Tournaments with Feedback Path Powers

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Abstract

A minimum feedback arc set of a digraph is a smallest sized set of arcs whose reversal makes the resulting digraph acyclic. Given an acyclic digraph D, we seek a smallest sized tournament T having $A(D)$ as a minimum feedback arc set. The reversing number of a digraph D equals $|V(T)| - |V(D)|$. We investigate the reversing number of the kth power of directed Hamiltonian path P_n^k, when k is fixed and n tends to infinity. We show that even for small values of k, where $\left|A\left(P_n^k\right)\right|$ is much closer to $|A(P_n)|$ than $|A(T_n)|$, the opposite relationship holds for the reversing number.

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1 Introduction

A *tournament* is a digraph whose underlying undirected graph is complete. A *minimum feedback arc set* of a digraph is a smallest sized set of arcs whose reversal makes the resulting digraph acyclic. Given an acyclic digraph D, we seek a smallest sized tournament T having $A(D)$ as a minimum feedback arc set. Barthélemy et al. [2] showed that every acyclic digraph D arises as a minimum feedback arc set of some tournament T and defined the *reversing number of a digraph* to be $r(D) = |V(T)| - |V(D)|$. Reversing numbers have been studied for many classes of digraphs [2], [3], [4], [5], [6], and [7].

We investigate the reversing number of the k-th power of a directed Hamiltonian path, denoted P_n^k. Recall P_n^k is the digraph containing the directed Hamiltonian path with vertices $v_1, ..., v_n$ and arcs v_iv_j where $i > j$ and $|i - j| \le k$. Note that in our statement of the definition if vertices $v_1, ..., v_n$ are ordered from left to right, the arcs of P_n^k are directed from right to left. This arrangement will be convenient since $A\left(P_n^k\right)$ will be the set of feedback arcs.

It is helpful to consider minimum feedback arc sets in the context of player rankings. Given a ranking of players an *inconsistency* arises whenever one player defeats another and the winner is ranked below the loser. Determining the size of a minimum feedback arc set is then equivalent to finding a player ranking that minimizes the number of inconsistencies. If players are ranked in this manner then $n + r(T_n)$ is the size of a smallest (round robin) tournament having an optimal ranking with a set of n players that are all ranked inconsistently with respect to each other. We seek $n+r(P_n^k)$ which is the size of a smallest tournament having an optimal ranking where most, but not all of the players are ranked inconsistently. An interesting property of the tournaments with $A\left(P_n^k\right)$ as a minimum feedback arc set is that the corresponding rankings do not have inconsistencies where the winner and loser are placed excessively far apart.

We continue by restating an elementary, but important result [2].

Lemma 1 *Let D and D' be digraphs on n vertices. Then $D' \subseteq D \Rightarrow r(D') \leq r(D)$.*

Reversing numbers have been investigated for extreme cases of weakly connected digraphs: directed paths and acyclic tournaments [2] and [7]. In particular it was established that $r(P_n) = n - 1$ and $2n - 4\log_2 n \leq r(T_n) \leq 2n - 4$. Note that P_n^k includes both directed paths and acyclic tournaments.

We investigate $r\left(P_n^k\right)$ for fixed values of k as n tends to infinity. By Lemma 1, $r(P_n) \leq r(P_n^k) \leq r\left(T_n\right)$ but it is not clear where $r(P_n^k)$ lies between these bounds. When k is much smaller than n, $\left|A\left(P_n^k\right)\right| - |A(P_n)| < |A(T_n)| - \left|A\left(P_n^k\right)\right|$ and when k is close to n the opposite inequality holds. It is tempting to think that the same relationship holds for the reversing number. However we show that suprisingly even for small values of k, $r(T_n) - r\left(P_n^k\right) < r\left(P_n^k\right) - r(P_n)$. This provides an easy construction of pairs of digraphs whose arc sets differ greatly in cardinality yet have little or no difference in their reversing numbers.

In the final section of the paper we present and discuss some related open problems.

2 Background

Values of $r(P_n^k)$ have been determined for $k \leq 7$ [1] and [4]. This is restated in our next lemma.

