Ranks of Line Graphs of Regular Graphs

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Abstract

We consider the rank of the adjacency matrix of the line graph for some classes of regular graphs. In particular, we study the line graphs of cycles, paths, complete graphs, complete bipartite and multipartite graphs, circulant graphs of degrees three and four and some Cartesian graph products.

1 Introduction and Preliminary Results

Let $P_G(\lambda)$ denote the characteristic polynomial of the adjacency matrix of a graph G. The characteristic polynomials of the graphs in this paper can be found in [1] unless otherwise cited. Note that if G is a regular graph of degree r with n vertices, then G has $m = \frac{1}{2}nr$ edges. The line graph of G, $L(G)$, is regular of degree $(2r-2)$, has m vertices and $\frac{1}{2}nr(r-1)$ edges. The following lemma, found in [1], relates the eigenvalues of a regular graph to those of its line graph.

Lemma 1 *If G is a regular graph of degree r with n vertices and $m = \frac{1}{2}nr$ edges, then the following relation holds:* $P_{L(G)}(\lambda) = (\lambda+2)^{m-n}P_G(\lambda-r+2)$

Having the characteristic polynomial of the line graph, finding its eigenvalues is a straightforward procedure. The rank is then found by subtracting the number of zero eigenvalues from the number of vertices. The following alternative statement of Lemma 1 is sometimes more useful in directly computing eigenvalues.

Lemma 2 *If G is a regular graph of degree r having n vertices and m edges, and $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of G, then the eigenvalues of $L(G)$ are $m-n$ values of -2 and $\lambda_i + r - 2$ for $i = 1, \ldots, n$.*

In the sections that follow, we investigate when zero eigenvalues can occur for various classes of graphs. We consider cycles, paths, complete

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graphs, complete bipartite and multipartite graphs, circulant graphs of degrees three and four and some Cartesian products of graphs.

2 Some Well Known Graphs

The rank of a cycle C_n is given by $\text{rank}(C_n) = \left\{ \begin{array}{cl} n-2 & \text{if } n \equiv 0 \bmod 4 \\ n & \text{otherwise} \end{array} \right\}$. Since the line graph of a cycle is a cycle of the same order, we can make the following observation.

Observation 1 *For $n \geq 3$, $rank(L(C_n)) = rank(C_n)$.*

Similarly, the rank of a path P_n is given by $\text{rank}(P_n) = \left\{ \begin{array}{cl} n-1 & \text{if } n \text{ is odd} \\ n & \text{if } n \text{ is even} \end{array} \right\}$. Since the line graph of a path of order n is a path of order $n-1$, we have a second observation.

Observation 2 *For $n \geq 2$, $rank(L(P_n)) = rank(P_{n-1})$.*

The rank of the complete graph K_n is n. The characteristic polynomial reveals not only the complete eigenvalue structure of K_n but also that of its line graph. Since

$$P_{K_n}(\lambda) = (\lambda - n + 1)(\lambda + 1)^{n-1}$$

and since K_n is regular of degree $(n-1)$, Lemma 1 gives

$$\begin{aligned} P_{L(K_n)}(\lambda) &= (\lambda+2)^{m-n} P_{K_n}(\lambda - (n-1) + 2) \\ &= (\lambda+2)^{m-n}(\lambda - 2n + 4)(\lambda - n + 4)^{n-1} \end{aligned}$$

implying that there are $(m-n)$ eigenvalues equal to -2, $(n-1)$ eigenvalues equal to $(n-4)$, and one eigenvalue equal to $(2n-4)$. Thus the rank is full unless $n = 2$ or $n = 4$; in which case the rank is $(m-1) = 1 - 1 = 0$, or $(m-3) = 6 - 3 = 3$, respectively. Hence, since $m = \frac{1}{2}n(n-1)$, we have the following theorem.

Theorem 1 *For $n \geq 2$, $rank(L(K_n)) = \left\{ \begin{array}{cl} 0 & \text{if } n = 2 \\ 3 & \text{if } n = 4 \\ \frac{1}{2}n(n-1) & \text{otherwise} \end{array} \right\}$.*

We close this section by examining three famous regular graphs, Petersen's Graph, the "Cocktail Party" Graph, and the n-dimensional hypercube.

Petersen's Graph has 10 vertices, is regular of degree 3, has 15 edges, and its characteristic polynomial (found in [5]) is

$$P_P(\lambda) = (\lambda - 3)(\lambda - 1)^5(\lambda + 2)^4.$$

As

$$P_{L(P)}(\lambda) = (\lambda + 2)^5(\lambda - 4)(\lambda - 2)^5(\lambda + 1)^4$$

from Lemma 1, there will be no zero eigenvalues, and we have a third observation.

Observation 3 *rank*$(L(P)) = 15$, *where* P *is Petersen's Graph.*

The so-called "Cocktail Party" Graph $CP(n)$ is the only $(n-2)$-regular graph on n vertices. It consists of a complete graph on an even number of vertices with a perfect matching removed. Having $n = 2k$ vertices and $2k^2 - 2k$ edges, its characteristic polynomial is

$$\begin{aligned} P_{CP(n)}(\lambda) &= (\lambda - n + 2)\lambda^{n/2}(\lambda + 2)^{n/2-1} \\ &= (\lambda - 2k + 2)\lambda^k(\lambda + 2)^{k-1} \end{aligned}$$

Therefore,

$$P_{L(CP(n))}(\lambda) = (\lambda + 2)\lambda^{2k^2-4k}(\lambda - 4k + 6)(\lambda - 2k + 4)^k(\lambda - 2k + 6)^{k-1}$$

Hence, there will be no zero eigenvalues unless $k = 2$ or $k = 3$. If $k = 2$, then there will be $k = 2$ zero eigenvalues. If $k = 3$, then there will be $k - 1 = 2$ zero eigenvalues. Thus we have the following.

Observation 4 *For* n *even,* $n \geq 4$, *let* $CP(n)$ *be the complete graph* K_n *with a perfect matching removed. Then,*

$$rank(L(CP(n)) = \left\{ \begin{array}{cc} 2 & \textit{if } n = 4 \\ 10 & \textit{if } n = 6 \\ n^2/2 - n & \textit{otherwise} \end{array} \right\}.$$

Finally we examine Q_n, the n-dimensional hypercube graph defined recursively by:

$$\begin{aligned} Q_1 &= K_2 \\ Q_{n+1} &= Q_n \times K_2 \; for \; n \geq 1 \end{aligned}$$

It follows that Q_n has 2^n vertices., is regular of degree n and thus has $n2^{n-1}$ edges.

