Forcing Full Domination in Graphs

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Abstract

For each vertex v in a graph G, let there be associated a particular type of a subgraph F_v of G. In this context, the vertex v is said to dominate F_v. A set S of vertices of G is called a full dominating set if every vertex of G belongs to a subgraph F_v of G for some $v \in S$ and every edge of G belongs to a subgraph F_w of G for some $w \in S$. The minimum cardinality of a full dominating set of G is its full domination number $\gamma_F(G)$. A full dominating set of G of cardinality $\gamma_F(G)$ is called a γ_F-set of G. We study three types of full domination in graphs: full star domination, where F_v is the maximum star centered at v, full closed domination, where F_v is the subgraph induced by the closed neighborhood of v, and full open domination, where F_v is the subgraph induced by the open neighborhood of v.

A subset T of a γ_F-set S in a graph G is a forcing subset for S if S is the unique γ_F-set containing T. The forcing full domination number of S in G is the minimum cardinality of a forcing subset for S, and the forcing full domination number $f_{\gamma_F}(G)$ of the graph G is the minimum forcing full domination number among all γ_F-sets of G. We present several realization results concerning forcing parameters in full domination.

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1 Introduction

A set S of vertices in a graph G is a *dominating set* for G if every vertex of G is either an element of S or is adjacent to an element of S. Thus, a vertex v in a dominating set of G is said to *dominate* itself as well as its neighbors. The minimum cardinality of a dominating set is the *domination number* $\gamma(G)$. A dominating set of cardinality $\gamma(G)$ is called a γ-set. A set S of vertices in a graph G is an *open dominating set* if every vertex of G is adjacent to at least one vertex of S. In this case, a vertex v in an open dominating set of G is said to *openly dominate* its neighbors but not itself. The minimum cardinality of an open dominating set is the *open domination number* $\gamma_t(G)$ of G. An open dominating set of cardinality $\gamma_t(G)$ is a γ_t-set for G. The open domination number is also referred to as the *total domination number*. A thorough treatment of domination in graphs can be found in the book by Haynes, Hedetniemi, and Slater [7].

In domination, a vertex dominates a set of vertices (according to some rule); while in covering, a vertex covers the edges incident with it. These concepts have been combined and extended to describe another variation of domination in [1]. For a graph G, let there be a function F that maps each vertex v of G into a particular subgraph F_v of G (or possibly $F_v = \emptyset$). Then the vertex v is said to *dominate* F_v. In this context, each vertex and each edge of F_v is said to be dominated by v. A set S of vertices of G is called a *full dominating set* if every vertex and every edge of G is dominated by some vertex of S; that is, every vertex of G belongs to a subgraph F_v of G for some $v \in S$ and every edge of G belongs to a subgraph F_w of G for some $w \in S$. For each full dominating set S of G and $v \in V(G) - S$, the set $S \cup \{v\}$ is also a full dominating set. If G has no isolated vertices, then we need only be concerned with each edge of G being dominated by some vertex of S. The minimum cardinality of a full dominating set of G is its *full domination number* (with respect to the function F) and is denoted by $\gamma_F(G)$. A full dominating set of G of cardinality $\gamma_F(G)$ is called a γ_F*-set* of G. Certainly, $\gamma_F(G)$ is defined for a graph G if and only if $V(G)$ is a full dominating set for G. The following three examples of full domination were studied in [1]:

1. *full star domination*, where F_v is the maximum star S_v centered at v;
2. *full closed domination*, where $F_v = \langle N[v] \rangle$ the subgraph induced by the closed neighborhood of v;
3. *full open domination*, where $F_v = \langle N(v) \rangle$, the subgraph induced by the open neighborhood of v.

For a graph G, the corresponding domination numbers are called the *full* *star domination number* $\gamma_{FS}(G)$, the *full closed domination number* $\gamma_{FC}(G)$, and the *full open domination number* $\gamma_{FO}(G)$ of G, respectively.

Let S be a full dominating set of a graph G with respect to a function F, where the subgraph F_v of G corresponds to $v \in S$. If $|S| = \gamma_F(G)$, then, of course, S is a γ_F-set. It may very well occur that G contains several γ_F-sets. However, there is always some subset T of S that determines S as the unique γ_F-set containing T. Such "forcing subsets" will be considered in this paper. More formally, a subset $T \subseteq S$ is called a *forcing subset* for S if S is the unique γ_F-set containing T. The *forcing full domination number* of S, denoted by $f_{\gamma_F}(S)$, is the minimum cardinality of a forcing subset for S. The *forcing full domination number* of the graph G is

$$f_{\gamma_F}(G) = \min\{f_{\gamma_F}(S)\},$$

where the minimum is taken over all γ_F-sets S in G. For every graph G, it follows that $f_{\gamma_F}(G) \leq \gamma_F(G)$. Forcing concepts have been studied previously for such diverse parameters as the chromatic number [4], the graph reconstruction number [6], and the domination number [2]. A survey of graphical forcing parameters is discussed in [5]. The following observation is useful.

