there are many relations among the E_{ij}. The next lemmas describe some of these relations.

For the first lemma, we use the following notation. The automorphism group of $\mathcal{B}$ contains the permutation $\sigma = (23)(45)(67)$, a product of transpositions.

For each $X \in \mathcal{A}$, put $X^* = \sigma(X)$. For example, if $X = 2$, then $X^* = 2^* = 3$.

We state the first lemma only for representatives under the automorphism group of $\mathcal{B}$.

Lemma 2.5 *(a) If* $2 \in \mathcal{A}$ *, or if* $12 \in \mathcal{A}$*, then* $S - 2^* = S - 3 \in \mathcal{A}$*, and* $S - \{6,7\}, S - \{4,5\}, S - \{4,5,6,7\} \in \mathcal{A}$*, and* $S - \{2^*,4,5\}, S - \{2^*,6,7\} \in \mathcal{A}$.

(b) If $24 \in \mathcal{A}$*, then* $S - 2^*, S - 4^*, S - \{2^*,4^*\} \in \mathcal{A}$ *and also* $S - \{4^*,6,7\}, S - \{2^*,6,7\}, S - \{67\} \in \mathcal{A}$.

(c) If $234 \in \mathcal{A}$*, then* $S - 4^*, S - \{4^*,6,7\}, S - \{6,7\} \in \mathcal{A}$.

(d) If $246 \in \mathcal{A}$*, then* $\mathcal{A}$ *contains all of the following sets:*
$S - 2^*, S - 4^*, S - 6^*, S - \{4^*,6^*\}, S - \{2^*,6^*\}, S - \{2^*,4^*\}$

Proof: We are assuming $\mathcal{A} \uplus \mathcal{B} \subseteq \mathcal{A}$. All the results above follow from this. We do (a) as an example: Suppose $2 \in \mathcal{A}$. The members of $\mathcal{B}$ are 123, 145, 167, 12345, 12367, 14567, 1234567; any one of these that contains 3 also contains 2 and vice versa; only 145, 167, 14567 do not already contain 2, and so we get 1245, 1267, 124567 (without 3, and not in $\mathcal{B}$), and 123, 12345, 12367, 1234567 (with 3, and in $\mathcal{B}$). Since all members of $\mathcal{B}$ contain 1, we get the same list of sets if $12 \in \mathcal{A}$.

¿From this lemma, we see that for example, there is an injection from E_{12} to E_{61}, which implies that $|E_{61}| - |E_{12}| \geq 0$; and similarly for some of the other E_{ij}. The next lemma lists these injections.

Lemma 2.6 *There are injections from* E_{ij} *to* E_{rs} *for the following sets:*
(1) $E_{12} \longrightarrow E_{61}$
(2) $E_{21} \longrightarrow E_{61}$
(3) $E_{22} \longrightarrow E_{51}$
(4) $E_{23} \longrightarrow E_{52}$

(5) $E_{31} \longrightarrow E_{41}$
(6) $E_{32} \longrightarrow E_{52}$

There are other relations also among the E_{ij}; the next lemma lists the most useful of these.

Lemma 2.7 *(i) If* $1 \in \mathcal{A}$, *or if* $|E_{21}| \geq 4$, *then* $E_{32} \longrightarrow E_{42}$.

(ii) If $|E_{21}| = 3$ *then* $|E_{42}| - |E_{32}| \geq -1$; *if* $|E_{21}| = 2$ *then* $|E_{42}| - |E_{32}| \geq -2$; *and if* $|E_{21}| = 1$ *then* $|E_{42}| - |E_{32}| \geq -4$.

(iii) $|E_{41}| \geq 2\ max(|E_{12}|, |E_{21}|)$

(iv) If $|E_{31}| \geq 9$, *then* $|E_{53}| \geq 5$, *if* $|E_{31}| \geq 7$, *then* $|E_{53}| \geq 4$, *and if* $|E_{31}| \geq 5$, *then* $|E_{53}| \geq 1$.

(v) If $|E_{32}| \geq 3$, *then* $|E_{53}| \geq 1$; *if* $|E_{32}| = 5$, *then* $|E_{53}| \geq 3$; *if* $|E_{32}| = 6$, *then* $|E_{53}| \geq 4$; *and if* $|E_{32}| \geq 7$, *then* $|E_{53}| = 6$.

