THE CONSTRUCTION OF 2-FOLD TRIANGULATIONS

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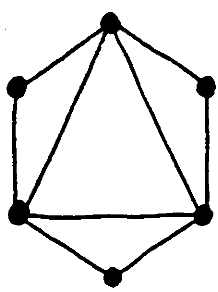
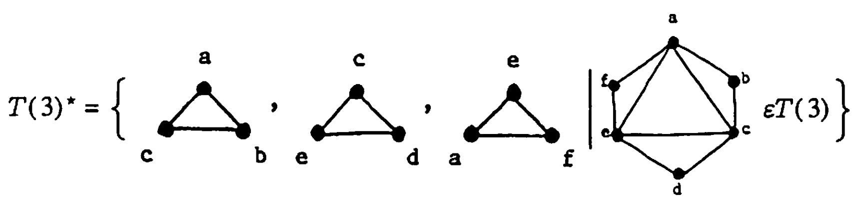
1. Introduction.

A *Steiner triple system* (more simply, triple system) is a pair (K_n, T) where K_n is the complete undirected graph on n vertices and T is a collection of edge-disjoint triangles (triples) which *partition* K_n. The *number* n is called the *order* of the triple system (K_n, T). It is well-known that a necessary and sufficient condition for the existence of a triple system (K_n, T) is $n \equiv 1$ or $3 \pmod 6$ and in this case $|T| = n(n-1)/6$. A *triangulation* is a triple $(K_n, T(3), D)$ where $T(3)$ is a collection of edge-disjoint copies of the graph



(called a 3-triangle)

and D is a collection of edge-disjoint triangles such that $T(3) \cup D$ is an edge-disjoint *partition* of K_n. The collection of triangles D is called the *deficiency* of the triangulation $(K_n, T(3), D)$. If we set



then $(K_n, T(3)^* \cup D)$ is a Steiner triple system.

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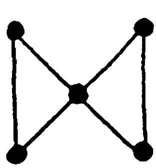
Now given a triple system (K_n, T), the problem of finding a triangulation $(K_n, T(3), D)$ such that $T = T(3)^* \cup D$ with deficiency D as small as possible is immediate. For our purposes "small" means $|D| = 0, 1$, or 2. Whether or not such a triangulation exists for an arbitrary triple system is undoubtably an extremely difficult problem. A more tractable problem is the *construction*, for each $n \equiv 1$ or $3 \pmod 6$, of a triple system of order n having a triangulation with deficiency D, where $|D| = 0, 1$, or 2: that is to say, the construction for each $n \equiv 1$ or $3 \pmod 6$ of a triangulation $(K_n, T(3), D)$ with $|D| = 0, 1$, or 2. In [3] R.C. Mullin, A.L. Poplove, and L. Zhu gave a complete solution of this problem by constructing a triangulation $(K_n, T(3), D)$ with: $|D| = 0$ for every $n \equiv 1$ or $9 \pmod{18}$, $|D| = 1$ for every $n \equiv 3$ or $7 \pmod{18}$, and $|D| = 2$ for every $n \equiv 13$ or $15 \pmod{18}$. In the case with deficiency $|D| = 2$, two triangulations are given for every $n \equiv 13$ or $15 \pmod{18}$: one where the deficiency consists of a pair of disjoint triangles and one consisting of a pair of triangles with exactly one vertex in common (the only two possibilities). It is worth remarking that P. Horak and A. Rosa [1] have studied the decomposition of triple systems into subgraphs other than 3-triangles.

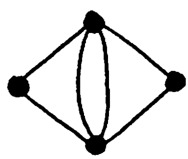
A *2-fold triple system* is a pair $(2K_n, T)$ where T is a collection of edge-disjoint triangles which partition $2K_n$. It is well-known that a 2-fold triple system of order n exists precisely when $n \equiv 0$ or $1 \pmod 3$ and that $|T| = n(n-1)/3$. A *2-fold triangulation* is a triple $(2K_n, T(3), D)$ where $T(3)$ and D are defined as before and such that $T(3) \cup D$ is an edge-disjoint partition of $2K_n$. If $T(3)^*$ is defined as before, then $(2K_n, T(3)^* \cup D)$ is a 2-fold triple system.