Observation 1.1 *Let G be a graph. Then $f_{\gamma_F}(G) = 0$ if and only if G has a unique γ_F-set, $f_{\gamma_F}(G) = 1$ if and only if G has at least two γ_F-sets, one of which is a unique γ_F-set containing some element of S, and $f_{\gamma_F}(G) = \gamma_F(G)$ if and only if no γ_F-set of G is the unique γ_F-set containing any of its proper subsets.*

We illustrate these concepts with full star domination for the graph G of Figure 1. Since G has no isolated vertices, $\gamma_{FS}(G) = \alpha_o(G)$, the vertex covering number of G (the minimum number of vertices that cover all edges of G). A well-known theorem of Gallai [3] states that if G is a graph of order n without isolated vertices, then $\alpha_o(G) + \beta_o(G) = n$, where $\beta_o(G)$ is the vertex independence number of G. Therefore, $\gamma_{FS}(G) = n - \beta_o(G)$ for every graph G of order n without isolated vertices. Since the graph G of Figure 1 has order 6 and $\beta_o(G) = 2$, it follows that $\gamma_{FS}(G) = 4$.

Next we show that $f_{\gamma_{FS}}(G) = 2$. Every γ_{FS}-set in G is of the type $V(G) - I$, where I is a maximum independent set of vertices in G. Since the complement $\overline{G}$ of G is the path $P_6 : v, z, x, w, u, y$, it follows that G contains five γ_{FS}-sets, namely $S_1 = \{u, w, x, y\}$, $S_2 = \{u, v, w, y\}$, $S_3 = \{u, v, y, z\}$, $S_4 = \{v, x, y, z\}$, and $S_5 = \{v, w, x, z\}$. Since every vertex of G belongs to at least two distinct γ_{FS}-sets, $f_{\gamma_{FS}}(G) \geq 2$. Furthermore, since S_1 is the unique γ_{FS}-set of G containing the 2-element subset $\{u, x\}$, it follows that $f_{\gamma_{FS}}(S_1) = 2$ and $f_{\gamma_{FS}}(G) = 2$. On the other hand, S_2 is not a unique

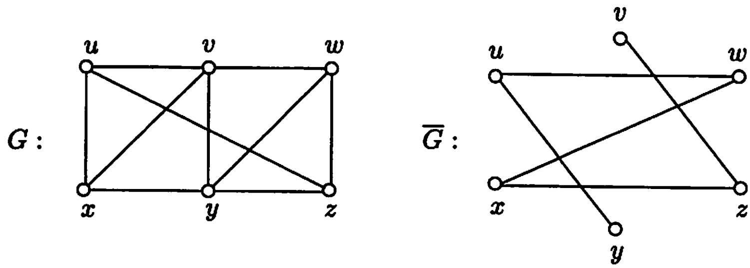

Figure 1: A graph G with $f_{\gamma_{FS}}(G) = 2$ and $\gamma_{FS}(G) = 4$

γ_{FS}-set containing any of its 2-element subset, but S_2 is the unique γ_{FS}-set containing the 3-element subset $\{u, v, w\}$; so $f_{\gamma_{FS}}(S_2) = 3$. It can be shown that $f_{\gamma_{FS}}(S_5) = 2$ and $f_{\gamma_{FS}}(S_3) = f_{\gamma_{FS}}(S_4) = 3$. Therefore, $f_{\gamma_{FS}}(G) = 2$.

We now review some of the results (see [1]) related to the full star domination number $\gamma_{FS}(G)$, the full closed domination number $\gamma_{FC}(G)$, and the full open domination number $\gamma_{FC}(G)$ of a graph G, beginning with $\gamma_{FS}(G)$. Since every full star dominating set of a graph is also a dominating set, it follows that $\gamma(G) \leq \gamma_{FS}(G)$ and so

$$1 \leq \gamma(G) \leq \gamma_{FS}(G) \leq n - 1$$

for every graph G of order n with at most $n-2$ isolated vertices. Certainly, $\gamma_{FS}(G) = n - 1$ if and only if $G = K_n$, which implies that $\gamma(G) = 1$. On the other hand, the independent domination number $i(G)$ satisfies

$$\gamma(G) \leq i(G) \leq \beta_o(G) = n - \gamma_{FS}(G)$$

for graphs with no isolated vertices. This implies that $\gamma(G) + \gamma_{FS}(G) \leq n$, thereby obtaining Ore's [10] well-known inequality $\gamma(G) \leq n/2$ for graphs G of order n without isolated vertices. The following realization result appeared in [1].

Theorem A *For every triple a, b, n of integers with $n \geq 3$, $1 \leq a \leq b \leq n - 2$, and $a + b \leq n$, there exists a graph G of order n without isolated vertices such that $\gamma(G) = a$ and $\gamma_{FS}(G) = b$.*

A set S of vertices in a graph G is a ***full closed dominating set*** if every vertex and every edge of G belongs to $\langle N[v] \rangle$ for some $v \in S$. The minimum cardinality of a full closed dominating set is the ***full closed domination number*** $\gamma_{FC}(G)$. A full closed dominating set of cardinality $\gamma_{FC}(G)$ is referred to as a *γ_{FC}-set.* This parameter was first introduced by Sampathkumar and Neeralagi in [9], where it was called the neighborhood number of a graph, and further studied by Jayaram, Kwong, and Straight in [8]. The following result appeared in [9].