Proof: (i) Every possible member of E_{32} contains just one of 2, 3; one of 4 , 5; and one of 6, 7. Suppose e.g. 246 is in $\mathcal{A}$. Then if any of 1, 12, 14, 16 is also in $\mathcal{A}$, then $1246 \in \mathcal{A}$ also. If $|E_{21}| \geq 4$, then E_{21} must contain either both of 12, 13 or both or 14, 15, or both of 16,17, and so for every member X of E_{32}, one of the pairs in E_{21} has a common element with X, so that $\{1\} \cup X \in E_{42}$.

(ii) Every possible member of E_{32} contains just one of 2, 3; one of 4 , 5; and one of 6, 7. Then given any three 2-sets $1a, 1b, 1c$, there is at most one member of E_{32} which does not contain one of a, b, c. Given $1a, 1b$, there are at most two members of E_{32} which do not contain one of a, b, and given just $1a$, there are at most four members of E_{32} which do not contain a.

(iii) follows from Lemma 2.5.

(iv) We prove only the second statement; the others are similar. If E_{31} contains four 3-sets with a common pair (one of 23, 45, 67) then $|E_{53}| \geq 4$. If not, and if $|E_{31}| = 7$, then E_{31} must contain three 3-sets with a common pair, say 234, 235, 236, and either three 3-sets with common pair 45 and one 3-set with 67, or vice versa, or two 3-sets with common pair 45 and two with common pair 67. Checking all possibilities, we find that $|E_{53}| \geq 4$.

(v) In the set $T = \{246, 247, 256, 257, 346, 347, 356, 357\}$, each pair appears exactly twice; of any three of these sets, two of them will have a common element but not a common pair, and their union is a 5-set. Each possible member of E_{53} can be written as a union of two of the members of E_{32}, in two different ways, e.g. 34567 is the union of 346 and 357, and also of 356, 347. Taking out, say, 356, 357 we lose 34567, and also 23567 (since the remaining members containing 3 are 346, 347, both of which contain 4), but we have all the rest, so that any six sets in T will generate at least four 5-sets in E_{53}. The rest of (v) is done similarly.

Using these lemmas, we can show that S_1, S_2, S_4 are all non-negative, and get partial results for S_2.

Theorem 2.8 $S_1 \geq 0$.

Proof:
We have $|E_{51}| \geq |E_{22}|$ and $|E_{52}| \geq |E_{23}|$, from Lemma 2.6.
If $1 \notin \mathcal{A}$, or if $234567 \in \mathcal{A}$, then $S_1 \geq (1/4)(|E_{62}| - |E_{11}| \geq 0$.
If $1 \in \mathcal{A}$, and $234567 \notin \mathcal{A}$, then $\mathcal{A}$ contains $\mathcal{B}$ (so $|E_{51}| = 3$), but cannot contain all of 23, 45, 67, and then $|E_{51}| - |E_{22}| \geq 1$, and $S_1 \geq 0$.

Theorem 2.9 *If* $|E_{21}| = 5, 6$ *then* $S_2 \geq 0$.

Proof: We have $S_2 = (1/8)[|E_{41}| + |E_{42}| + |E_{53}| - |E_{21}| - |E_{31}| - |E_{32}|]$. From Lemma 2.6 and Lemma 2.7, we have $|E_{41}| - |E_{31}| \geq 0$, and $|E_{42}| - |E_{32}| \geq 0$.
Suppose $|E_{21}| = 5$. Then $|E_{41}| \geq 10$ by Lemma 2.7. If $|E_{31}| \leq 5$, then $S_2 \geq 0$. If $|E_{31}| = 6$, then $|E_{53}| \geq 1$ and $S_2 \geq (1/8)(10 - 6 + 1 - 5) \geq 0$. If $|E_{31}| = 7, 8$, then $|E_{53}| \geq 4$, and $S_2 \geq (1/8)(10 - 8 + 4 - 5) \geq 0$. If $|E_{31}| = 9, 10$, then $|E_{53}| \geq 5$, and $S_2 \geq (1/8)(10 - 10 + 5 - 5) \geq 0$. Finally, if $|E_{31}| = 11, 12$, then $|E_{53}| = 6$ and $S_2 \geq 0$.
The proof for $|E_{21}| = 6$ is similar.