The object of this paper is to give a complete solution of the 2-fold triangulation problem. In particular, we construct a 2-fold triangulation $(2K_n, T(3), D)$ with: $|D| = 0$ for every $n \equiv 0$ or $1 \pmod 9$, $|D| = 1$ for every $n \equiv 4$ or $6 \pmod 9$, and $|D| = 2$ for every $n \equiv 3$ or $7 \pmod 9$. In the case with deficiency $|D| = 2$, there are four possibilities for the deficiency D:

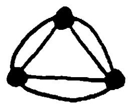
disjoint sets of vertices



one vertex in common



two vertices in common

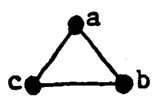


three vertices in common.

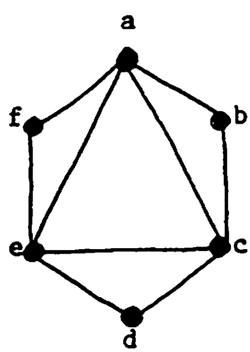
We handle all four possibilities.

2. 2-fold triangulations with $|D| = 0$.

In what follows, we will abbreviate "2-fold triangulation with $|D| = 0$" to simply "2-fold triangulation". Additionally we will denote the triangle



by $\{a, b, c\}$, and the 3-triangle



consisting of the 3-triangles

$\{a, b, c\}$, $\{c, d, e\}$, and $\{e, f, a\}$ by any cyclic shift of (a^*, b, c^*, d, e^*, f) or (a^*, f, e^*, d, c^*, b).

Example 2.1: Each of the following is a 2-fold triangulation of the given order n.

$n = 9$.

$$\{(1^*, 3, 0^*, 7, 5^*, \infty) + i \pmod 8 \mid i \in Z_8\}, \text{ where } \infty + i = \infty.$$

$n = 10$.

$$\{((0,0)^*, (0,1), (4,1)^*, (2,1), (2,0)^*, (4,0)) + (i,0) \pmod 5 \mid i \in Z_5\} \cup$$
$$\{((0,0)^*, (1,1), (3,1)^*, (4,0), (2,1)^*, (1,0)) + (i,0) \pmod 5 \mid i \in Z_5\}.$$

$n = 18$.

$$\{(0^*, 11, 3^*, 13, 7^*, \infty) + i \pmod{17} \mid i \in Z_{17}\} \cup$$
$$\{(0^*, 1, 2^*, 16, 4^*, 9) + i \pmod{17} \mid i \in Z_{17}\}, \text{ where } \infty + i = \infty.$$

$n = 19$.

$$\{(0^*, 18, 5^*, 9, 2^*, 10) + i \pmod{19} \mid i \in Z_{19}\} \cup$$
$$\{(0^*, 1, 14^*, 10, 17^*, 9) + i \pmod{19} \mid i \in Z_{19}\}.$$

Let $H = \{h_1, h_2, \ldots, h_k\}$ be a partition of the set Q. In what follows we will call the sets $h_i \in H$ *holes*. Now let (Q, o) be a quasigroup with the property that for each hole $h_i \in H$, (h_i, o) is a subquasigroup. Such a quasigroup (Q, o) is called quasigroup with holes H.

The $9k$ construction. Let (Q, o) be a quasigroup of order $3k$ with k holes $H = \{h_1, h_2, \ldots, h_k\}$ of size 3 and set $S = Q \times \{1, 2, 3\}$. Now define $(S, T(3), D = \phi)$ as follows:

(1) For each hole $h \in H$, let $(h \times \{1, 2, 3\}, T(3h), D = \phi)$ be a 2-fold triangulation of order 9 and place the eight 3-triangles of $T(3h)$ in $T(3)$; and

(2) for each x and y belonging to different holes of H place the two 3-triangles $((x, 1)^*, (y, 1), (x \, o \, y, 2)^*, (x, 2), (y, 3)^*, (x \, o \, y, 3))$, and $((y, 1)^*, (x, 1), (y \, o \, x, 2)^*, (y, 2), (x, 3)^*, (y \, o \, x, 3))$ in $T(3)$.