Theorem B *For every graph G, $\gamma(G) \leq \gamma_{FC}(G) \leq \gamma_{FS}(G)$. Moreover, if G is a triangle-free graph, then $\gamma_{FC}(G) = \gamma_{FS}(G)$.*

If $\gamma(G) = 1$, then $\gamma_{FC}(G) = 1$ while $1 \leq \gamma_{FS}(G) \leq n-1$. For each integer k with $1 \leq k \leq n-1$, the graph H obtained by deleting the edges of a complete subgraph of order $n-k$ from K_n has $\gamma(H) = \gamma_{FC}(H) = 1$ and $\gamma_{FS}(H) = k$. For $\gamma(G) \geq 2$, the following realization result appeared in [8].

Theorem C *For every triple a, b, c of integers with $2 \leq a \leq b \leq c$, there exists a graph G with $\gamma(G) = a$, $\gamma_{FC}(G) = b$, and $\gamma_{FS}(G) = c$.*

A vertex v in a graph G ***openly dominates*** the subgraph $\langle N(v) \rangle$ induced by the (open) neighborhood $N(v)$ of v, but not v and any edge incident with v. A set S of vertices in G is a ***full open dominating set*** if every vertex and every edge of G belongs to $\langle N(v) \rangle$ for some $v \in S$. The minimum cardinality of a full open dominating set is the ***full open domination number*** $\gamma_{FO}(G)$. A full open dominating set of cardinality $\gamma_{FO}(G)$ is referred to as a *γ_{FO}-set.* Note that a graph G has a full open dominating set if and only if every edge of G lies on a triangle in G. It was shown in [1] that every vertex in a full open dominating set S in a graph G belongs to some triangle all of whose vertices belong to S. Thus every full open dominating set of a graph G must contain at least three vertices and so $\gamma_{FO}(G) \geq 3$. Certainly, every full open dominating set of a graph G is also a full closed dominating set and so $\gamma_{FO}(G) \geq \gamma_{FC}(G)$. Therefore, for a graph G in which every edge belongs to a triangle,

$$\gamma_{FO}(G) \geq \max\{3, \gamma_{FC}(G)\}.$$

The following result was established in [1].

Theorem D *For each pair a, b of integers with $1 \leq a \leq b$, there exists a connected graph G with $\gamma_{FC}(G) = a$ and $\gamma_{FO}(G) = b$ unless $(a, b) \in \{(1,1), (1,2), (2,2)\}$.*

Certainly, every full open dominating set of a graph G is also an open dominating set. Thus if G is a graph without isolated vertices in which every edge is in a triangle, then $\gamma_{FO}(G) \geq \gamma_t(G)$. In fact, more can be say as we state the following result which appeared in [1].

Theorem E *If G is a graph without isolated vertices in which every edge is in a triangle, then $\gamma_{FO}(G) > \gamma_t(G)$. Moreover, for every pair a, b of integers with $2 \leq a < b$, there exists a graph G with $\gamma_t(G) = a$ and $\gamma_{FO}(G) = b$.*

2 Forcing Full Domination Numbers of Graphs

For ordinary domination, the problem of determining all integers a, b with $0 \leq a \leq b$ for which there exists a graph G with $f_\gamma(G) = a$ and $\gamma(G) = b$ was solved in [2]. In this paper, we consider the corresponding problems for full star domination, full closed domination, and full open domination, beginning with star domination.

Proposition 2.1 *For every pair a, b of integers with $0 \leq a \leq b$ and $b \geq 1$, there exists a connected graph G with $f_{\gamma_{FS}}(G) = a$ and $\gamma_{FS}(G) = b$.*

Proof. For $a = 0$, let G be the complete bipartite graph $K_{b,b+1}$ with partite sets V_1 and V_2, where $|V_1| = b$ and $|V_2| = b+1$. Since V_1 is the unique γ_{FS}-set for G, it follows that $f_{\gamma_{FS}}(G) = 0$ and $\gamma_{FS}(G) = b$. For $a = 1$, let $G = K_{b,b}$ with partite sets V_1 and V_2. Then $\gamma_{FS}(G) = b$. Since G has two distinct γ_{FS}-sets, it follows that $f_{\gamma_{FS}}(G) \geq 1$. On the other hand, for every $u \in V(G)$, the partite set containing u is the unique γ_{FS}-set of G containing u. Thus, $f_{\gamma_{FS}}(G) = 1$.

We now assume that $a \geq 2$. If $a = b$, then let $G = K_{a+1}$. So $\gamma_{FS}(G) = a$. Since every γ_{FS}-set of G has the form $V(G) - \{v\}$ for some vertex v in G, it follows that $f_{\gamma_{FS}}(G) = a$ as well. Next assume that $a < b$. Now let the graph G be obtained from the graph K_{a+1} and the path P_{b-a} : $w_1 w_2 \cdots w_{b-a}$ by first adding the edge uw_1, where $u \in V(K_{a+1})$, and then adding $2(b-a)$ new vertices u_i, v_i $(1 \leq i \leq b-a)$ and the edges $u_i w_i$, $v_i w_i$ $(1 \leq i \leq b-a)$. The graph G is shown in Figure 2.