Theorem 2.10 $S_3 \geq 0$.

Proof: This follows from Lemma 2.6; we have $|E_{61}| - |E_{12}| \geq 0$.

Theorem 2.11 $S_4 \geq 0$.

Proof: Our assumptions on $\mathcal{A}$ imply that $1234567 \in \mathcal{A}$, and so $|E_7| - |E_0| \geq 1 - |E_0| \geq 0$.

In general, the sum S_2 does not have to be non-negative, but as we show in the next series of lemmas, the sum $S_1 + S_2$ is always non-negative. Each lemma deals with one value of $|E_{53}|$. In view of Theorem 2.9, we need only consider values of $|E_{21}|$ which are no more than four.

Lemma 2.12 *If* $|E_{53}| = 6$, *then* $S_1 + S_2 \geq 0$.

Proof: We have $S_2 = (1/8)[(|E_{41}| - |E_{31}|) + (|E_{53}| - |E_{21}|) + (|E_{42}| - |E_{32}|)]$. Also $(|E_{41}| - |E_{31}|) \geq 0$ (from Lemma 2.6) and $(|E_{53}| - |E_{21}|) \geq 0$. If $|E_{21}| \geq 4$, then $(|E_{42}| - |E_{32}|) \geq 0$ by Lemma 2.7, and so $S_2 \geq 0$.
If $|E_{21}| = 3$, then $(|E_{53}| - |E_{21}|) = 3$ and $(|E_{42}| - |E_{32}|) \geq -1$ (by Lemma 2.7), so $S_2 \geq 0$.

The argument is similar if $|E_{21}| = 1, 2$.

Now suppose $|E_{21}| = 0$. If $1 \in \mathcal{A}$, then $(|E_{42}| - |E_{32}|) \geq 0$ and $S_2 \geq 0$.

Since $|E_{53}| = 6$, then $|E_{62}| = 1$, and so if $1 \notin \mathcal{A}$, then $S_1 \geq 1/4$. Also, $|E_{53}| - |E_{32}| \geq (6 - 8) = -2$, so $S_2 \geq (1/8)(-2) = -1/4$, and then $S_1 + S_2 \geq 0$.

Lemma 2.13 *If* $|E_{53}| = 5$, *then* $S_1 + S_2 \geq 0$.

Proof: Suppose $\mathcal{A}$ contains all but 23456 of the 5-subsets of 234567. Then $\mathcal{A}$ can contain at most one 4-subset of 23456, and at most four 3-subsets of 23456; thus $|E_{31}| + |E_{32}| \leq 14$, and $|E_{31}| \leq 10$ and $|E_{32}| \leq 6$.

If $|E_{21}| = 4$ then $(|E_{41}| - |E_{31}|) \geq 0$, $(|E_{53}| - |E_{21}|) \geq 0$ and $(|E_{42}| - |E_{32}|) \geq 0$, so $S_2 \geq 0$.

If $|E_{21}| = 1, 2, 3$, the argument is similar to that in Lemma 2.12.

Suppose $|E_{21}| = 0$. If $1 \in \mathcal{A}$ then $(|E_{42}| - |E_{32}|) \geq 0$ and $S_2 \geq 0$.

If $1 \notin \mathcal{A}$, since we do have that $\mathcal{A}$ contains 234567, we get $S_1 \geq 1/4$. Since $|E_{32}| \leq 6$, then $S_2 \geq (1/8)(-6 + 5) \geq -1/8$, and $S_1 + S_2 \geq 0$.

Lemma 2.14 *If* $|E_{53}| = 4$, *then* $S_1 + S_2 \geq 0$.

Proof: If $\mathcal{A}$ is missing two of the 5-subsets of 234567, then $|E_{31}| + |E_{32}| \leq 11$, $|E_{31}| \leq 8$, and $|E_{32}| \leq 6$.

If $1 \in \mathcal{A}$, or if $|E_{21}| = 4$, then $|E_{42}| \geq |E_{32}|$ and $S_2 \geq 0$. So we suppose that $1 \notin \mathcal{A}$, and $|E_{21}| \leq 3$. Since $|E_{53}| = 4$, then $234567 \in \mathcal{A}$, and $S_1 \geq 1/4$.