Lemma 2 *Let P_n^k denote the k-th power of the directed Hamiltonian path P_n. Then*

$$r(P_n) = n - 1,$$
$$r(P_n^2) = r(P_n^3) = \left\lceil \tfrac{3n-4}{2} \right\rceil,$$
$$r(P_n^4) = r(P_n^5) = \left\lceil \tfrac{5n-7}{3} \right\rceil, \text{ and}$$
$$r(P_n^6) = r(P_n^7) = \left\lceil \tfrac{7n-12}{4} \right\rceil.$$

These values suggest that $r\left(P_n^{2k}\right) = r(P_n^{2k+1})$ and can be approximated by $\left(\frac{2k+1}{k+1}\right)n$. We investigate this extension of Lemma 2. We obtain new results involving $r\left(P_n^k\right)$ by establishing new lower bounds for $r\left(P_n^k\right)$ and combining them with the best known upper bounds for tournaments. For the following bounds we will use $c_j(k)$ to denote a constant that is only dependent on k, the given power of the directed path. In particular we show $\left(\frac{2k-1}{k}\right)n - c_0(k) \leq r\left(P_n^{2k-1}\right) \leq 2n - 4$ and $\left(\frac{2k+1}{k+1}\right)n - c_1(k) \leq r(P_n^{2k}) \leq 2n - 4$ when $n \geq 4k$. Note that in both cases the lower and upper bounds follow the same asymptotic behavior.

As mentioned earlier, $r(T_n)$ was studied by Isaak [3]. Because P_n^k contains subtournaments we can directly apply known methods used to study tournaments to investigate $r\left(P_n^k\right)$. In the next example we illustrate methods used to obtain lower bounds for the reversing number of tournaments [3].

Example 3

Let T_3 denote the acyclic tournament on three players as shown in Figure 1. In the context of player rankings $r(T_3)$ represents the minimum number of extra players that are needed to obtain a larger tournament with an optimal ranking where the original three players are all ranked inconsistently with respect to each other.

In general we will add extra players that defeated at least one of the original players or lost to one of the original players. The reasoning is that extra players that defeated all of the original players or lost to all of the original players have no effect on the number of inconsistencies in an optimal ranking. That is when considering inconsistencies among the original players it does not help to add extra players that beat all of the original players or lost to all of the original players.

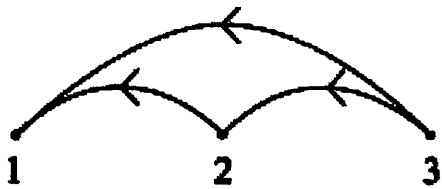

Figure 1. The acyclic tournament T_3.

We use U_1 to represent extra players that lost to player 1 but defeated players 2 and 3, and U_2 represents extra players that lost to players 1 and 2 but defeated player 3. This is shown in Figure 2 where the players are ranked from left to right. We let $x_i = |U_i|$ and seek to minimize $x_1 + x_2$ over all optimal rankings.

An optimal ranking must have at least one extra player ranked in between any two original players. If not we could simply interchange the ranks of the two original players and reduce the number of inconsistencies, contradicting the optimality. Hence $x_1 \geq 1$ and $x_2 \geq 1$. Then as mentioned above, any potentially optimal ordering must have the form $1U_1 2U_2 3$ as illustrated in Figure 2. Comparing this (optimal) ordering which has three inconsistencies with the ordering $U_1 321 U_2$ that has $x_1 + x_2$ inconsistencies we obtain the inequality $x_1 + x_2 \geq 3$. Hence $r(T_3) \geq 3$. ■

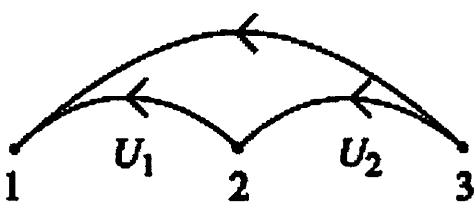

Figure 2. All arcs not drawn are directed from left to right.

Building upon ideas from the previous example, we continue with a detailed description of the methods used to investigate $r\left(P_n^k\right)$. We define a tournament $T\left(\sigma, \overrightarrow{x}, P_n^k\right)$ with minimum feedback arc set P_n^k, having extra players $\overrightarrow{x}$, and an ordering of the tournament vertices σ where each arc in P_n^k is directed from right to left.