Theorem 2 *rank*$(L(Q_n)) = n(2^{n-1} - 1)$, *where* Q_n *is the* n*-dimensional hypercube graph..*

Proof. The spectrum of Q_n consists of the numbers $(n-2k)$, each with multiplicity $\binom{n}{k}, k = 0, 1, ..., n$. Since Q_n is regular of degree n, there are $\frac{1}{2}n(2^n) = 2^{n-1}n$ vertices in $L(Q_n)$. Hence, the spectrum of $L(Q_n)$ consists of $2^{n-1}(n-2)$ values of -2 and the numbers $(\lambda_i+n-2) = n-2k+n-2 = 2(n-k-1)$, each with multiplicity $\binom{n}{k}, k = 0, 1, ..., n$. Setting $(\lambda_i+n-2) = 0$ gives $2(n-k-1) = 0$, whose only solution is $k = (n-1)$. The multiplicity of this eigenvalue is $\binom{n}{n-1} = n$. Therefore rank$(L(Q_n))$= $n(2^{n-1}-1)$.■

3 Complete Multipartite Graphs

We now investigate the complete bipartite graph $K_{n,n}$. This graph has $2n$ vertices, n^2 edges and is regular of degree n. Since

$$P_{K_{n,n}}(\lambda) = (\lambda^2 - n^2)\lambda^{2n-2},$$

we have from Lemma 1

$$\begin{aligned} P_{L(K_{n,n})}(\lambda) &= (\lambda+2)^{n^2-2n}((\lambda-n+2)^2 - n^2)(\lambda-n+2)^{2n-2} \\ &= (\lambda+2)^{n^2-2n}(\lambda-n+2-n)(\lambda-n+2+n)\cdot \\ &\quad\; (\lambda-n+2)^{2n-2} \\ &= (\lambda+2)^{n^2-2n}(\lambda-2n+2)(\lambda+2)(\lambda-n+2)^{2n-2} \\ &= (\lambda+2)^{n^2-2n+1}(\lambda-2n+2)(\lambda-n+2)^{2n-2} \end{aligned}$$

Thus there will be zero eigenvalues only when $n = 2$, which implies a rank of $n^2 - 2n + 2 = 2$. This proves the following theorem.

Theorem 3 *rank*$(L(K_{n,n})) = \left\{ \begin{array}{ll} 0 & \text{if } n = 1 \\ 2 & \text{if } n = 2 \\ n^2 & \text{otherwise} \end{array} \right\}$.

The complete bipartite graph $K_{m,n}$ is only slightly more complicated. This graph is not regular, but semiregular with degrees n and m. More generally, if G is a semi-regular bipartite graph with n_1 mutually non-adjacent vertices of degree r_1 and n_2 mutually non-adjacent vertices of degree r_2, where $n_1 > n_2$, then

$$P_{L(G)}(\lambda) = (\lambda+2)^{n_1r_1-n_1-n_2}\sqrt{\left(-\frac{\alpha_1}{\alpha_2}\right)^{n_1-n_2} P_G(\sqrt{\alpha_1\alpha_2})P_G(-\sqrt{\alpha_1\alpha_2})}$$

where $\alpha_1 = \lambda - r_1 + 2$ and $\alpha_2 = \lambda - r_2 + 2$. For the complete bipartite graph $K_{m,n}$,

$$P_{K_{m,n}}(\lambda) = (\lambda^2 - mn)\lambda^{m+n-2}$$

and the graph has n vertices of degree m, and m vertices of degree n so that $\alpha_1 = \lambda - n + 2$, and $\alpha_2 = \lambda - m + 2$. In our attempt to simplify the expression, we first examine the product $P_G(\sqrt{\alpha_1\alpha_2})P_G(-\sqrt{\alpha_1\alpha_2})$. Upon substitution, this becomes

$$(\sqrt{\alpha_1\alpha_2}^2 - mn)(\sqrt{\alpha_1\alpha_2})^{m+n-2}((-\sqrt{\alpha_1\alpha_2})^2 - mn)(-\sqrt{\alpha_1\alpha_2})^{m+n-2}$$

which simplifies to

$$(\alpha_1\alpha_2 - mn)^2(-\alpha_1\alpha_2)^{m+n-2}.$$

Now letting $\gamma = (\lambda + 2)^{mn-m-n}$, we have

$$\begin{aligned} P_{L(K_{m,n})}(\lambda) &= \gamma\sqrt{\left(-\frac{\alpha_1}{\alpha_2}\right)^{m-n}(\alpha_1\alpha_2 - mn)^2(-\alpha_1\alpha_2)^{m+n-2}} \\ &= \gamma(\alpha_1\alpha_2 - mn)\sqrt{(-1)^{2m-2}\alpha_1^{m-n}\alpha_2^{-m+n}\alpha_1^{m+n-2}\alpha_2^{m+n-2}} \\ &= \gamma(\lambda+2)(\lambda+2-m-n)\alpha_1^{2m-2}\alpha_2^{2n-2} \\ &= (\lambda+2)^{mn-m-n+1}(\lambda+2-m-n)\alpha_1^{m-1}\alpha_2^{n-1} \end{aligned}$$

Therefore, there will be zero eigenvalues only when $\alpha_1 = \lambda - n + 2 = 0$ or $\alpha_2 = \lambda - m + 2 = 0$, which implies zero eigenvalues for $n = 2$ or $m = 2$. Hence we have a proof of the following theorem.

Theorem 4 *For $m \neq n$, rank$(L(K_{m,n}))$* $= \left\{ \begin{array}{ll} mn - (m-1) & \textit{if } n = 2 \\ mn - (n-1) & \textit{if } m = 2 \\ mn & \textit{otherwise} \end{array} \right\}.$

It turns out that the rank of the line graph of the complete multipartite graph $K_{n,n,\ldots,n}$ with nk vertices neatly divides into two cases. Note that having nk vertices implies that n appears as a subscript in $K_{n,n,\ldots,n}$ exactly k times.