If $|E_{21}| = 3$, then by Lemma 2.7, $|E_{42}| - |E_{32}| \geq -1$, and $|E_{53}| - |E_{21}| = 1$, and $S_2 \geq 0$.

If $|E_{21}| = 2$, then $|E_{42}| - |E_{32}| \geq -2$, and $|E_{53}| - |E_{21}| = 2$, and $S_2 \geq 0$.

If $|E_{21}| = 1$, then $|E_{42}| - |E_{32}| \geq -4$, and $|E_{53}| - |E_{21}| = 3$, and $S_2 \geq -1/8$, so $S_1 + S_2 \geq 0$.

If $|E_{21}| = 0$, then $|E_{42}| - |E_{32}| \geq -6$, and $|E_{53}| - |E_{21}| = 4$; then $S_2 \geq -1/4$, so $S_1 + S_2 \geq 0$.

Lemma 2.15 *If* $|E_{53}| = 3$, *then* $S_1 + S_2 \geq 0$.

Proof: If $|E_{53}| = 3$, then $|E_{31}| + |E_{32}| \leq 10$, $|E_{31}| \leq 6$ (by Lemma 2.7 (iv)) and $|E_{32}| \leq 5$.

If $|E_{21}| \geq 4$, then $|E_{42}| - |E_{32}| \geq 0$,and by Lemma 2.7 (iii) , we get $S_2 \geq (1/8)(2|E_{21}| - 6 + 3 - |E_{21}|) \geq 0$.

If $|E_{21}| \leq 3$ and $1 \in \mathcal{A}$, then $S_2 \geq |E_{53}| - |E_{21}| \geq 0$.

If $|E_{21}| \leq 3$ and $1 \notin \mathcal{A}$, then $S_1 \geq 1/4$, and (applying Lemma 2.7 (ii)) $S_2 \geq -1/4$, so $S_1 + S_2 \geq 0$.

Lemma 2.16 *If* $|E_{53}| = 2$, *then* $S_1 + S_2 \geq 0$.

Proof: If $|E_{53}| = 2$, then $|E_{31}| + |E_{32}| \leq 10$, $|E_{31}| \leq 6$ (by Lemma 2.7 (iv)) and $|E_{32}| \leq 4$.

If $|E_{21}| \geq 4$, then $|E_{42}|-|E_{32}| \geq 0$,and $S_2 \geq (1/8)(2|E_{21}|-6+2-|E_{21}|) \geq 0$. So we assume that $|E_{21}| \leq 3$.

Suppose first that $|E_{22}| = 3$. Then $|E_{32}| = 0$ and $|E_{31}| \leq 4$. (To see this, for example, if $\mathcal{A}$ contains 246, 23, 45, 67, then it also contains 23456, 23467, and 24567.)

Then if $|E_{21}| = 3$, we have $S_2 \geq (1/8)((6-4)+(2-3)+0 \geq 0$, and if $|E_{21}| \leq 2$, $S_2 \geq 0+(2-|E_{21}|)+0 \geq 0$.

Now suppose that $|E_{22}| \leq 2$. If $1 \in \mathcal{A}$ then $S_1 \geq 1/4$, and if $1 \notin \mathcal{A}$, then $S_1 \geq 1/2$.

If $1 \in \mathcal{A}$, then $S_2 \geq (1/8)(|E_{53}|-|E_{21}|) \geq -1/8$, and so $S_1+S_2 \geq 0$.

$1 \notin \mathcal{A}$ then $S_1 \geq 1/2$, and by Lemma 2.7 (ii), we have the following.

If $E_{21} = 3$, then $S_2 \geq (1/8)(0+(2-3)+(-1) = -1/4$; if $E_{21} = 2$, then $S_2 \geq (1/8)(0+(2-2)+(-2) = -1/4$; if $E_{21} = 1$, then $S_2 \geq (1/8)(0+(2-1)+(-4) = -3/8$; if $E_{21} = 0$, then $S_2 \geq (1/8)(0+(2-0)+(-4) = -1/4$; and so in all cases $S_1+S_2 \geq 0$.