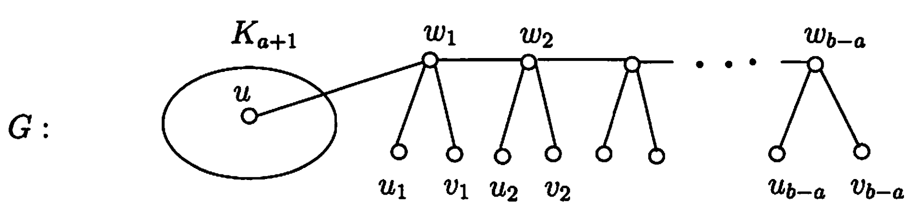

Figure 2: The graph G for $2 \leq a < b$

Every γ_{FS}-set of G has the form

$$(V(K_{a+1}) - \{v\}) \bigcup \{w_1, w_2, \cdots, w_{b-a}\},$$

where $v \in V(K_{a+1})$ and so $\gamma_{FS}(G) = b$. Since each γ_{FS}-set is uniquely determined by the subset $V(K_{a+1}) - \{v\}$ for some vertex v in K_{a+1}, it follows that $f_{\gamma_{FS}}(G) = a$. ■

Next we consider the forcing full closed domination and full closed domination numbers of a graph.

Theorem 2.2 *For every pair a, b of integers with $0 \le a \le b$ and $b \ge 1$, there exists a connected graph G with $f_{\gamma_{FC}}(G) = a$ and $\gamma_{FC}(G) = b$.*

Proof. For $a = 0$, let $G = K_{b,b+1}$; while for $a = 1$, let $G = K_{b,b}$. This gives us the desired results when $a \in \{0, 1\}$. For $2 \le a \le b$, we consider two cases.

Case 1. $2 \le a = b$. We construct a graph G from K_{a+1} as follows. For each edge e in K_{a+1}, we add two new vertices, each joined to the two incident vertices of e. The graph G for $a = 2$ is shown in Figure 3. Every γ_{FC}-set of G has the form $V(K_{a+1}) - \{v\}$ for some $v \in V(K_{a+1})$. Hence $f_{\gamma_{FC}}(G) = \gamma_{FC}(G) = a$.

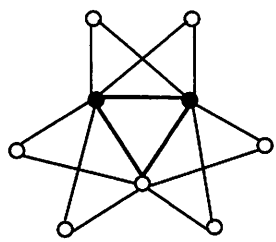

Figure 3: The graph G in Case 1 for $a = 2$

Case 2. $2 \le a < b$. We now construct a graph G' from the graph G defined in Case 1 and the path $P_{b-a} : w_1 w_2 \cdots w_{b-a}$ by first adding the edge uw_1, where $u \in V(K_{a+1})$ and then adding $2(b-a)$ new vertices u_i, v_i $(1 \le i \le b-a)$ and the edges $u_i w_i, v_i w_i$ $(1 \le i \le b-a)$. Every γ_{FC}-set of G has the form $(V(K_{a+1}) - \{v\}) \bigcup \{w_1, w_2, \cdots, w_{b-a}\}$. An argument similar to the one employed in the proof of Theorem 2.2 shows that $f_{\gamma_{FC}}(G) = a$ and $\gamma_{FC}(G) = b$. ■

Finally, we consider the forcing full open domination number and full open domination number of a graph. Recall that a graph G has a full open dominating set if and only if G contains no isolated vertices and every edge of G lies on a triangle. Moreover, every full open dominating set of a graph G must contain at least three vertices and so $\gamma_{FO}(G) \ge 3$. For the complete graph K_n, where $n \ge 4$, each 3-element subset of $V(K_n)$ is a γ_{FO}-set. Thus $f_{\gamma_{FO}}(K_n) = \gamma_{FO}(K_n) = 3$. In fact, every integer $b \ge 3$ is simultaneously realizable as the forcing full open domination number and full open domination number of some connected graph, as we show next.

Theorem 2.3 *For every integer $b \geq 3$, there exists a connected graph G with*

$$f_{\gamma_{FO}}(G) = \gamma_{FO}(G) = b.$$

Proof. Since $f_{\gamma_{FO}}(K_n) = \gamma_{FO}(K_n) = 3$ for each integer $n \geq 4$, we assume that $b \geq 4$. For each integer $b \geq 4$, we construct a graph G_b from the complete graph K_{b+1} as follows. For each 3-element subset T of $V = V(K_{b+1}) = \{1, 2, \cdots, b+1\}$, we add a new vertex T to K_{b+1} and join T to each vertex in T. Thus the order of G_b is $(b+1) + \binom{b+1}{3}$. Then $V(G_b) = V \cup U$, where

$$U = V(G_b) - V = \{T : T \subseteq V, |T| = 3\}.$$

The graph G_4 is shown in Figure 4, where each vertex $\{i, j, k\}$ in U (with $i < j < k$) is denoted by ijk.