As mentioned earlier any tournament $T\left(\sigma, \overrightarrow{x}, P_n^k\right)$ having $A\left(P_n^k\right)$ as a minimum feedback arc set must have its extra players $\overrightarrow{x}$ placed in between the ranks of the original players. To see this, note that any extra players that are placed ahead all of the original players or behind all of the original players may be removed without affecting a minimum feedback arc set of

$T(\sigma, \vec{x}, P_n^k)$. Let $V(P_n^k) = \{v_1, v_2, ..., v_n\}$ and $A(P_n^k) = \{(v_j, v_i) \mid i < j\}$. Then $V(T(\sigma, \vec{x}, P_n^k)) = V(P_n^k) \cup \{u_{i,j} \mid 1 \leq i \leq n-1, 1 \leq j \leq x_i\}$ and $A(T(\sigma, \vec{x}, P_n^k)) = A(P_n^k) \cup \{(u_{i,j}, u_{s,t}) : i < s \text{ or } i = s \text{ and } j < t\}$ $\cup \{(v_i, u_{s,t}) \mid i \leq s\} \cup \{(u_{i,j}, v_s) \mid i < s\}$. That is $V(T(\sigma, \vec{x}, P_n^k))$ is composed of $V(P_n^k)$ along with a set of extra vertices dependent upon P_n^k, and $A(T(\sigma, \vec{x}, P_n^k))$ is composed of arcs consistent with the ordering:

$$v_1, u_{1,1}, ..., u_{1,x_1}, v_2, u_{2,1}, ..., u_{2,x_2}, v_3, ..., v_{n-1}, u_{n-1,1}, ..., u_{n-1,x_{n-1}}, v_n$$

except for arcs between v_i and v_j, $i < j$, which are inconsistent with the ordering. Since we are concerned with the number of extra vertices between each v_i and v_{i+1} we may combine each set of x_i vertices into a single set U_i.

Then $r(P_n^k)$ equals the minimum value of $\sum_{i=1}^{n-1} x_i$ such that a tournament on $n + \sum_{i=1}^{n-1} x_i$ vertices has $A(P_n^k)$ as a minimum feedback arc set. We investigate inequalities involving the number of extra vertices specified by $\vec{x}$. In Example 3 we established $x_i \geq 1$ for $1 \leq i \leq n$ and $x_i + x_{i+1} \geq 3$, for $1 \leq i \leq n-1$. As we will see in our next lemma, the same technique can be used to study larger tournaments and establish bounds such as $x_i + 2x_{i+1} + x_{i+2} \geq 6$ when $1 \leq i \leq n-2$, and $x_i + 2x_{i+1} + 2x_{i+2} + x_{i+3} \geq 10$ when $1 \leq i \leq n-3$. The following known bound [3] can be applied to all subtournaments contained in P_n^k.

Lemma 4 *Let x_i be as defined above. Then $\sum_{i=1}^{\lceil \frac{t-1}{2} \rceil} i x_i + \sum_{i=\lceil \frac{t-1}{2} \rceil + 1}^{t-1} (t-i) x_i \geq \binom{t}{2}$.*

This bound is obtained by comparing the orderings $1, U_1, 2, ..., U_{t-1}, t$ and $U_1, \ldots, U_{\lfloor \frac{t}{2} \rfloor}, t, t-1, \ldots, 1, U_{\lfloor \frac{t}{2} \rfloor + 1}, ..., U_{t-1}$.

In our next example, we illustrate the main ideas that will be used later in the general case.