Lemma 2.17 *Suppose that* $1 \in \mathcal{A}$, *and* $234567 \notin \mathcal{A}$ *and* $|E_{53}| = 1$. *Then* $|E_{41}|-|E_{31}|-|E_{21}| \geq -1$ *and* $S_2 \geq 0$.

Proof: Suppose $\mathcal{A}$ contains 23456. Then (since $234567 \notin \mathcal{A}$) any set X in $\mathcal{A}$ which contains 7, must also contain 1. In particular, E_{31} and E_{32} consist of 3-subsets of 23456.

Since E_{31} must be contained in the set of 3-subsets of 23456 containing one of 23, 45, and since $1 \in \mathcal{A}$, E_{41} contains one set matching each member of E_{31}; none of these contain 7. For every singleton j or doubleton $1j$ in $\mathcal{A}$, if $j \neq 6,7$, we will also get the 4-set $1j67$. If $17 \in \mathcal{A}$, we also get the 4-sets 1237, 1457.

Thus $|E_{41}|-|E_{31}| \geq |E_{21}-\{16\}| \geq |E_{21}|-1$. Since $1 \in \mathcal{A}$, we have $|E_{42}|-|E_{32}| \geq 0$, and then $S_2 \geq (1/8)(-1+0+1) = 0$.

Lemma 2.18 *If* $|E_{53}| = 1$, *then* $S_1+S_2 \geq 0$.

Proof: If $|E_{53}| = 1$, then $|E_{31}|+|E_{32}| \leq 10$, $|E_{31}| \leq 6$ (by Lemma 2.7 (iv)) and $|E_{32}| \leq 4$.

First suppose that $E_{22} = 3$. Then $|E_{31}| \leq 2$ and $|E_{32}| = 0$, as in the preceding Lemma. So $S_2 \geq (1/8)[|E_{41}|+|E_{42}|+1-|E_{21}|-2]$. If $|E_{21}| \geq 1$, then since $|E_{41}| \geq 2|E_{21}|$, we have $S_2 \geq 0$. If $|E_{21}| = 0$, then $S_2 \geq (1/8)[|E_{41}|-|E_{31}|+|E_{42}|+1] \geq 0$.

Now suppose that $E_{22} \leq 2$. By Lemma 2.17, we may assume that either $1 \notin \mathcal{A}$, or $1, 234567 \in \mathcal{A}$; in either case, $S_1 \geq 1/4$.

If $1 \in \mathcal{A}$, then for $|E_{21}| \geq 4$ we have $S_2 \geq (1/8)(2|E_{21}|-6+1-|E_{21}| \geq -1/8$, and for $|E_{21}| \leq 3$, $S_2 \geq (1/8)(1-|E_{21}| \geq -1/4$. So $S_1+S_2 \geq 0$.

So, suppose $1 \notin \mathcal{A}$. If $|E_{21}| \geq 4$, then $|E_{42}|-|E_{32}| \geq 0$, and the argument above gives $S_2 \geq -1/8$.

For $|E_{21}| \leq 3$, using Lemma 2.7(ii), we can get $S_2 \geq -1/2$, and since if $|E_{22}| \leq 1$ we would have $S_1 \geq 1/2$, we assume that $|E_{22}| = 2$. Suppose that $\mathcal{A}$

contains 23456, and 23, 45. Note that $E_{32} \cup E_{31}$ must be a subset of the set of 3-sets of 23456, and every member of E_{32} contains the element 6. As in Lemma 2.17, we have $|E_{41}| - |E_{31}| \geq |E_{21}| - 1$.

Let $|E_{21}| = 3$; so that $\mathcal{A}$ contains at least one of the pairs 12, 13, 14, 15. If $\mathcal{A}$ contains 12, then it also contains 1267, and 267 cannot be a member of E_{31} since $45 \in \mathcal{A}$; 267 would give 24567 in $\mathcal{A}$. So $|E_{41}| - |E_{31}| \geq 1$. Then $S_2 \geq (1/8)(1 + (-1) + (1 - 3)) \geq -1/4$, and so $S_1 + S_2 \geq 0$.