Theorem 5 *For $n \geq 2$ and $k \geq 3$,*
rank$(L(K_{n,n,\ldots,n}))$ $= \left\{ \begin{array}{ll} 10 & \textit{if } n = 2 \textit{ and } k = 3 \\ \frac{1}{2}n^2k(k-1) & \textit{otherwise} \end{array} \right\}$, *where $K_{n,n,\ldots,n}$ has nk vertices.*

Proof. Let $n \geq 2$ and $k \geq 3$. The graph of $K_{n,n,\ldots,n}$ has nk vertices and is regular of degree $n(k-1)$. Thus $L(K_{n,n,\ldots n})$ has $\frac{1}{2}n^2k(k-1)$ vertices. The characteristic polynomial of $K_{n,n,\ldots,n}$ is

$$P_{K_{n,n,\ldots,n}}(\lambda) = \lambda^{k(n-1)}(\lambda + n - nk)(\lambda + n)^{k-1}$$

We see immediately that the rank of $K_{n,n,\ldots,n}$ can never be full, as there will always be $k(n-1)$ zero eigenvalues, making its rank $nk - k(n-1) = k$. Hence the characteristic polynomial of the line graph

$$\begin{aligned} P_{L(K_{n,n,\ldots,n})}(\lambda) &= (\lambda+2)^{\frac{1}{2}n^2k(k-1)-nk}(\lambda - n(k-1)+2)^{k(n-1)} \cdot \\ &\quad (\lambda - n(k-1)+2+n-nk)(\lambda - n(k-1)+2+n)^{k-1} \\ &= (\lambda+2)^{\frac{1}{2}n^2k(k-1)-nk}(\lambda+n-nk+2)^{k(n-1)} \cdot \\ &\quad (\lambda+2(n-nk+1))(\lambda+2n-nk+2)^{k-1} \end{aligned}$$

Since $n \geq 2$ and $k \geq 3$, zero eigenvalues can only occur when $-2n+nk-2=0$, which implies $n=2$ and $k=3$. In that case, the rank will be 10 and the result holds.■

4 3-Circulant Graphs

For n even, a 3-circulant graph G with n vertices determined by the set $S = \{a, \frac{n}{2}, n-a\}$ has vertex set $V = \{0,1,2,\ldots,n-1\}$, and vertex i is adjacent to vertex j if and only if $|i-j| \bmod n \in S$. For information on 3-circulant graphs, including their ranks and eigenvalues, see [2]. Hence all 3-circulant graphs are 3-regular. In this section we will assume that n is even and $n \geq 4$.

Lemma 1 implies that $L(G)$ has $m = \frac{3}{2}n$ vertices, and that $P_{L(G)}(\lambda) = (\lambda+2)^{n/2}P_G(\lambda-1)$. Thus, the line graph $L(G)$ will have a zero eigenvalue if and only if the original circulant graph G has an eigenvalue of -1. As there is a formula for the eigenvalues of a 3-circulant see [4], some algebraic conditions follow describing when such zero λ can exist.

Theorem 6 *If A is an $n \times n$ circulant matrix with first row $[c_1, c_2, \ldots, c_n]$, then the eigenvalues of A are given by $\lambda_p = \sum_{i=1}^{n} c_i\omega^{(i-1)p}$, $p = 0,1,2,\ldots,(n-1)$, where $\omega = e^{2\pi i/n}$.*

Applying the above theorem to the line graph case, we have the following.

Lemma 3 *Let G be the 3-circulant graph determined by $\{a, \frac{n}{2}, n-a\}$. Then if λ_p is an eigenvalue of G, $\lambda_p = \omega^{ap} + \omega^{\frac{n}{2}p} + \omega^{(n-a)p} = -1$ if and only if p satisfies*

$$ap \equiv \frac{n}{4} \bmod \frac{n}{2} \quad \text{for } p \text{ odd} \qquad (1)$$

$$ap \equiv \frac{n}{2} \bmod n \quad \text{for } p \text{ even} \qquad (2)$$

Proof: Since $\omega = e^{2\pi i/n} = \cos(2\pi/n) + i\sin(2\pi/n)$, we have that $\omega^{ap} + \omega^{\frac{n}{2}p} + \omega^{(n-a)p} = -1$ implies

$$\begin{aligned} 2\cos(2\pi ap/n) + \cos(\pi p) &= -1 \\ \sin(n\pi) &= 0 \end{aligned}$$

The second relationship is an identity, and the first relationship naturally splits into the two cases of p odd and p even. For p odd, $2\cos(2\pi ap/n) = 0$ if and only if ap is an odd multiple of $\frac{n}{4}$. For p even, $2\cos(2\pi ap/n) = -2$ if and only if ap is an odd multiple of $\frac{n}{2}$.■

Recall that a 3-circulant is connected if and only if $\gcd(a, \frac{n}{2}) = 1$. We are now able to determine the rank of the line graph of any connected 3-circulant.

Theorem 7 *Let n be even, $n \geq 4$ and G be a connected 3-circulant with n vertices defined by $\{a, \frac{n}{2}, n-a\}$. Then*

$$rank(L(G)) = \left\{ \begin{array}{llll} (i) & m & if & \frac{n}{2} \text{ odd} \\ (ii) & m-1 & if & \frac{n}{2} \equiv 0 \bmod 4 \\ (iii) & m-3 & if & \frac{n}{2} \equiv 2 \bmod 4 \end{array} \right\},$$

where $L(G)$ has $m = \frac{3}{2}n$ vertices.

Proof. (i) Suppose $\frac{n}{2}$ is odd. Using Lemma 3, if p is odd, then $ap \equiv \frac{n}{4} \bmod \frac{n}{2}$ has no solutions since $\frac{n}{4}$ is not an integer. Similarly, if p is even, $ap \equiv \frac{n}{2} \bmod n$ has no solutions since ap is even and $\frac{n}{2}$ is odd. Thus, $L(G)$ has no zero eigenvalues, and has rank m.

(ii) Suppose $\frac{n}{2} \equiv 0 \bmod 4$. If p is odd, $ap \equiv \frac{n}{4} \bmod \frac{n}{2}$ can have no solution since $\gcd(a, \frac{n}{2}) = 1$ and $\frac{n}{4}$ is even. If p is even, $ap \equiv \frac{n}{2} \bmod n$ has exactly one solution, again since $\gcd(a, \frac{n}{2}) = 1$. In fact, this solution is $p = \frac{n}{2}$. Thus, $\text{rank}(L(G)) = m - 1$.