Example 5 *Let $D = P_{10}^5$.*

Using Lemma 4 we construct the inequalities:

$$\begin{array}{llllllllll}
x_1 & + 2x_2 + & 3x_3 + & 2x_4 + & x_5 & & & & & \geq 15 \\
& x_2 & + 2x_3 + & 3x_4 + & 2x_5 + & x_6 & & & & \geq 15 \\
& & x_3 & + 2x_4 + & 3x_5 + & 2x_6 + & x_7 & & & \geq 15 \\
& & & x_4 & + 2x_5 + & 3x_6 + & 2x_7 + & x_8 & & \geq 15 \\
& & & & x_5 & + 2x_6 + & 3x_7 + & 2x_8 + & x_9 & \geq 15 \\
x_1 & + 2x_2 + & 2x_3 + & x_4 & & + x_6 & + 2x_7 + & 2x_8 + & x_9 & \geq 20 \\
x_1 & + 2x_2 + & x_3 & & & & + x_7 & + 2x_8 + & x_9 & \geq 12 \\
2x_1 + & 2x_2 & & & & & & + 2x_8 + & 2x_9 & \geq 12 \\
4x_1 & & & & & & & & + 4x_9 & \geq 8
\end{array}$$

Summing these inequlaties yields $9\sum_{i=1}^{9} x_i \geq 127 \Rightarrow \sum_{i=1}^{9} x_i \geq \lceil \frac{127}{9} \rceil = 15$. Hence $r\left(P_{10}^5\right) \geq 15$. The combination of this lower bound with a known upper bound [3] yields $r\left(P_{10}^5\right) \leq r(T_{10}) = 15$.

Next, we take a closer look at the inequalities from our previous example. We will show how the inequalities used to establish one lower bound can be extended to obtain lower bounds for larger digraphs. For example the inequalities used to obtain $r\left(P_{10}^5\right) \geq 15$ can be used to establish a lower bound for $r\left(P_{11}^5\right)$. We add the inequality $x_6 + 2x_7 + 3x_8 + 2x_9 + x_{10} \geq 15$ and shift the indicies of inequalities in the lower right of the table to obtain a new set of inequalities:

$$\begin{array}{lllllllllll}
x_1 & + 2x_2 + & 3x_3 + & 2x_4 + & x_5 & & & & & & \geq 15 \\
& x_2 & + 2x_3 + & 3x_4 + & 2x_5 + & x_6 & & & & & \geq 15 \\
& & x_3 & + 2x_4 + & 3x_5 + & 2x_6 + & x_7 & & & & \geq 15 \\
& & & x_4 & + 2x_5 + & 3x_6 + & 2x_7 + & x_8 & & & \geq 15 \\
& & & & x_5 & + 2x_6 + & 3x_7 + & 2x_8 + & x_9 & & \geq 15 \\
& & & & & x_6 & + 2x_7 + & 3x_8 + & 2x_9 + & x_{10} & \geq 15 \\
x_1 & + 2x_2 + & 2x_3 + & x_4 & & & + x_7 & + 2x_8 + & 2x_9 + & x_{10} & \geq 20 \\
x_1 & + 2x_2 + & x_3 & & & & & + x_7 & + 2x_9 + & x_{10} & \geq 12 \\
2x_1 + & 2x_2 & & & & & & & + 2x_9 + & 2x_{10} & \geq 12 \\
4x_1 & & & & & & & & & + 4x_9 & \geq 8
\end{array}$$

Thus $r\left(P_{11}^5\right) = \sum_{i=1}^{10} x_i \geq \left\lceil \frac{127+\binom{6}{2}}{9} \right\rceil = 16$.

This extension is described in Figure 3.

(i) (ii)

Figure 3. Extending one lower bound to establish another lower bound.

The coefficients of each x_i stay fixed at 9. To see that the column sums stay fixed note that the position of the left triangular block in the table is unchanged, the second triangular block simply shifts to the right, and the new column has column sum 9. In general this property will hold whenever the power of the directed path is less than or equal to $\frac{n}{2}$, (that is when the two 'triangular blocks' do not overlap). Given a lower bound of the form $r\left(P_n^{2k}\right) \geq \left(\frac{2k+1}{k+1}\right) n - c_1(k)$ where $n \geq 4k$, we use the above methods to obtain lower bounds for all larger values of n. To simplify the proof of the general case we observe that the sum of the right side of the inequalities that correspond to the two triangular blocks, is a constant $c_j(k)$.

3 A general lower bound

We obtain lower bounds for $\sum_{i=1}^{n-1} x_i$ and $r\left(P_n^k\right)$ by summing a set of carefully chosen inequalities. We use methods from the previous section to formulate a general lower bound for $r\left(P_n^k\right)$ for all $n \geq 2k$.