Let $|E_{21}| = 2$; so that $\mathcal{A}$ contains at least one of the pairs 12, 13, 14, 15, 16. If $\mathcal{A}$ contains the pair 16, then $|E_{42}| \geq |E_{32}|$ and then $S_2 \geq (1/8)(1-2) \geq -1/8$. Otherwise, as above, $|E_{41}| - |E_{31}| \geq 1$. Then $S_2 \geq (1/8)(1 + (-2) + (1 - 2)) \geq -1/4$. Either way, $S_1 + S_2 \geq 0$.

Let $|E_{21}| = 1$. If $\mathcal{A}$ contains 17, then E_{41} contains 1237, 1457, and neither 237 nor 457 is in E_{31}, and so $|E_{41}| - |E_{31}| \geq 2$ and $S_2 \geq (1/8)(2-4+0) = -1/4$. If $\mathcal{A}$ contains 16, then $|E_{42}| \geq |E_{32}|$ and $S_2 \geq 0$. If $\mathcal{A}$ contains 15, then $|E_{42}| - |E_{32}| \geq -2$ and $S_2 \geq (1/8)(0 - 2 + 0) = -1/4$. In all cases, $S_1 + S_2 \geq 0$.

Let $|E_{21}| = 0$. Then $S_2 \geq (1/8)(-|E_{32}| + 1) \geq -3/8$. Suppose first that $|E_{32}| = 4$. Then $E_{32} = \{246, 256, 346, 356\}$, and since $\mathcal{A}$ contains 123, 145, 167, it follows that $|E_{52}| \geq 8$ (E.g. from 246, we get 12346, 12456, 12467 in $|E_{52}|$.)

Thus, we now consider $|E_{52}| - |E_{23}|$. If $|E_{23}| \leq 7$, then $S_1 \geq 3/8$, and $S_1 + S_2 \geq 0$. If $|E_{23}| = 8$, then $\mathcal{A}$ contains all the 2-subsets of 23456, including 26, 36, 46, 56, and then $\mathcal{A}$ also contains 1267, 1367, 1467, 1567. Since none of 267, 367, 467, 567 are in E_{31}, then $|E_{41}| - |E_{31}| \geq 4$ and $S_2 \geq 0$.

Finally, if $|E_{32}| \leq 3$, then $S_2 \geq -1/4$, and $S_1 + S_2 \geq 0$.

This proves the result.

Lemma 2.19 *If* $|E_{53}| = 0$, *then* $S_1 + S_2 \geq 0$.

Proof: Note that if $\mathcal{A}$ contains 2 or 12, it also has 1245 and 1267; since $|E_{53}| = 0$, E_{31} cannot contain both 245 and 267; similarly for all values $j = 3, 4, 5, 6, 7$. That is, for every pair in E_{21}, E_{41} picks up a new (different) 4-set. Thus $|E_{41}| \geq |E_{31}| + |E_{21}|$. Then $S_2 \geq (1/8)(|E_{42}| - |E_{32}|)$.

Suppose first that $|E_{22}| = 3$. Then $|E_{32}| = |E_{31}| = 0$. (For instance, if $\mathcal{A}$ has 23, 45, 67 and 246 (resp. 234), then it also has 23456 (resp. 23467).) Then $S_2 \geq (1/8)(|E_{42}|) \geq 0$.

Next suppose $|E_{22}| = 2$; say $\mathcal{A}$ contains 23 and 45. Then any 3-set containing either 6 or 7 (or both) must also contain 1 (otherwise we would have $|E_{53}| \geq 1$), and so $|E_{32}| = 0$ and then $S_2 \geq 0$.

If $|E_{22}| \leq 1$, then $S_1 \geq 1/4$ and by Lemma 2.7, $|E_{32}| \leq 2$. Then $S_2 \geq (1/8)(|E_{42}| - |E_{32}|) \geq -1/4$, and $S_1 + S_2 \geq 0$.