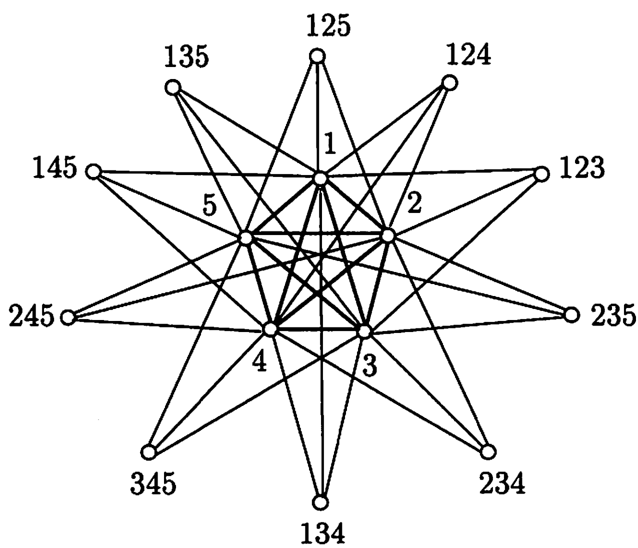

Figure 4: The graph G_4

First we show that every set of the form $V - \{i\}$, where $1 \leq i \leq b+1$, is a full open dominating set of G_b. It suffices to show that every edge of G_b is openly dominated by some vertex of $V - \{i\}$. Since $\gamma_{FO}(K_{b+1}) = 3$, it follows that every edge of K_{b+1} is openly dominated by some vertex of $V - \{i\}$. Let xy be an edge of G_b that does not belong to K_{b+1}. Thus we can assume that $x = T$ for some 3-element subset of V and $y \in T$. Since

$|T| = 3$, there exists $z \in T - \{y, i\}$. Because xy is openly dominated by z, it follows that $V - \{i\}$ is a full open dominating set of G_b and, consequently, $\gamma_{FO}(G_b) \leq b$.

Next we show that any set $S \subseteq V(G_b)$ containing at most $b-1$ elements of V is not a full open dominating set. Then there exist integers i and j with $1 \leq i < j \leq b+1$ such that $i, j \notin S$. Consider the 3-element set $T = \{i, j, k\}$, where $k \neq i, j$ and $1 \leq k \leq b+1$. Since the edge Tk is not openly dominated by any element of S, it follows that S is not a full open dominating set. This implies both that $\gamma_{FO}(G_b) = b$ and that the only γ_{FO}-sets of G_b are of the type $V - \{i\}$ for some integer i with $1 \leq i \leq b+1$. Furthermore, every $(b-1)$-element subset of V is contained in two γ_{FO}-sets and so $f_{\gamma_{FO}}(G_b) = b$. ∎

Next we show that every pair a, b of integers with $0 \leq a \leq b-1$ and $b \geq 3$ is realizable as the forcing full open domination number and full open domination number of some connected graph.

Theorem 2.4 *For every pair a, b of integers with $0 \leq a \leq b-1$ and $b \geq 3$, there exists a connected graph G with $f_{\gamma_{FO}}(G) = a$ and $\gamma_{FO}(G) = b$.*

Proof. Assume first that a is an even integer. For $a = 0$, let F_0 be a copy of the complete graph K_b with $V(F_0) = \{u_1, u_2, \cdots, u_b\}$. For each i with $1 \leq i \leq b$, let $F_i : x_i y_i$ be a copy of K_2. Let G be the graph obtained from the graphs F_i $(0 \leq i \leq b)$ by adding the $4b$ new edges $u_{i+1}x_i$, $u_i x_i$, $u_i y_i$, and $u_{i+1} y_i$ for all $1 \leq i \leq b$, where each subscript is expressed as one of the integers $1, 2, \cdots, b$ modulo b. Since $V(F_0)$ is the unique γ_{FO}-set of G, it follows that $f_{\gamma_{FO}}(G) = 0$ and $\gamma_{FO}(G) = |V(F_0)| = b$.

Next we assume that $a > 0$. Then $a = 2k$ and $b = a + \ell$, where $k, \ell \geq 1$. We consider three cases, according to whether $\ell = 1$, $\ell = 2$, or $\ell \geq 3$.

Case 1. $\ell = 1$. If $k = 1$, then let $G = C_4 + K_1$ be the wheel, where $\deg v = 4$. Since every γ_{FO}-set contains v and any two adjacent vertices of C_4, it follows that $f_{\gamma_{FO}}(G) = 2$ and $\gamma_{FO}(G) = 3$. For $k \geq 2$, let $G = kK_3 + K_1$, where $\deg v = 3k$. Then every γ_{FO}-set contains v and any two vertices from each copy of K_3. Since there are k copies of K_3 in G, it follows that $f_{\gamma_{FO}}(G) = 2k$ and $\gamma_{FO}(G) = 2k+1$.

Case 2. $\ell = 2$. If $k = 1$, then let F be the graph of Figure 5. Since every γ_{FO}-set of F consists of u, v, one of x_1, x_2, and one of y_1, y_2, it follows that $f_{\gamma_{FO}}(F) = 2$ and $\gamma_{FO}(F) = 4$. For $k \geq 2$, let G be the graph obtained from the graph F of Figure 5 and $(k-1)K_3$ by joining all vertices of $(k-1)K_3$ to the vertex u in F. Since every γ_{FO}-set of G contains u, v, one of x_1, x_2, one of y_1, y_2, and two vertices from each copy of K_3, it follows that $f_{\gamma_{FO}}(G) = 2 + 2(k-1) = 2k$ and $\gamma_{FO}(G) = 4 + 2(k-1) = 2k+2$.