In particular we show $r\left(P_n^{2k-1}\right) \geq \left(\frac{2k-1}{k}\right) n - c_0(k)$ for $n \geq 4k-2$ and $r(P_n^{2k}) \geq \left(\frac{2k+1}{k+1}\right) n - c_1(k)$ when $n \geq 4k$. An important property is that both lower bounds follow the same asymptotic behavior as the best known general upper bound for tournaments, $r(T_n) \leq 2n-4$ [2].

Theorem 6 *Let* $n \geq 4k-2$. *Then* $\left(\frac{2k-1}{k}\right) n - c_0(k) \leq r\left(P_n^{2k-1}\right) \leq 2n-4$.

Proof. Since the upper bound is taken directly from the upper bound for $r(T_n)$ we need only justify the lower bound. Let $n \geq 4k-2$. We consider

$$\sum_{t=0}^{n-2k}\left(\sum_{i=1}^{k} ix_{i+t} + \sum_{i=k+1}^{2k-1}(2k-i)x_{i+t}\right)$$

and

$$\sum_{i=1}^{2k-2} a_i(x_i + x_{n-i})$$

where the positive integers a_i satisfy:

$$\sum_{t=0}^{n-2k}\left(\sum_{i=1}^{k} ix_{i+t} + \sum_{i=k+1}^{2k-1}(2k-i)x_{i+t}\right) + \sum_{i=1}^{2k-2} a_i(x_i + x_{n-i}) = k^2\sum_{i=1}^{n-1} x_i.$$

Then combining the following two inequalities,

$$\sum_{t=0}^{n-2k}\left(\sum_{i=1}^{k} ix_{i+t} + \sum_{i=k+1}^{2k-1}(2k-i)x_{i+t}\right) \geq (n-2k+1)\binom{2k}{2}$$

and

$$\sum_{i=1}^{2k-2} a_i(x_i + x_{n-i}) \geq 2\sum_{i=1}^{2k-2} a_i$$

gives us

$$\begin{aligned}\sum_{t=0}^{n-2k}&\left(\sum_{i=1}^{k} ix_{i+t} + \sum_{i=k+1}^{2k-1}(2k-i)x_{i+t}\right)\\ &+\sum_{i=1}^{2k-2} a_i(x_i + x_{n-i}) = k^2\sum_{i=1}^{n-1} x_i\\ &\geq (n-2k+1)\binom{2k}{2} + 2\sum_{i=1}^{2k-2} a_i\end{aligned}$$

which implies

$$\sum_{i=1}^{n-1} x_i \geq \left\lceil \frac{1}{k^2}\left[n\binom{2k}{2} + (-2k+1)\binom{2k}{2} + 2\sum_{i=1}^{2k-2} a_i\right]\right\rceil$$

Since $a_i < k^2$ for all i it follows that

$$2\sum_{i=1}^{2k-2} a_i < 2(2k-2)(k^2) = 4k^3 - 4k^2.$$

Then $(-2k+1)\binom{2k}{2} + (4k^3-4k^2) = (-4k^3+4k^2-k) + (4k^3-4k^2)$
$= -k < 0$

$\Rightarrow (-2k+1)\binom{2k}{2} + 2\sum_{i=1}^{2k-2} a_i < 0.$

Since this last term only depends on k, we have

$$\begin{aligned}\sum_{i=1}^{n-1} x_i &\geq \left\lceil \frac{1}{k^2}\left[n\binom{2k}{2} + (-2k+1)\binom{2k}{2} + 2\sum_{i=1}^{2k-2} a_i\right]\right\rceil \\ &\geq \left(\frac{2k-1}{k}\right) n - c(k).\end{aligned}$$

Hence

$$r\left(P_n^{2k-1}\right) = \sum_{i=1}^{n-1} x_i \geq \left(\frac{2k-1}{k}\right) n - c(k).$$

■

The next result is an an analog of the previous theorem and is proved in a similar manner. For completeness we include the details.