Claim: $(S, T(3), D = \phi)$ is a triangulation. It suffices to show that every edge of the form $\{(a, i), (b, j)\}$, a and b in different holes of H, belongs to two 3-triangles of $T(3)$. We handle the case $\{(a, 1), (b, 3)\}$, the other cases being similar or trivial. Let $a \, o \, c = b$ *and* $a \, o \, b = d$. Then (by part (2) of the construction) the two 3-triangles

$$((a, 1)^*, (c, 1), (a \, o \, c, 2)^*, (a, 2), (c, 3)^*, (a \, o \, c, 3)) =$$
$$((a, 1)^*, (c, 1), (b, 2)^*, (a, 2), (c, 3)^*, (b, 3)) \text{ and}$$
$$((a, 1)^*, (b, 1), (a \, o \, b, 2)^*, (a, 2), (b, 3)^*, (a \, o \, b, 3)) =$$
$$((a, 1)^*, (b, 1), (d, 2)^*, (a, 2), (b, 3)^*, (d, 3)) \text{ belong to } T(3).$$

∎

The $9k + 1$ construction. Let (Q, o) be a quasigroup of order $3k$ with holes $H = \{h_1, h_2, \ldots, h_k\}$ of size 3 and set $S = \{\infty\} \cup (Q \times \{1, 2, 3\})$. Define $(S, T(3), D = \phi)$ by:

(1) For each hole $h \in H$, let $(\{\infty\} \cup (h \times \{1, 2, 3\}), T(3h), D = \phi)$ be a 2-fold triangulation of order 10 and place the ten 3-triangles of $T(3h)$ in

$T(3)$; and

(2) the same as part (2) in the $9k$ construction.

Then $(S, T(3), D = \phi)$ is a 2-fold triangulation of order $9k+1$. ∎

Theorem 2.2. *A 2-fold triangulation of order n exists precisely when $n \equiv 0$ or $1 \pmod 9$.*

Proof: Let $k \geq 3$ and let (Q, o) be a quasigroup of order $3k$ with holes of size 3. (Take the direct product of an idempotent quasigroup of order k and a quasigroup of order 3.) Then the $9k$ and $9k+1$ constructions produce a 2-fold triangulation of every order $n \equiv 0$ or $1 \pmod 9 \geq 27$. The cases $n = 9$, 10, 18, and 19 are taken care of by Example 2.1. ∎

3. 2-fold triangulations with $|D| = 1$.

A necessary condition for the existence of a 2-fold triangulation with $|D| = 1$ is $n \equiv 4$ or $6 \pmod 9$. We handle the case $n \equiv 6 \pmod 9$ first. We begin with three examples.

Example 3.1: Each of the following is a 2-fold triangulation with $|D| = 1$ of the given order n.

$n = 6$.

$T(3) = \{(2^*, 1, 3^*, 6, 4^*, 5), (3^*, 1, 4^*, 2, 6^*, 5), (5^*, 4, 1^*, 6, 2^*, 3)\}$ and $D = \{\{5, 1, 6\}\}$.

$n = 15$.

In [3] a triangulation $(K_{15}, T(3), D)$ is constructed with deficiency $D = \{\{1,2,3\}, \{1,4,5\}\}$. Let $\alpha = (15)(26)$. Then $(K_{15}, T(3)\alpha, D\alpha)$ is a triangulation with deficiency $D\alpha = \{\{3,5,6\}, \{1,4,5\}\}$, and $(2K_{15}, T(3) \cup T(3)\alpha \cup \{(1^*, 2, 3^*, 6, 5^*, 4)\}, \{1,4,5\})$ is a 2-fold triangulation with deficiency 1.

$n = 24$.