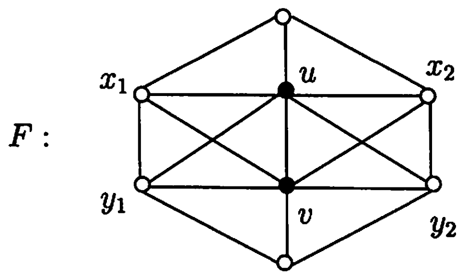

Figure 5: The graph F in Case 2

Case 3. $\ell \geq 3$. Suppose, first, that ℓ is odd. Then $\ell = 2h + 1$ for some integer $h \geq 1$. Let $G = (kK_3 \cup hK_2) + K_1$, where $\deg v = 3k + 2h$. The graph G is shown in Figure 6(a) for $k = 1$ and $h = 2$. Since every γ_{FO}-set consists of v, all vertices in hK_2, and any two vertices from each copy of K_3, it follows that $f_{\gamma_{FO}}(G) = 2k = a$ and $\gamma_{FO}(G) = 2k + 1 + 2h = b$. Next assume that ℓ is even. Then $\ell = 2h$ for some integer $h \geq 2$. Let $G = (kK_3 \cup P_5 \cup (h-2)K_2) + K_1$, where $P_5 : v_1 v_2 \cdots v_5$ and $\deg v = 3k + 2h + 1$. The graph G is shown in Figure 6(b) for $k = 1$ and $h = 3$. Now every γ_{FO}-set consists of v, v_2, v_3, v_4, all vertices of $(h-2)K_2$, and any two vertices from each copy of K_3. Therefore, $f_{\gamma_{FO}}(G) = 2k = a$ and $\gamma_{FO}(G) = 2k + 4 + 2(h-2) = 2k + 2h = b$.

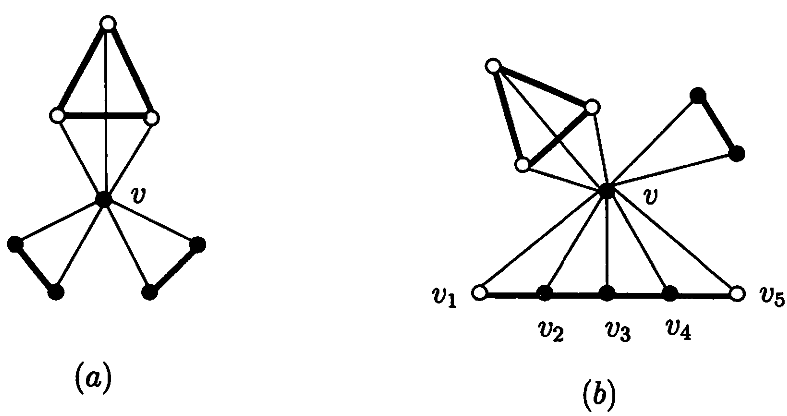

Figure 6: The graphs $(K_3 \cup 2K_2) + K_1$ and $(K_3 \cup K_2 \cup P_5) + K_1$ in Case 3

Assume next that a is odd. For $a = 1$, we consider consider two cases, according to whether b is even or b is odd.

Case 1. *b is even.* Then $b = 2h$ for some integer $h \geq 2$. If $h = 2$, then let $F_1 = P_5 + \overline{K}_2$, where $P_5 : v_1v_2 \cdots v_5$ and $V(\overline{K}_2) = \{u, v\}$. Since $\{u, v_2, v_3, v_4\}$ and $\{v, v_2, v_3, v_4\}$ are the only two γ_{FO}-sets in F_1, it follows that $f_{\gamma_{FO}}(F_1) = 1$ and $\gamma_{FO}(F_1) = 4$. If $h \geq 3$, then let $F_2 = (h-2)K_2$ and let G be the graph obtained from F_1 and F_2 by joining all vertices of F_2 to v_2 in F_1. Since $\{u, v_2, v_3, v_4\} \cup V(F_2)$ and $\{v, v_2, v_3, v_4\} \cup V(F_2)$ are the only two γ_{FO}-sets in G, it follows that $f_{\gamma_{FO}}(G) = 1$ and $\gamma_{FO}(G) = 2h = b$.

Case 2. *b is odd.* Then $b = 2h + 1$ for some integer $h \geq 1$. If $h = 1$, then let $H_1 = P_4 + \overline{K}_2$, where $P_4 : v_1v_2v_3v_4$ and $V(\overline{K}_2) = \{u, v\}$. Since $\{u, v_2, v_3\}$ and $\{v, v_2, v_3\}$ are the only two γ_{FO}-sets in H_1, it follows that $f_{\gamma_{FO}}(H_1) = 1$ and $\gamma_{FO}(H_1) = 3$. If $h \geq 2$, let $H_2 = (h-1)K_2$. Then let G be the graph obtained from H_1 and H_2 by joining all vertices of H_2 to v_2 in H_1. Since $\{u, v_2, v_3\} \cup V(H_2)$ and $\{v, v_2, v_3\} \cup V(H_2)$ are the only two γ_{FO}-sets in G, it follows that $f_{\gamma_{FO}}(G) = 1$ and $\gamma_{FO}(G) = 2h + 1 = b$.