(iii) Suppose $\frac{n}{2} \equiv 2 \bmod 4$, so that $n = 8k + 4$ for some integer k. If p is odd, $ap \equiv \frac{n}{4} \bmod \frac{n}{2}$ implies $ap \equiv (2k+1) \bmod (4k+2)$. This equation has exactly two solutions, $p = (2k+1)$ and $(6k+3)$, or $p = \frac{n}{4}$ and $\frac{3n}{4}$. This is because $\gcd(a, \frac{n}{2}) = \gcd(a, (4k+2)) = 1$. Similarly, if p is even, $ap \equiv \frac{n}{2} \bmod n$ implies $ap = (4k+2) \bmod (8k+4)$. This equation has exactly one solution, and it is $p = (4k+2) = \frac{n}{2}$, since a and $(4k+2)$ have no common factors. Thus, $\text{rank}(L(G)) = m - 3$.■

The previous theorem gives the rank of the line graph of any connected 3-circulant. If the circulant is not connected, it splits into isomorphic connected components. The following result describes this case, and further details can be found in [2].

Theorem 8 *Let n be even, $n \geq 4$ and $S = \{a, \frac{n}{2}, n-a\}$. If $\gcd(a, \frac{n}{2}, n) = d$, then the circulant graph with n vertices formed by S has d compo-*

nents each isomorphic to the circulant graph on $\frac{n}{d}$ vertices formed by $S' = \{\frac{a}{d}, \frac{n}{2d}, \frac{n-a}{d}\}$.

We now give the rank of the line graph of a disconnected 3-circulant.

Corollary 1 *Let n be even, $n \geq 4$ and G be a 3-circulant with n vertices defined by $\{a, \frac{n}{2}, n-a\}$, with $d = \gcd(a, \frac{n}{2})$. Then*

$$rank(L(G)) = \left\{ \begin{array}{llll} (i) & m & if & \frac{n}{2d}\ odd \\ (ii) & m-d & if & \frac{n}{2d} \equiv 0 \bmod 4 \\ (iii) & m-3d & if & \frac{n}{2d} \equiv 2 \bmod 4 \end{array} \right\}, \text{ where } L(G) \text{ has}$$

$m = \frac{3}{2}n$ vertices.

Proof. If $d = 1$, the result follows from Theorem 7. If $d > 1$, then G has d isomorphic connected components, each isomorphic to the 3-circulant defined by $\{\frac{a}{d}, \frac{n}{2d}, \frac{n-a}{d}\}$. If $\frac{n}{2d}$ is odd, each component has a line graph of full rank, and rank$(L(G)) = m$. If $\frac{n}{2d} \equiv 0 \bmod 4$, then each component has a line graph rank deficient by 1, and rank$(L(G)) = m - d$. If $\frac{n}{2d} \equiv 2 \bmod 4$, then each component has a line graph rank deficient by 3, and rank$(L(G)) = m - 3d$.■

5 4-Circulant Graphs

A 4-circulant graph G with n vertices determined by the set $S = \{a, b, n-a, n-b\}$, denoted $4C_n(a, b)$, has vertex set $V = \{0, 1, 2, ..., n-1\}$, and vertex i is adjacent to vertex j if and only if $|i - j| \bmod n \in S$. Thus all 4-circulant graphs G are 4-regular. In this section we will assume that $n \geq 5$.

Lemma 1 implies that $L(G)$ has $m = 2n$ vertices, and that $P_{L(G)}(\lambda) = (\lambda + 2)^n P_G(\lambda - 2)$. Thus, the line graph $L(G)$ will have a zero eigenvalue if and only if the original circulant graph G has an eigenvalue of -2. As there is a formula for the eigenvalues of a 4-circulant [4], some algebraic conditions follow describing when such zero λ can exist (see [4]).

By applying Theorem 6, we have the following.

Theorem 9 *The eigenvalue λ_p of $L(4C_n(a, b))$ is zero if and only if one or more of the following holds:*

$$ap \equiv \frac{n}{2} \bmod n \text{ and } bp \equiv \frac{n}{4} \bmod \frac{n}{2} \quad (3)$$

$$ap \equiv \frac{n}{4} \bmod \frac{n}{2} \text{ and } bp \equiv \frac{n}{2} \bmod n \quad (4)$$

$$ap \equiv \frac{n}{3} \bmod n \text{ and } bp \equiv \pm\frac{n}{3} \bmod n \quad (5)$$

$$ap \equiv \frac{2n}{3} \bmod n \text{ and } bp \equiv \pm\frac{n}{3} \bmod n \quad (6)$$

Proof. From Lemma 1, $L(4C_n(a,b))$ has a zero eigenvalue if and only if $\lambda_p = -2$ in $4C_n(a,b)$. Thus, $\lambda_p = \omega^{ap} + \omega^{bp} + \omega^{(n-b)p} + \omega^{(n-a)p} = -2$. Since $\omega^x = \cos(\frac{2\pi x}{n}) + \sin(\frac{2\pi x}{n})$, $\lambda_p = -2$ if and only if

$$\begin{aligned}\cos(\frac{2\pi ap}{n}) + \cos(\frac{2\pi bp}{n}) + \cos(\frac{2\pi(n-b)p}{n}) + \cos(\frac{2\pi(n-a)p}{n}) &= -2 \\ \sin(\frac{2\pi ap}{n}) + \sin(\frac{2\pi bp}{n}) + \sin(\frac{2\pi(n-b)p}{n}) + \sin(\frac{2\pi(n-a)p}{n}) &= 0\end{aligned}$$

The equation in sines reduces to an identity; the equation in cosines reduces to $\cos\alpha + \cos\beta = -1$, where $\alpha = \frac{2\pi ap}{n}$ and $\beta = \frac{2\pi bp}{n}$. The only rational multiples of π with $\alpha, \beta \in [0, 2\pi]$ that solve this equation are

α	π	$\frac{\pi}{2}, \frac{3\pi}{2}$	$\frac{2\pi}{3}$	$\frac{4\pi}{3}$
β	$\frac{\pi}{2}, \frac{3\pi}{2}$	π	$\frac{2\pi}{3}, \frac{4\pi}{3}$	$\frac{2\pi}{3}, \frac{4\pi}{3}$

Supposing $\alpha = \pi + 2k\pi$ for $k \in Z$ implies $2ap = n(2k+1)$. Hence, $ap \equiv \frac{n}{2} \bmod n$. This also implies $\beta = \frac{\pi}{2}(2j+1)$ for $j \in Z$. Hence, $2bp = \frac{n}{2}(2j+1)$ and $bp \equiv \frac{n}{4} \bmod \frac{n}{2}$. Similarly, if $\beta = \pi + 2k\pi$, then $bp \equiv \frac{n}{2} \bmod n$ and $ap \equiv \frac{n}{4} \bmod \frac{n}{2}$.