Let $(\{\infty\} \cup Q, T)$ be a Steiner triple system of order 9 and let $H = \{\{\infty\} \cup h_1, \{\infty\} \cup h_2, \{\infty\} \cup h_3, \{\infty\} \cup h_4\}$ be the 4 triples containing ∞. Set $S = Q \times \{1,2,3\}$ and define $(S, T(3), D)$ as follows:

(1) For each "hole" h_i, let $(h_i \times \{1,2,3\}, T(3h_i), D_i)$ be a 2-fold triangulation with $|D_i| = 1$ and require $T(3h_i) \subseteq T(3)$ and $D_i \subseteq D$; and

(2) for each triple $\{a, b, c\} \in T \backslash H$ let

$$t(a,b,c) = \begin{cases} ((a,1)^*, (b,1), (c,2)^*, (a,2), (b,3)^*, (c,3)), \\ ((c,1)^*, (a,1), (b,2)^*, (c,2), (a,3)^*, (b,3)), \text{ and} \\ ((b,1)^*, (c,1), (a,2)^*, (b,2), (c,3)^*, (a,3)\}, \text{ and} \end{cases}$$

place *two* copies of the 3-triangles belonging to $t(9)$ in $T(3)$. Then $(S, T(3), D)$ is a 2-fold triangulation with deficiency $|D| = 4$. We modify $(S, T(3), D)$ as follows. We can assume $D_2 = \{(a,1), (a,2), (a,3)\}$,

$D_3 = \{(b,1),(b,2),(b,3)\}$, and $D_4 = \{(c,1),(c,2),(c,3)\}$ and that $\{a,b,c\} \in T\backslash H$. Set $T(3)^\star = (T(3)\backslash t(a,b,c)) \cup t(a,b,c)^\star$ and $D^\star = D_1$, where

$$t(a,b,c)^\star = \begin{cases} ((a,1)^\star,(b,1),(c,1)^\star,(c,2),(c,3)^\star,(b,2)), \\ ((a,2)^\star,(b,2),(c,2)^\star,(a,3),(b,1)^\star,(c,3)), \\ ((a,3)^\star,(c,3),(b,3)^\star,(b,1),(b,2)^\star,(c,1)), \text{ and} \\ ((a,1)^\star,(a,3),(a,2)^\star,(c,1),(b,3)^\star,(c,2)). \end{cases}$$

Since $t(a,b,c) \cup D_2 \cup D_3 \cup D_4$ and $t(a,b,c)^\star$ are mutually balanced (= cover the same edges) $(S, T(3)^\star, D^\star)$ is a 2-fold triangulation with $|D^\star| = 1$. ∎

Before plunging into the $9k + 6$ construction we need one more preliminary result.

Lemma 3.2. *If $k \geq 3$, there exists a quasigroup of order $3k + 2$ with* one *hole of size 2 and k holes of size 3.*

Proof: Let (Q, o) be an idempotent quasigroup of order k having a transversal T disjoint from the main diagonal. (A pair of orthogonal quasigroups guarantees such a quasigroup for all $k \geq 3$, $k \neq 6$, and $k = 6$ is handled easily by example.) Then the *singular direct product* (see [2], for example) constructed from (Q, o) using the transversal T and a quasigroup of order 5 with a subquasigroup of order 2, and a quasigroup of order 3, gives a quasigroup of order $3k + 2$ with *one* hole of size 2 and k holes of size 3. ∎

The $9k+6$ construction. Let (Q, o) be a quasigroup of order $3k+2$ with holes $H = \{t, h_1, h_2, \ldots, h_k\}$, where $|t| = 2$ and $|h_i| = 3$. Set $S = Q \times \{1,2,3\}$ and define $(S, T(3), D)$ as follows:

(1) Let $(t \times \{1,2,3\}, T(3t), D(t))$ be a 2-fold triangulation of order 6 with $|D(t)| = 1$ and require $T(3t) \subseteq T(3)$ and $D(t) = D$;
(2) for each hole $h_i \in H$, $|h_i| = 3$, let $(h_i \times \{1,2,3\}, T(3h_i), D_i = \phi)$ be a 2-fold triangulation of order 9 and require $T(3h_i) \subseteq T(3)$; and
(3) for each x and y belonging to different holes of H place the two 3-triangles $((x,1)^\star,(y,1),(x \circ y,2)^\star,(x,2),(y,3)^\star,(x \circ y,3))$, and $((y,1)^\star,(x,1),(y \circ x,2),(y,2),(x,3)^\star,(y \circ x,3))$ in $T(3)$.