Corollary 2.20 *The UC family generated by* $\{123, 145, 167\}$ *is an FC family.*

Theorem 2.21 *If a UC family $\mathcal{A}$ contains three 3-sets with a common element, then $\mathcal{A}$ is an FC family, and in particular the Frankl conjecture is satisfied for $\mathcal{A}$.*

Proof: Suppose that $\{1ab, 1cd, 1ef\}$ are in $\mathcal{A}$. In [3], we showed that if $|\{1, a, b, c, d, e, f\}| \leq 6$, then $\{1ab, 1cd, 1ef\}$ is an FC family, and from Corollary 2.20, the same is true if $|\{1, a, b, c, d, e, f\}| = 7$. Since the union of three 3-sets with a common element cannot involve more than a total of seven elements, then $\mathcal{A}$ contains an FC family, and hence itself is an FC family.

Corollary 2.22 *If $\mathcal{A}$ is a UC family in $\mathcal{P}(n)$ for which the Frankl conjecture is not satisfied, then $\mathcal{A}$ cannot contain more than $2n/3$ 3-sets.*

Proof: If the Frankl conjecture fails, then $\mathcal{A}$ cannot contain any FC families, and so no element can appear in more than two of the 3-sets of $\mathcal{A}$; the result follows.

3 A Possible Improvement

In [3], we described a computational method which produces, for any collection T of sets of cardinality at least 3, with $|\cup T| = n$, a certain number, called $NUM(T)$. We have found (among other things) that if $4 \leq |\cup T| \leq 6$ and $NUM(T) \leq 1$, then T generates an FC family, and in the preceding section we found this also to be true for $T = \{123, 145, 167\}$. However, we have not yet verified this for any other collections T.

In view of the possible connection between the value of $NUM(T)$, and whether or not T generates an FC family, it is interesting to consider the behavior of those T which do (or don't) have $NUM(T) \leq 1$. So far, we have enough calculations for the case when T consists entirely of 3-sets, to make some general statements.

Definition 3.1 *Let T be a collection of 3-sets, with $\cup T = \{1, 2, \ldots, n\}$. We say that T is* **proper** *if T does not properly contain any collection S with $NUM(S) \leq 1$. Let $Y(n,3)$ be the set of all collections T in $\mathcal{P}(n)$ such that for some $S \subseteq T$, $NUM(S) \leq 1$, and let $O(n,3)$ be the set of all proper collections T in $\mathcal{P}(n)$ such that $NUM(T) > 1$. Finally, let $M(n,3)$ be the largest integer k such that $O(n,3)$ contains a collection T with $|T| = k$.*

We will show that $M(n,3) \leq n/2$. For this, we need the lists of sets in $O(n,3)$ for $4 \leq n \leq 9$. (We give only the isomorphism types of these sets.)

$O(4,3)$: $T_4 = \{123, 124\}$
$O(5,3)$: $T_5 = \{123, 145\}$
$O(6,3)$: $T_6 = \{123, 124, 356\}$
$O(7,3)$: $T_{71} = \{123, 124, 567\}$; $T_{72} = \{123, 145, 267\}$
$O(8,3)$: $T_{81} = \{123, 124, 567, 568\}$; $T_{82} = \{123, 124, 356, 678\}$;
$O(9,3)$: $T_{91} = \{123, 124, 567, 789\}$; $T_{92} = \{123, 145, 267, 789\}$;
$T_{93} = \{123, 124, 356, 789\}$

Lemma 3.2 *Let* $T = \{123, 124, 356, 678, abc\}$, *and* $|\cup T| = n$. *If any of* a, b, c *is in* $\{1,2,3,4,5,6,7,8\}$, *then* T *is in* $Y(n,3)$.

Proof: Let t_i denote the number of members of T containing i. We have $t_1 = t_2 = t_3 = t_6 = 2$, so if any of a, b, c is in $\{1,2,3,6\}$, we would have a proper subset S of T with $NUM(S) \leq 1$. If (say) $a = 4$, then $S = \{123, 124, 356, 4bc\}$ has $|\cup S| = j \leq 7$, and none of the members of $O(j,3)$ has this form, so $S \in Y(j,3)$. If two of a, b, c are ≤ 8, so that $|\cup S| = j \leq 9$, then again T contains a proper subset S with $NUM(S) \leq 1$. So the only possible values for a, b, c (up to isomorphism) are $5, 9, 10$, and $7, 9, 10$, and $8, 9, 10$, and for each of these we compute $NUM(T) \leq 1$.

Corollary 3.3 *If* $T \in O(n,3)$ *and if* $T = S \cup T_{82}$, *then no member of* S *contains any of* $\{1,2,3,4,5,6,7,8\}$.