Now supposing $\alpha = \frac{2\pi}{3} + 2k\pi$ for $k \in Z$ implies $2ap = \frac{2n}{3}(3k+1)$. Hence, $ap \equiv \frac{n}{3} \bmod n$. This also implies $\beta = \pm\frac{2\pi}{3} + 2j\pi$ where $j \in Z$. Hence, $bp = nj \pm \frac{n}{3}$ and $bp \equiv \pm\frac{n}{3} \bmod n$. Similarly, if $\alpha = \frac{4\pi}{3} + 2k\pi$, then $ap \equiv \frac{2n}{3} \bmod n$ and $bp \equiv \pm\frac{n}{3} \bmod n$.■

Corollary 2 *If $n \neq 0 \bmod 3$ and $n \neq 0 \bmod 4$, then $rank(L(4C_n(a,b))) = 2n$.*

Proof. By Theorem 9, at least one of conditions (3) through (6) must be satisfied for there to be any deficiency in rank. However, if n is not divisible by 4, neither (3) nor (4) are satisfied, and if n is not divisible by 3, neither (5) nor (6) are satisfied. Thus, there can be no rank deficiency and rank $L(C_n(a,b)) = 2n$.■

To examine the rank of the line graph of a 4-circulant determined by $\{a, b, n-b, n-a\}$, the quantities $d_1 = \gcd(a,n)$ and $d_2 = \gcd(b,n)$ play an important role. The following lemma from number theory is an important tool in this study.

Lemma 4 *If $d_1 = \gcd(r,n)$ and $d_2 = \gcd(s,n)$ where r, s and n are integers, then*

$$\gcd(\frac{n}{d_1}, \frac{n}{d_2}) = \frac{n}{\text{lcm}(d_1, d_2)}.$$

Now there naturally arise four cases: $n \equiv (1,2,5,7,10,11) \bmod 12$, $n \equiv (3,6,9) \bmod 12$, $n \equiv (4,8) \bmod 12$, and $n \equiv 0 \bmod 12$. In the following, we assume connected graphs; so $\gcd(a,b,n) = 1$.

Theorem 10 *Let $4C_n(a,b)$ be connected with $n \equiv 0 \bmod 3$, $d_1 = \gcd(a,n)$ and $d_2 = \gcd(b,n)$. If 3 divides (d_2-d_1) or (d_2-2d_1), then rank$(L(4C_n(a,b)))$ $\leq 2n-2$. Equality holds if $n \equiv (3,6,9) \bmod 12$ and if 3 divides (d_2-d_1) or 3 divides (d_2-2d_1). Furthermore, if $n \equiv (3,6,9) \bmod 12$ and 3 does not divide (d_2-d_1) and 3 does not divide (d_2-2d_1), then rank$(L(4C_n(a,b))) = 2n$.*

Proof. Let $4C_n(a,b)$ be connected with $n \equiv 0 \bmod 3$, $d_1 = \gcd(a,n)$ and $d_2 = \gcd(b,n)$. By Theorem 9, conditions (5) and (6) could be satisfied. Thus, we determine the number of simultaneous solutions of (5) and (6). Assume first that each of the two congruencies in (5) have solutions. Then we can reduce the system by the technique of Corollary 3 in [3] and then apply the Generalized Chinese Remainder Theorem. Thus, the system has $\gcd(d_1,d_2) = \gcd(a,b,n) = 1$ solution if and only if $\gcd(\frac{n}{d_1},\frac{n}{d_2})$ divides $\frac{n(d_2-d_1)}{3d_1d_2}$ or if $\gcd(\frac{n}{d_1},\frac{n}{d_2})$ divides $\frac{n(d_2-2d_1)}{3d_1d_2}$. However, note that d_1 and d_2 are relatively prime; and so by Lemma 4 $\gcd(\frac{n}{d_1},\frac{n}{d_2}) = \frac{n}{\mathrm{lcm}(d_1,d_2)} = \frac{n}{d_1d_2}$. So there is one solution to (5) if and only if $\frac{n}{d_1d_2}$ divides $\frac{n}{d_1d_2}(\frac{d_2-d_1}{3})$ or if $\frac{n}{d_1d_2}$ divides $\frac{n}{d_1d_2}(\frac{d_2-2d_1}{3})$. This implies that there is one solution to (5) if and only if 3 divides (d_2-d_1) or if 3 divides (d_2-2d_1)

We can proceed similarly with (6), resulting in one solution if and only if 3 divides $2(d_2-d_1)$ or $(2d_2-d_1)$.

Next, if (5) has a solution and if 3 divides (d_2-d_1), then 3 of course divides $2(d_2-d_1)$. Also, if (5) has a solution and if 3 divides (d_2-2d_1), then 3 also divides $(2d_2-d_1)$ since d_1 and d_2 are relatively prime. So if (5) has a solution, then (6) also has a solution. Thus, there are two solutions, and rank $L(4C_n(a,b)) \leq 2n-2$. Now, if $n \equiv 0 \bmod 12$, there could be more solutions given by (3) and (4) which implies a greater rank deficiency. So if $n \equiv (3,6,9) \bmod 12$, (3) or (4) could not be satisfied and there is no further possibility of rank deficiency; thus if $n \equiv (3,6,9) \bmod 12$ then rank $L(4C_n(a,b)) = 2n-2$.■

Corollary 3 *Let $4C_n(a,b)$ be connected with $n \equiv 0 \bmod 3$, $d_1 = \gcd(a,n)$ and $d_2 = \gcd(b,n)$. If 3 divides a or 3 divides b, then neither (5) nor (6) are satisfied.*

Proof. Let $4C_n(a,b)$ be connected with $n \equiv 0 \bmod 3$, $d_1 = \gcd(a,n)$ and $d_2 = \gcd(b,n)$. Let 3 divide a. Then 3 divides d_1 and 3 does not divide d_2. Thus 3 does not divide (d_2-d_1) nor (d_2-2d_1). If 3 divides b, then 3 does not divide d_1 and 3 divides d_2. Thus again, 3 does not divide (d_2-d_1) nor (d_2-2d_1). Hence, neither (5) nor (6) are satisfied.■

The importance of Corollary 3 for rank is that if $n \equiv (3,6,9) \bmod 12$ and 3 divides a or 3 divides b, then rank $L(4C_n(a,b)) = 2n$.