It is immediate that $(S, T(3), D)$ is a 2-fold triangulation of order $9k+6$ with deficiency $|D| = 1$. ∎

Theorem 3.3. *A 2-fold triangulation of order n with deficiency $|D| = 1$ exists for every $n \equiv 6 \pmod 9$.*

Proof: $n = 6$, 15, and 24 are taken care of by Example 3.1 and the $9k + 6$ construction takes care of the remaining cases. ∎

We now shift to the case of $n \equiv 4 \pmod 9$. As with the case $n \equiv 6 \pmod 9$ we begin with some *necessary* examples.

Example 3.4: Each of the following is a 2-fold triangulation with $|D| = 1$ of the given order n.

$n = 13$.

In [3] a triangulation $(K_{13}, T(3), D)$ is constructed with deficiency $D = \{\{1,2,3\}, \{1,4,5\}\}$. Let $\alpha = (15)(26)$. Then $(K_{15}, T(3)\alpha, D\alpha)$ is a triangulation with deficiency $D\alpha = \{\{3,5,6\}, \{1,4,5\}\}$. Then $(2K_{15}, T(3) \cup T(3)\alpha \cup \{(1^*, 2, 3^*, 6, 5^*, 4)\}, \{1,4,5\})$ is a 2-fold triangulation with deficiency = 1.

$n = 22$.

In what follows we will denote by $(K_9, t(a,b,c)^*, \phi)$ the triangulation of order 9 defined in Example 3.1 and by $(K_9, t(a,b,c), D(a,b,c))$ the triangulation of order 9 with deficiency $|D| = 3$ defined in Example 3.1. Finally, let $(K_7, T(x,y), \{(x,1),(x,2),(x,3)\})$ be a triangulation of order 7 with deficiency 1 defined on $\{\infty\} \cup (\{x,y\} \times \{1,2,3\})$. Then $(2K_{22}, T(3), \{(4,1), (4,2), (4,3)\})$ is a 2-fold triangulation of order 22 with deficiency 1 where

$$\begin{aligned} T(3) = {} & t(1,4,6)^* \cup t(2,4,6)^* \cup t(2,3,6)^* \cup t(1,3,4)^* \cup t(1,3,5)^* \\ & \cup t(3,5,6)^* \cup t(2,4,7)^* \cup t(4,5,7)^* \cup t(1,6,7)^* \cup t(2,5,7)^* \\ & \cup t(1,2,5) \cup T(1,7) \cup T(2,1) \cup T(3,2) \cup T(3,4) \cup T(4,5) \\ & \cup T(5,6) \cup T(7,6) \cup C(7,3), \end{aligned}$$

where $(2K_6, C(7,3), \{(7,1),(7,2),(7,3)\})$ is a 2-fold triangulation of order 6 with deficiency 1 based on $\{3,7\} \times \{1,2,3\}$.

$n = 31$.

In [3] a triangulation $(K_{31}, T(3), D)$ is constructed with deficiency $D = \{\{1,2,3\}, \{4,5,6\}\}$. If $\alpha = (15)(26)$ then $2(K_{31}, T(3) \cup T(3)\alpha \cup \{(1^*, 2, 3^*, 6, 5^*, 4)\}, D^* = \{\{1,4,5\}\})$ is a 2-fold triangulation with deficiency $|D^*| = 1$.