We now assume that $a \geq 3$. Then $a = 2k + 1$ for some integer $k \geq 1$, we consider two cases, according to whether $b = a + 1$ or $b \geq a + 2$.

Case 1. $b = a + 1$. First we show that the graph H shown in Figure 7 has full open domination number 4 and forcing full open domination number 3. Since $\{v, v_1, v_2, v_7\}$ is a full open dominating set in H, it follows that $\gamma_{FO}(H) \leq 4$. Observe that for each i with $1 \leq i \leq 8$, the edge v_iv_{i+2} (where the addition $i+2$ is performed modulo 8) is openly dominated only by v_{i+1} and v. This implies that every full open dominating set of H contains v or contains all vertices v_i with $1 \leq i \leq 8$. Since $\gamma_{FO}(H) \leq 4$, every γ_{FO}-set of H contains v. Thus we may assume that there is a γ_{FO}-set S containing v and v_1. Since the edges vv_1, vv_4, vv_5, vv_6 are neither openly dominated by v or v_1 nor by any other vertex of H, it follows that $\gamma_{FO}(H) = |S| \geq 4$. Therefore, $\gamma_{FO}(H) = 4$.

Next we show that $f_{\gamma_{FO}}(H) = 3$. Since v belongs to every γ_{FO}-set of H, it follows that $f_{\gamma_{FO}}(H) \leq 3$. Observe that

(i) $\{v, v_1\}$ is a subset of $S_1 = \{v, v_1, v_2, v_4\}$ and $S_2 = \{v, v_1, v_3, v_5\}$,
(ii) $\{v_1, v_2\}$ is a subset of $S_3 = \{v, v_1, v_2, v_5\}$ and $S_4 = \{v, v_1, v_2, v_7\}$,
(iii) $\{v_1, v_3\}$ is a subset of $S_5 = \{v, v_1, v_3, v_5\}$ and $S_6 = \{v, v_1, v_3, v_7\}$,
(iv) $\{v_1, v_4\}$ is a subset of $S_7 = \{v, v_1, v_2, v_4\}$ and $S_8 = \{v, v_1, v_3, v_4\}$,
(v) $\{v_1, v_5\}$ is a subset of $S_9 = \{v, v_1, v_3, v_5\}$ and $S_{10} = \{v, v_1, v_5, v_7\}$.

Since S_i $(1 \leq i \leq 10)$ is a γ_{FO}-set, $f_{\gamma_{FO}}(H) = 3$. Therefore, the graph H of Figure 7 has $f_{\gamma_{FO}}(H) = 3$ and $\gamma_{FO}(H) = 4$, as desired.

Next, let $a = 2k + 1$ and $b = 2k + 2$, where $k \geq 2$. Let G be the graph obtained by joining each vertex of $(k-1)K_3$ to the vertex v in the graph H of Figure 7. Then each γ_{FO}-set of G consists of a γ_{FO}-set of the

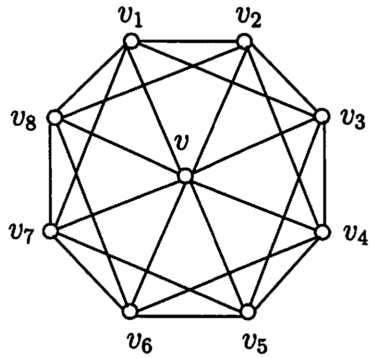

Figure 7: The graph H with $f_{\gamma_{FO}}(H) = 3$ and $\gamma_{FO}(H) = 4$

subgraph H in G together with two vertices from each copy of K_3. Thus $\gamma_{FO}(G) = 4+2(k-1) = 2k+2 = b$ and $f_{\gamma_{FO}}(G) = 3+2(k-1) = 2k+1 = a$, as desired.

Case 2. $b \geq a+2$. There are two subcases, according to whether $a = 3$ or $a \geq 5$.

Subcase 2.1. $a = 3$. For $b = 5$, let $G = (K_3 \cup P_3) + K_1$, where $P_3 : xyz$ and $\deg u = 6$. Then every γ_{FO}-set consists of u, y, one of x, z, and any two of vertices of K_3. Thus $f_{\gamma_{FO}}(G) = 3 = a$ and $\gamma_{FO}(G) = 5 = b$. For $b = 6$, let F_0 be a copy of K_3 with $V(F_0) = \{u_1, u_2, u_3\}$ and let $F_i : x_iy_iz_i$ be a copy of P_3 for each i with $1 \leq i \leq 3$. Then the graph G is obtained by joining u_i to (1) the vertices of F_i for each i with $1 \leq i \leq 3$ and (2) x_{i+1} and z_{i-1}, where addition is performed modulo 3 (see Figure 8). Every γ_{FO}-set in G consists of $V(F_0)$ and one vertex from each set $\{x_i, z_i\}$ for $1 \leq i \leq 3$. Hence $f_{\gamma_{FO}}(G) = 3$ and $\gamma_{FO}(G) = 6$.