Corollary 3.4 *If* $T \in O(n,3)$ *and if* $T = \{123, 124, 356\} \cup S$, *then* $t_4 = 1$, *and either* $t_5 = 1$ *or* $t_6 = 1$.

Lemma 3.5 *Suppose* $T \in O(n,3)$ *and* T *contains* 123, *and* $t_1 = t_2 = t_3 = 2$. *Then* T *contains (an isomorphic copy of)* T_6.

Proof: Suppose first that T also contains some $12a$. Since $t_3 = 2$, there must be some set $3bc \in T$; then $S = \{123, 12a, 3bc\}$ has $|\cup S| \leq 6$. Since we assumed $T \in O(n,3)$, S must be isomorphic to T_6.

Now suppose that none of $1, 2, 3$ appear together in a member of T. We may assume that T contains $S = \{123, 145, 2ab, 3cd\}$. If any of a, b, c, d is in $\{1,2,3,4,5\}$, we would have $|\cup S| \leq 8$, and none of the T_{ij} for $i \leq 8$ are isomorphic to S, so that $NUM(S) \leq 1$, a contradiction. So it must be that a, b, c, d are distinct, and $|\cup S| = 9$. Of the three T_{9j}, only one of them contains a 3-set abc such that $t_a = t_b = t_c = 2$, and that one is T_{93}, which contains T_6.

Theorem 3.6 *For* $n \geq 4$, $M(n,3) \leq n/2$.

Proof: The proof is by induction on n. The statement is true for $4 \leq n \leq 9$, by computation, and we assume it is true for all j, $7 \leq j \leq n-1$.

Let $T \in O(n,3)$, and $|T| = k$. Let a be the number of integers i such that $t_i = 2$, and b the number of i such that $t_i = 1$. Then (since $|\cup T| = n$) $a + b = n$, and $2a + b = 3k$, and so $a = 3k - n$ and $b = 2n - 3k$. If $b \geq k$, then $b = 2n - 3k \geq k$ gives $2n \geq 4k$, so that $k \leq n/2$.

Now suppose that $b < k$. If $t_i = 1$, then i appears in only one of the 3-sets of T; then at least one 3-set in T, say 123, has $t_1 = t_2 = t_3 = 2$, and so by Lemma 3.5, we may assume without loss of generality, that T contains (up to isomorphism) $123, 124, 356$, and $t_4 = t_5 = 1$.

Suppose that n is even. Let $S = T - \{123, 124\}$. Then $|\cup S| \leq n - 3$ (we have removed 1,2,4), and $|S| = k - 2$. Then $k - 2 \leq M(n-3, 3)$, so that by the inductive hypothesis, $k - 2 \leq (n-3)/2$, that is, $k \leq (n+1)/2$. Since n is even, then we get $k \leq n/2$, as required.

If n is odd, consider the two cases: $t_6 = 1$, and $t_6 = 2$. We already have $t_5 = 1$. If $t_6 = 1$, then let $S = T - \{356\}$. Then $|\cup S| \leq n - 2$, and $|S| = k - 1$, and as above, $k - 1 \leq (n-2)/2$, and $k \leq n/2$. If $t_6 = 2$, then T must contain (up to isomorphism) $123, 124, 356, 678$, which is T_{82}. In particular, by Corollary 3.3, $t_7 = t_8 = 1$. Then if we put $S = T - \{678\}$, the same argument applies, so that $k \leq n/2$ in both cases.

If we knew that every $T \in Y(n,3)$, $7 \leq n \leq 10$, were an FC family, it would follow that a UC family $\mathcal{A}$ with more than $n/2$ 3-sets satisfies the Frankl conjecture. However, direct verification is extremely tedious even for the smallest values of n, so it is to be hoped that some other proof will be forthcoming.

References

[1] Bjorn Poonen, *Union-Closed Families* , J. Combinatorial Theory, Series A 59 (1992) 253-268.

[2] Theresa P. Vaughan, *Families Implying the Frankl Conjecture* (to appear, Eur. J. Comb.)

[3] Theresa P. Vaughan, *A Note on the Frankl Conjecture* (to appear, J. Comb. Math. and Comb. Computing)