Corollary 4 *Let $4C_n(a,b)$ be connected with $n \equiv (3,6,9) \bmod 12$, $d_1 = \gcd(a,n)$ and $d_2 = \gcd(b,n)$, then*

$$rank(L(4C_n(a,b))) = \left\{ \begin{array}{ll} 2n-2 & \text{if } 3 \text{ divides } d_2 - d_1 \text{ or } d_2 - 2d_1 \\ 2n & \text{otherwise} \end{array} \right\}.$$

Now our attention turns to the case where n is divisible by 4.

Corollary 5 *Let $4C_n(a,b)$ be connected with $n \equiv 0 \bmod 4$, $d_1 = \gcd(a,n)$ and $d_2 = \gcd(b,n)$. If 4 divides a or 4 divides b, then neither (3) nor (4) are satisfied.*

Proof. Let $4C_n(a,b)$ be connected with $n \equiv 0 \bmod 4$, $d_1 = \gcd(a,n)$ and $d_2 = \gcd(b,n)$. Let 4 divide a. Then in (3) from Theorem 9, 4 also divides d_1, so d_1 does not divide $\frac{n}{2}$ and (3) has no solutions. If 4 divides b, then in (4), 4 also divides d_2, so d_2 does not divide $\frac{n}{2}$ and (4) has no solutions.■

The importance of Corollary 5 for rank is that if $n \equiv (4,8) \bmod 12$ and 4 divides a or 4 divides b, then rank $L(4C_n(a,b)) = 2n$.

Theorem 11 *Let $4C_n(a,b)$ be connected with $n \equiv 0 \bmod 4$, $d_1 = \gcd(a,n)$ and $d_2 = \gcd(b,n)$. If a and b are both odd, rank $L(4C_n(a,b)) = 2n$. If 4 divides $(2d_2 - d_1)$ or $(d_2 - 2d_1)$, then rank $L(4C_n(a,b)) \leq 2n-2$; equality holds if $n \equiv (4,8) \bmod 12$. Furthermore, if $n \equiv (4,8) \bmod 12$ and 4 does not divide $(2d_2 - d_1)$ and 4 does not divide $(d_2 - 2d_1)$, then rank $L(4C_n(a,b)) = 2n$.*

Proof. Let $4C_n(a,b)$ be connected with $n \equiv 0 \bmod 4$, $d_1 = \gcd(a,n)$ and $d_2 = \gcd(b,n)$. Then we determine the number of simultaneous solutions of (3) and (4) in Theorem 9. Notice that in (3) we have $bp \equiv \frac{n}{4} \bmod \frac{n}{2}$, which is actually two congruencies; one congruent to $\frac{n}{4} \bmod n$ and one congruent to $\frac{3n}{4} \bmod n$. Therefore, we only need to consider $bp \equiv \frac{n}{4} \bmod n$, since the number of solutions to this is half the number of solutions to the original congruence. Hence, we can simplify our examination of (3) and (4):

$$ap \equiv \frac{n}{2} \bmod n \text{ and } bp \equiv \frac{n}{4} \bmod n \tag{7}$$

$$ap \equiv \frac{n}{4} \bmod n \text{ and } bp \equiv \frac{n}{2} \bmod n \tag{8}$$

Now we determine the number of simultaneous solutions of (7) and (8). Assuming each congruence in (7) has a solution, we can reduce the congruencies and apply the Generalized Chinese Remainder Theorem. Condition (7) has $\gcd(d_1, d_2) = \gcd(a,b,n) = 1$ solution if and only if $\gcd(\frac{n}{d_1}, \frac{n}{d_2})$

divides $\frac{n(2d_2-d_1)}{4d_1d_2}$. But by Lemma 4, $\gcd(\frac{n}{d_1}, \frac{n}{d_2}) = \frac{n}{\mathrm{lcm}(d_1,d_2)} = \frac{n}{d_1d_2}$ since d_1 and d_2 are relatively prime. Thus (7) has one solution if and only if $\frac{n}{d_1d_2}$ divides $\frac{n}{d_1d_2}(\frac{2d_2-d_1}{4})$. Thus, (7) has one solution if and only if 4 divides $2d_2 - d_1$.

Similarly with (8), we have one solution if and only if 4 divides $d_2 - 2d_1$.

Now if a and b are both odd, then d_1 and d_2 are odd so $(2d_2 - d_1)$ is odd and $(d_2 - 2d_1)$ is odd. Thus (7) and (8) have no simultaneous solutions, and rank $L(4C_n(a,b)) = 2n$.

Since (7) or (8) could have one solution, the original congruencies (3) or (4) could have two solutions. Thus, there are two solutions if 4 divides $(2d_2 - d_1)$ or $(d_2 - 2d_1)$. Therefore, rank $L(C_n(a,b)) \leq 2n - 2$. Now if $n \equiv 0 \bmod 12$, there could be more solutions given by (5) and (6) which implies a greater rank deficiency. So if $n \equiv (4,8) \bmod 12$, (5) or (6) could not be satisfied and there is no further possibility of rank deficiency; thus if $n \equiv (4,8) \bmod 12$ then rank $L(4C_n(a,b)) = 2n - 2$.