Theorem 7 *Let* $n \geq 4k$. *Then* $\left(\frac{2k+1}{k+1}\right) n - c_1(k) \leq r\left(P_n^{2k}\right) \leq 2n-4.$

Proof. Again the upper bound is the best known upper bound for $r(T_n)$ so we need only show the lower bound. Let $n \geq 4k$. Consider

$$\sum_{t=0}^{n-2k-1}\left(\sum_{i=1}^{k} i x_{i+t} + \sum_{i=k+1}^{2k}(2k-i)x_{i+t}\right)$$

and

$$\sum_{i=1}^{2k-1} b_i(x_i + x_{n-i})$$

where each b_i satisfies

$$\sum_{t=0}^{n-2k-1}\left(\sum_{i=1}^{k} i x_{i+t} + \sum_{i=k+1}^{2k}(2k-i)x_{i+t}\right) + \sum_{i=1}^{2k-1} b_i(x_i + x_{n-i}) = (k^2+k)\sum_{i=1}^{n-1} x_i.$$

We then can combine

$$\sum_{t=0}^{n-2k-1}\left(\sum_{i=1}^{k} i x_{i+t} + \sum_{i=k+1}^{2k}(2k-i)x_{i+t}\right) \geq (n-2k)\binom{2k+1}{2}$$

and

$$\sum_{i=1}^{2k-1} b_i(x_i + x_{n-i}) \geq 2\sum_{i=1}^{2k-1} b_i$$

to get

$$\sum_{t=0}^{n-2k-1}\left(\sum_{i=1}^{k} ix_{i+t}+\sum_{i=k+1}^{2k}(2k-i)x_{i+t}\right)$$
$$+\sum_{i=1}^{2k-1} b_i(x_i+x_{n-i})=(k^2+k)\sum_{i=1}^{n-1} x_i$$
$$\geq (n-2k)\binom{2k+1}{2}+2\sum_{i=1}^{2k-1} b_i.$$

This gives the inequality

$$\sum_{i=1}^{n-1} x_i \geq \left\lceil \frac{1}{k^2+k}\left[n\binom{2k+1}{2}-2k\binom{2k+1}{2}+2\sum_{i=1}^{2k-1} b_i\right]\right\rceil.$$

Next observe $b_i < k^2+k$ for all i. Then the statements

$$2\sum_{i=1}^{2k-1} b_i < 2(2k-1)(k^2)=4k^3-2k^2$$

and

$$-2k\binom{2k+1}{2}=-4k^3-2k^2$$

can be combined to get

$$2\sum_{i=1}^{2k-1} b_i-2k\binom{2k+1}{2}<-4k^2<0.$$

Since $(-2k)\binom{2k+1}{2}+2\sum_{i=1}^{2k-1} b_i$ does not depend on n we have

$$\sum_{i=1}^{n-1} x_i \geq \left\lceil \frac{1}{k^2+k}\left[n\binom{2k+1}{2}-2k\binom{2k+1}{2}+2\sum_{i=1}^{2k-1} b_i\right]\right\rceil$$
$$\geq \left(\frac{2k+1}{k+1}\right)n-c_1(k).$$

Hence

$$r\left(P_n^{2k}\right)=\sum_{i=1}^{n-1} x_i \geq \left(\frac{2k+1}{k+1}\right)n-c_1(k).$$

■

The combination of Theorems 6 and 7 yields a general lower bound for $r\left(P_n^k\right)$ for all $n \geq 2k$.

4 Conclusion

We pose the following problem which would be an improvement of the results contained in this paper.

Problem 8 *Determine* $r(P_n^k)$ *for all values of* k *and* n.

This problem would require determination of $r(T_n)$, which is only known for relatively few values of n. In addition it is not known if the integer programming approach used in this paper can be used to obtain $r(P_n^k)$, even if $c_j(k)$ is explicitly determined.

We also present an interesting subproblem. The fact that $r(P_n^k)$ approaches $r(T_n)$ quickly as k nears n suggests the following problem.

Problem 9 *For given* n *determine the smallest* k *such that* $r(P_n^k) = r(T_n)$.

A partial solution is given in [5] for $n = 2^s - 2^t$ and $s - t \geq 2$, but still remains open for many values of n, for which $r(T_n)$ is known. This would be interesting because it would identify a set of arcs that can be removed from T_n without affecting the reversing number.

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