As with the $9k + 6$ construction, we will need one preliminary lemma before stating the $9k + 4$ construction. ∎

Lemma 3.5. *If $k \geq 3$, there exists a quasigroup of order $3k + 4$ with one hole of size 4 and k holes of size 3.*

Proof: Let (Q, o) be an idempotent quasigroup of order $k + 1$ having a transversal T intersecting the main diagonal in precisely the cell (1,1). (Three mutually orthogonal quasigroups guarantee such a quasigroup for all $k \geq 3$, $k + 1 \neq 6$ or 10, and 6 and 10 can be handled without difficulty by example.) Then the *singular direct product* constructed from (Q, o) using the transversal T and a quasigroup of order 4 with a subquasigroup of order 1, and a quasigroup of order 3, gives a quasigroup of order $3k + 4$ with *one* hole of size 4 and k holes of size 3. ∎

The $9k+4$ construction. Let (Q, o) be a quasigroup of order $3k + 4$ with holes

$H = \{f, h_1, h_2, \ldots, h_k\}$, where $|f| = 4$ and $|h_i| = 3$. Set $S = \{\infty\} \cup (Q \times \{1,2,3\})$ and define $(S, T(3), D)$ as follows:

(1) Let $(\{\infty\} \cup (f \times \{1,2,3\}), T(3f), D(f))$ be a 2-fold triangulation of order 13 with $|D(f)| = 1$ and require $T(3f) \subseteq T(3)$ and $D(f) = D$;

(2) for each hole $h_i \in H$, $|h_i| = 3$, let $(\{\infty\} \cup (h_i \times \{1,2,3\}), T(3h_i), D_i = \phi)$ be a 2-fold triangulation of order 10 and require $T(3h_i) \subseteq T(3)$; and

(3) for each x and y belonging to different holes of H place the two 3-triangles $((x,1)^*, (y,1), (x \circ y, 2)^*, (x,2), (y,3)^*, (x \circ y, 3))$, and $((y,1)^*, (x,1), (y \circ x, 2)^*, (y,2), (x,3)^*, (y \circ x, 3))$ in $T(3)$.

Then $(S, T(3), D)$ is a 2-fold triangulation of order $9k + 4$ with deficiency $|D| = 1$. ∎

Theorem 3.6. *A 2-fold triangulation of order n with deficiency $|D| = 1$ exists for every $n \equiv 4 \pmod 6$.*

Proof: $n = 13$, 22, and 31 are taken care of by Example 3.5 and the $9k + 4$ construction handles the remaining cases. ∎

4. 2-fold triangulations with $|D| = 2$.

As mentioned, in the introduction there are four possibilities for the deficiency D here: a pair of triples having 0, 1, 2, or 3 vertices in common. We handle all four cases. We begin with some examples.

Example 4.1: For *each* n we give four 2-fold triangulations of order n and deficiency $|D| = 2$ covering all four possibilities for D.

$n = 7$.

In [3] a triangulation $(K_7, T(3), D)$ is given with $|D| = 1$. A suitable permutation (a la Examples 3.1 and 3.4) gives a 2-fold triangulation $(2K_7, T(3) \cup T(3)\alpha, D \cup D\alpha)$, where $D \cup D\alpha$ consists of a pair of triples with the prescribed number of vertices in common.

$n = 16$.

Set

$$T(3) = \{(0^*, 4, 1^*, 3, 9^*, 12) + i \mid i \in Z_{13}\}$$
$$\cup \{(0^*, \infty_1, 2^*, \infty_2, 7^*, \infty_3) + i \mid i \in Z_{13}\},$$

where $\infty_i + i = \infty_i$. Define

$$T(3)_1 = (T(3) \backslash \{(0^*, \infty_1, 2^*, \infty_2, 7^*, \infty_3)\}) \cup \{(\infty_3^*, \infty_1, \infty_2^*, 2, 7^*, 0)\}$$
$$T(3)_2 = (T(3) \backslash \{(0^*, \infty_1, 2^*, \infty_2, 7^*, \infty_3)\}) \cup \{(\infty_2^*, \infty_1, 2^*, 0, 7^*, \infty_3)\}$$
$$T(3)_3 = (T(3) \backslash \{(0^*, \infty_1, 2^*, \infty_2, 7^*, \infty_3)\}) \cup \{(\infty_1^*, 0, \infty_3^*, 7, \infty_2^*, 2)\}.$$