Corollary 6 *Let $4C_n(a,b)$ be connected with $n \equiv (4,8) \bmod 12$, $d_1 = \gcd(a,n)$ and $d_2 = \gcd(b,n)$, then*

$$rank(L(4C_n(a,b)) = \left\{ \begin{array}{cl} 2n-2 & if\ 4\ divides\ 2d_2 - d_1 \text{ or } d_2 - 2d_1 \\ 2n & otherwise \end{array} \right\}.$$

Corollary 7 *Let $4C_n(a,b)$ be connected with $n \equiv 0 \bmod 12$, $d_1 = \gcd(a,n)$ and $d_2 = \gcd(b,n)$, then*

$$rank(L(4C_n(a,b))) = \left\{ \begin{array}{cl} 2n-4 & if\ (3\ divides\ d_2 - d_1\ or\ d_2 - 2d_1)\ and \\ & \quad (4\ divides\ 2d_2 - d_1\ or\ d_2 - 2d_1) \\ 2n-2 & if\ (3\ divides\ d_2 - d_1\ or\ d_2 - 2d_1)\ or \\ & \quad (4\ divides\ 2d_2 - d_1\ or\ d_2 - 2d_1) \\ & \quad but\ not\ both \\ 2n & otherwise \end{array} \right\}$$

Proof: Let $4C_n(a,b)$ be connected with $n \equiv 0 \bmod 12$, $d_1 = \gcd(a,n)$ and $d_2 = \gcd(b,n)$. Then any of the conditions in Theorem 9 could be satisfied, depending on the divisibility conditions given in Theorems 10 or 11. If all conditions are satisfied, there are four solutions; if only two sets of conditions are satisfied, there are two solutions; and if none are satisfied, there are no solutions, and the rank is diminished accordingly.■

Putting the previous results together, we have the following summary theorem for the rank of the line graph of a connected 4-circulant graph.

Theorem 12 *$rank(L(4C_n(a,b)) =$*

$$\left\{\begin{array}{ll} 2n-4 & if\ n \equiv 0 \bmod 12\ and\ (3\ divides\ d_2-d_1\ or\ d_2-2d_1)\ and \\ & \quad (4\ divides\ 2d_2-d_1 \text{ or } d_2-2d_1) \\ 2n-2 & if\ (n \equiv 0 \bmod 3\ and\ [3\ divides\ d_2-d_1\ or\ d_2-2d_1)])\ or \\ & \quad (n \equiv 0 \bmod 4\ and\ [4\ divides\ 2d_2-d_1 \text{ or } d_2-2d_1]) \\ & \quad but\ not\ both \\ 2n & otherwise \end{array}\right\},$$

where $d_1 = \gcd(a,n)$ and $d_2 = \gcd(b,n)$.

6 Some Cartesian Graph Products

We begin by recalling several facts that will be useful in this section. These basic results are well known and can be found in various sources including [1].

Lemma 5 *(i) The eigenvalues of C_n are $\lambda_j = 2\cos\frac{2\pi j}{n}$ for $j = 1, 2, \ldots, n$.*
(ii) The eigenvalues of K_n are $-1, -1, \ldots, -1, n-1$.
(iii) The eigenvalues of P_n are $\mu_j = 2\cos\frac{\pi j}{n+1}$ for $j = 1, 2, \ldots, n$. Hence the eigenvalues of P_2 are ± 1.
(iv) If G has eigenvalues $\lambda_1, \ldots, \lambda_n$ and H has eigenvalues $\mu_1, \ldots, \mu_m$, then $G \times H$ has eigenvalues $\lambda_i + \mu_j$ for $1 \le i \le n$ and $1 \le j \le m$.

Our first Cartesian graph product is that of C_n with P_2.

Theorem 13 *For $n \ge 3$,*

$$rank(L(C_n \times P_2)) = \left\{\begin{array}{ll} 3n & n\ odd \\ 3n-1 & n \equiv 2 \bmod 4 \\ 3n-3 & n \equiv 0 \bmod 4 \end{array}\right\}.$$

Proof. The eigenvalues of $C_n \times P_2$ are $2\cos\frac{2\pi j}{n} - 1$ for $j = 1, 2, \ldots, n$ and $2\cos\frac{2\pi j}{n} + 1$ for $j = 1, 2, \ldots, n$. Since $C_n \times P_2$ is 3-regular with $2n$ vertices and $3n$ edges, the eigenvalues of $L(C_n \times P_2)$ are n values of -2, as well as $2\cos\frac{2\pi j}{n}$ for $j = 1, 2, \ldots, n$ and $2\cos\frac{2\pi j}{n} + 2$ for $j = 1, 2, \ldots, n$. These eigenvalues will equal zero only when either $\cos\frac{2\pi j}{n} = 0$ for $j = 1, 2, \ldots, n$ or $\cos\frac{2\pi j}{n} = 1$ for $j = 1, 2, \ldots, n$. For $\cos\frac{2\pi j}{n} = 0$, it must be true that $\frac{2\pi j}{n} = \frac{(2k+1)\pi}{2}$ for some integer k. Hence, n must be divisible by 4 and $j = \frac{n}{4}$ or $j = \frac{3n}{4}$. For $\cos\frac{2\pi j}{n} = -1$, it must be true that $\frac{2\pi j}{n} = (2k+1)\pi$ for some integer k. Hence, n must be even and $j = \frac{n}{2}$.

Now, if n is odd, neither of these equations will hold for any n. Hence, $L(C_n \times P_2)$ will be full rank (i.e. $3n$). If $n \equiv 2 \bmod 4$, n is even but not divisible by 4. So there will only be one eigenvalue equal to zero. If $n \equiv 0 \bmod 4$, then n is even and divisible by 4. So there will be three eigenvalues equal to zero. Therefore the result holds. ■

The case of the product of a complete graph with P_2 reduces somewhat further. The rank of this line graph is full except for two special cases.

Theorem 14 *For $n \geq 2$,*

$$rank(L(K_n \times P_2)) = \left\{ \begin{array}{ll} 2 & n = 2 \\ 13 & n = 4 \\ n^2 & otherwise \end{array} \right\}.$$

Proof. The eigenvalues of $K_n \times P_2$ are $(n-1)$ values each of -2 and 0 along with $(n-2)$ and n. Since $K_n \times P_2$ is n-regular with $2n$ vertices and n^2 edges, the eigenvalues of $L(K_n \times P_2)$ are $n^2 - 2n$ values of -2, $(n-1)$ values each of $(n-4)$ and $(n-2)$ along with $(2n-4)$ and $(2n-2)$. These eigenvalues are never zero whenever $n \neq 1, 2, 4$. Hence, in these cases, $L(K_n \times P_2)$ has full rank (i.e. n^2). When $n = 2$, the only nonzero eigenvalues are -2 and 2. Hence the rank of $L(K_2 \times P_2) = 2$. When $n = 4$, the only nonzero eigenvalues are $n^2 - 2n = 8$ values of -2, $n - 1 = 3$ values of $n - 2 = 2$ plus $2n - 4 = 4$ and $2n - 2 = 6$. Hence the rank of $L(K_4 \times P_2) = 13$. Therefore the result holds. ■

Our next case is that of the product of two cycles.