Now let $b \geq 7$. If $b = a + 2h$ for some integer $h \geq 2$, then let

$$G = (K_3 \cup P_3 \cup (h-1)K_2) + K_1.$$

If $b = a + 2h + 1$ for some integer $h \geq 2$, then let

$$G = (K_3 \cup P_3 \cup P_5 \cup (h-2)K_2) + K_1.$$

It is routine to verify that $f_{\gamma_{FO}}(G) = 3$ and $\gamma_{FO}(G) = b$.

Subcase 2.2. $a = 2k+1 \geq 5$. Suppose first that $b = a + 2h$ for some $h \geq 1$. For $k = 2$ and $h = 1$, let F be the graph shown in Figure 9. Then every γ_{FO}-set contains u, v, one of w_1, w_2, and two vertices from each set $\{x_i, y_i, z_i\}$ for $i = 1, 2$. Thus $f_{\gamma_{FO}}(F) = 5 = a$ and $\gamma_{FO}(F) = 7 = b$.

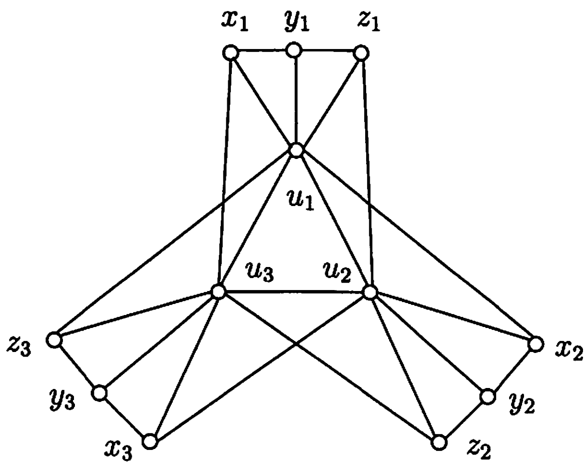

Figure 8: A graph G with $f_{\gamma_{FO}}(G) = 3$ and $\gamma_{FO}(G) = 6$

For $k \geq 3$ and $h \geq 1$, let G be obtained from the graph F of Figure 9 and $H = (h-1)K_2 \cup (k-2)K_3$ by joining each vertex in H with the vertex v in F. Then $f_{\gamma_{FO}}(F) = 5 + 2(k-2) = 2k+1 = a$ and $\gamma_{FO}(F) = (2k+1) + 2 + 2(h-1) = 2k+1+2h = b$.

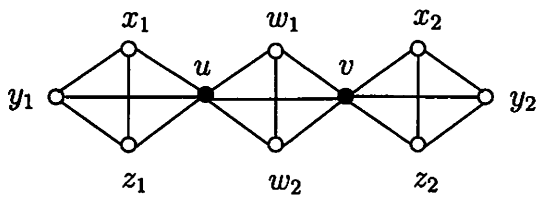

Figure 9: The graph F in Case 2

Finally, suppose that $b = a + 2h + 1$ for some $h \geq 1$. Then let G be obtained from $P_5 + \overline{K_2}$, where $P_5 : v_1 v_2 \cdots v_5$ and $V(\overline{K_2}) = \{u, v\}$, and $(h-1)K_2 \cup kK_3$ by joining all vertices in $(h-1)K_2 \cup kK_3$ to v_3. Then every γ_{FO}-set consists of v_2, v_3, v_4, all vertices of $(h-1)K_2$, one of u, v, and two vertices from each K_3. Therefore, $f_{\gamma_{FO}}(F) = 2k+1 = a$ and $\gamma_{FO}(F) = 3 + 2(h-1) + 1 + 2k = 2k + 1 + 2h + 1 = b$. ∎

For $b = 4$, the graph G constructed in the first paragraph of the proof of Theorem 2.4 is shown in Figure 10. As shown in the proof, $\gamma_{FO}(G) = 4$ and $f_{\gamma_{FO}}(G) = 0$. It is not difficult to see that $\gamma_{FC}(G) = 2$ and $f_{\gamma_{FC}}(G) = 1$, that is,

$$f_{\gamma_{FO}}(G) < f_{\gamma_{FC}}(G) < \gamma_{FC}(G) < \gamma_{FO}(G).$$

In particular, $f_{\gamma_{FO}}(G) < f_{\gamma_{FC}}(G)$, despite the fact that $\gamma_{FC}(G) \leq \gamma_{FO}(G)$ for *every* graph G.

G :

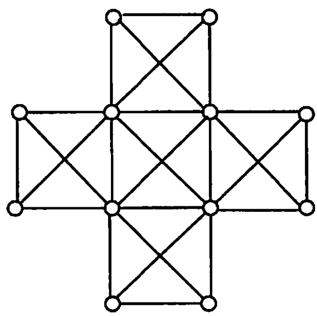

Figure 10: A graph G with $f_{\gamma_{FO}}(G) < f_{\gamma_{FC}}(G)$

Combining Theorems 2.3 and 2.4, we have the following realization result.

Corollary 2.5 *For every pair a, b of integers with $0 \leq a \leq b$ and $b \geq 3$, there exists a connected graph G with $f_{\gamma_{FO}}(G) = a$ and $\gamma_{FO}(G) = b$.*

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