Then

$(2K_{16}, T(3), \{\{\infty_1, \infty_2, \infty_3\}, \{\infty_1, \infty_2, \infty_3\}\})$,
$(2K_{16}, T(3)_1, \{\{2, 0, \infty_1\}, \{\infty_1, \infty_2, \infty_3\}\})$,
$(2K_{16}, T(3)_2, \{\{0, \infty_1, \infty_2\}, \{\infty_1, \infty_2, \infty_3\}\})$, and
$(2K_{16}, T(3)_3, \{\{0, 2, 7\}, \{\infty_1, \infty_2, \infty_3\}\})$,

are four 2-fold triangulations with deficiency 2 covering all four possibilities.

$n = 25$.

In [3] a triangulation $(K_{25}, T(3), D)$ is given with $|D| = 1$. As above, suitable permutation give the desired 2-fold triangulations with deficiency 2.

The $9k+7$ construction. Let (Q, o) be a quasigroup of order $3k+2$ with holes $H = \{t, h_1, h_2, \ldots, h_k\}$, where $|t| = 2$ and $|h_i| = 3$. Let $S = \{\infty\} \cup (Q \times \{1,2,3\})$ and define $(S, T(3), D)$ as follows:

(1) Let $(\{\infty\} \cup (t \times \{1,2,3\}), T(3t), D(t))$ be a 2-fold triangulation of order 7 with $|D(t)| = 2$ and require $T(3t) \subseteq T(3)$ and $D(t) = D$;
(2) same as the $9k+4$ construction; and
(3) same as the $9k+4$ construction.

Then $(S, T(3), D)$ is a 2-fold triangulation of order $9k+7$ with deficiency $|D| = 2$. Since the 2-fold triangulation in (1) can have deficiency any one of the four possibilities, so can $(S, T(3), D)$. ∎

Theorem 4.2. *A 2-fold triangulation of order n and deficiency $|D| = 2$, D any one of the four possibilities, exists for every $n \equiv 7 \pmod 9$.*

Proof: The cases $n = 7$, 16, and 25 are handled by Example 4.1 and the $9k+7$ construction takes care of the remaining cases. ∎

More examples!

Example 4.3: For *each* n we give four 2-fold triangulations of order n and deficiency $|D| = 2$ covering all four possibilities for D.

$n = 12$.

Let

$$T(3) = \{(10^*,1,8^*,2,6^*,3),(1^*,7,11^*,2,9^*,12),(7^*,1,5^*,8,3^*,6),$$
$$(1^*,10,5^*,2,12^*,6),(4^*,1,11^*,5,3^*,12),(4^*,7,2^*,5,9^*,3),$$
$$(7^*,10,2^*,8,12^*,3),(4^*,8,10^*,5,7^*,11),$$
$$(4^*,8,1^*,9,6^*,5),(3^*,11,8^*,7,9^*,10),(2^*,4,10^*,12,11^*,6),$$
$$(5^*,4,6^*,10,9^*,11),(6^*,7,12^*,10,11^*,8),(8^*,7,9^*,4,12^*,5)\}.$$

Define

$$T(3)_1 = (T(3)\backslash\{(10^*,1,8^*,2,6^*,3)\}) \cup \{(3^*,2,6^*,8,10^*,1)\},$$
$$T(3)_2 = (T(3)\backslash\{(10^*,1,8^*,2,6^*,3)\}) \cup \{(3^*,1,2^*,8,6^*,10)\}, \text{ and}$$

$T(3)_3 = (T(3)\backslash\{(10^\star,1,8^\star,2,6^\star,3)\}) \cup \{(1^\star,8,2^\star,6,3^\star,10)\}$.

Then

$$(2K_{12}, T(3), \{\{1,2,3\},\{1,2,3\}\}),$$
$$(2K_{12}, T(3)_1, \{\{1,2,3\},\{1,2,8\}\}),$$
$$(2K_{12}, T(3)_2, \{\{1,2,3\},\{1,8,10\}\}), \text{ and}$$
$$(2K_{12}, T(3)_3, \{\{1,2,3\},\{8,9,10\}\})$$

are four 2-fold triangulations with $|D| = 2$ covering all four possibilities.