Theorem 15 *For $m, n \geq 3$, $rank(L(C_m \times C_n)) =$*

$$\left\{ \begin{array}{ll} 2mn-8 & if\ m \equiv 0 \bmod 12\ and\ n \equiv 0 \bmod 12 \\ 2mn-6 & \begin{array}{l} if\ (m \equiv 6 \bmod 12\ and\ n \equiv 0 \bmod 12) \\ or\ (m \equiv 0 \bmod 12\ and\ n \equiv 6 \bmod 12) \end{array} \\ 2mn-4 & \begin{array}{l} if\ (m \equiv 0 \bmod 4\ and\ n \equiv 0 \bmod 4) \\ or\ (m \equiv 0 \bmod 3\ and\ n \equiv 0 \bmod 3) \end{array} \\ 2mn-2 & \begin{array}{l} if\ (m \equiv 2 \bmod 4\ and\ n \equiv 0 \bmod 4) \\ or\ (m \equiv 0 \bmod 4\ and\ n \equiv 2 \bmod 4) \end{array} \\ 2mn & otherwise \end{array} \right\}$$

Proof. Assume both $m, n \geq 3$. The spectrum of $L(C_m \times C_n)$ consists of mn values of -2 and mn values of $\lambda_j + 2$, $j = 1, ..., mn$, where λ_j is an eigenvalue of $C_m \times C_n$. However, $\lambda_j + 2 = 2\cos\frac{2\pi a}{m} + 2\cos\frac{2\pi b}{n} + 2$, with $a = 1, ..., m$ and $b = 1, ..., n$. Setting $\lambda_j + 2 = 0$ gives $\cos\frac{2\pi a}{m} + \cos\frac{2\pi b}{n} = -1$, which has already been solved. Reworking the method used in Theorem 9 in this new context reveals the following conditions.

An eigenvalue of $L(C_m \times C_n)$ will be zero if and only if:

$$a \equiv \frac{m}{2} \bmod m \text{ and } b \equiv \pm\frac{n}{4} \bmod n \quad (9)$$

$$or\ a \equiv \pm\frac{m}{3} \bmod m \text{ and } b \equiv \pm\frac{n}{3} \bmod n \quad (10)$$

where $a = 1, ..., m$ and $b = 1, ..., n$.

For any nonzero eigenvalues, (9) implies that both m and n must be even. Moreover, one of m or n must have an even factor of at least 4. If it is the case that, say, m's only even factor is 2 (so that $m = 2k, k$ odd) and n's is 4 or larger, then $\cos\frac{2\pi a}{m} = \cos\frac{\pi a}{k}$ will only be equal to -1 once (at $a = k$), and this value will combine with the two values of 0 given by $\cos\frac{2\pi b}{n}$ (at $b = \frac{n}{4}$ and $b = \frac{3n}{4}$). Thus there will be only two solutions that satisfy (9). More precisely, there will be two zero eigenvalues if and only if $m \equiv 2 \bmod 4$ and $n \equiv 0 \bmod 4$ (or if and only if $m \equiv 0 \bmod 4$ and $n \equiv 2 \bmod 4$).

However, if both m and n have at least an even factor of 4, then each cosine will give a value of zero and two values of -1, for a total of four solutions. Hence there will be four zero eigenvalues if and only if $m \equiv 0 \bmod 4$ and $n \equiv 0 \bmod 4$.

Now (10) implies that both m and n are divisible by 3, so then each cosine gives two values of $-\frac{1}{2}$. Hence, there will be four zero eigenvalues if and only if $m \equiv 0 \bmod 3$ and $n \equiv 0 \bmod 3$. Both (9) and (10) together imply six zero eigenvalues if and only if $m \equiv 6 \bmod 12$ and $n \equiv 0 \bmod 12$ (or if and only if $m \equiv 0 \bmod 12$ and $n \equiv 6 \bmod 12$); there will be eight zero eigenvalues if and only if $m \equiv 0 \bmod 12$ and $n \equiv 0 \bmod 12$. Thus, the result is established.■

Our final case involves the product of two complete graphs.

Theorem 16 *For $m, n \geq 3$,*

$$rank(L(K_m \times K_n)) = \left\{ \begin{array}{ll} 14 & if\ m = n = 3 \\ \frac{1}{2}mn(m+n-2) & otherwise \end{array} \right\}.$$

Proof. Assume that $m, n \geq 3$. The spectrum of $K_m \times K_n$ consists of one eigenvalue of $(m+n-2)$, $(n-1)$ eigenvalues of $(m-2)$, $(m-1)$ eigenvalues of $(n-2)$, and $(m-1)(n-1)$ eigenvalues of -2. Since $K_m \times K_n$ is regular of degree $(m+n-2), L(K_m \times K_n)$ has $\frac{1}{2}mn(m+n-2)$ vertices. Then the spectrum of $L(K_m \times K_n)$ consists of $\frac{1}{2}mn(m+n-4)$ eigenvalues of -2, and mn values of $\lambda_j + m + n - 2 - 2 = \lambda_j + m + n - 4$. Substituting the above four possible values for λ_j, we see these mn eigenvalues of the line graph are: one value of $2m + 2n - 6$, $(n-1)$ values of $2m + n - 6$, $(m-1)$ values of $m + 2n - 6$ and the remaining $(m-1)(n-1)$ values of $m + n - 6$. Setting all four expressions equal to zero gives that only $m + n - 6 = 0$ will have a solution since both m and n are at least 3. Hence there are exactly $(m-1)(n-1)$ zero eigenvalues if and only if $m = n = 3$, in which case, $\text{rank}(L(K_m \times K_n)) = \frac{1}{2}mn(m+n-2) - (m-1)(n-1) = 14$. Otherwise, the rank is full.■

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