$n = 21$.

In [3] a triangulation $(K_{21}, T(3), D)$ is given with $|D| = 1$. Suitable permutations then give four 2-fold triangulations with the desired deficiences $D^\star$, $|D^\star| = 2$.

$n = 30$.

Let Q be a set of size 9 with holes $H = \{h_1, h_2, h_3\}$, $|h_i| = 3$. Let $X = \{\infty\} \cup Q$, and let (X, T) be a 2-fold triple system of order 10 such that each of $(\{\infty\} \cup h_i, T)$ is a subsystem of order 4. Let $S = X \times \{1,2,3\}$ and define $(S, T(3), D)$ as follows:

(1) For each $h \in H$, let $((\{\infty\} \cup h) \times \{1,2,3\}, T(h), D(h))$ be a 2-fold triangulation , where $D(h) = \{\{(\infty,1),(\infty,2),(\infty,3)\},\{(\infty,1),(\infty,2),(\infty,3)\}\}$, and require that $T(h) \subseteq T(3)$ and $D(h) = D$;

(2) for each triple $\{a,b,c\} \in T\backslash H$ place the three 3-triangles in $t(a,b,c)$ (defined in Example 3.1) in $T(3)$.

Then $(S, T(3), D)$ is a 2-fold triangulation of order 30 with deficiency $|D| = 2$. Unplugging any one of three subsystems of order 12 in (1) and replacing it with a 2-fold triangulation with the required deficiency completes this example.

The $9k + 3$ construction. Let $(Q, \circ)$ be a quasigroup of order $3k + 1$ with holes $H = \{f, h_1, h_2, \ldots, h_{k-1}\}$ where $|f| = 4$ and $|h_i| = 3$. Let $S = Q \times \{1,2,3\}$ and define $(S, T(3), D)$ as follows:

(1) Let $(f \times \{1,2,3\}, T(3f), D(f))$ be a 2-fold triangulation of order 12 with $|D(f)| = 2$ and require $T(3f) \subseteq T(3)$ and $D(f) = D$;

(2) same as the $9k + 6$ construction; and

(3) same as the $9k + 6$ construction.

Then $(S, T(3), D)$ is a 2-fold triangulation of order $9k + 3$ with deficiency $|D| = 2$. Since the 2-fold triangulation in (1) can have deficiency any one of the four possibilities so can $(S, T(3), D)$. ∎

Theorem 4.4. *A 2-fold triangulation of order n and deficiency $|D| = 2$, D any one of the four possibilities, exists for every $n \equiv 3 \pmod 9$.*

Proof: The cases $n = 12, 21$, and 30 are handled by Example 4.3 and the $9k + 3$

construction takes care of the remaining cases. ∎

5. Summary and concluding remarks.

The following theorem is a summary of the results obtained in this paper.

Theorem 5.1. *There exists a 2-fold triangulation* $(2K_n, T(3), D)$ *with:* $|D| = 0$ *if and only if* $n \equiv 0$ *or* $1 \pmod 9$; $|D| = 1$ *if and only if* $n \equiv 4$ *or* $6 \pmod 9$; *and* $|D| = 2$, *with* D *any one of the four possibilities, if and only if* $n \equiv 3$ *or* $7 \pmod 9$.

Problem: It is easy to construct *at least one* 2-fold triangulation of an *arbitrary* 2-fold triple system $(2K_n, T)$. Simply take $(2K_n, T(3) = \phi, D = T)$. Not a very interesting 2-fold triangulation since D is as "large" as possible. Much more interesting is a triangulation with deficiency D as "small" as possible. The general problem here is to determine a function f so that an arbitrary 2-fold triple system has a 2-fold triangulation $(2K_n, T(3), D)$ where $|D| \leq f(n)$ and $f(n)$ is "small". The same problem, of course, can be asked for triangulations of Steiner triple systems. Neither problem seems easy.

References

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