MAXIMAL PARTIAL TRIPLE SYSTEMS WITH HEXAGONAL LEAVE

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1. Introduction. Maximal partial triple systems (MPTs) of order $v \leq 11$ were enumerated in [1]. Extending these tables up to $v \leq 13$, while most desirable, does not appear feasible, due to the existence of a large number of MPTs "halfway" through the spectrum. A more restricted task, such as an enumeration of MPTs of order 13 having quadratic leaves, for instance, is therefore more realistic.

A necessary and sufficient condition for the existence of an MPT with a quadratic leave was obtained in [2]. In particular, an MPT(v) whose leave is C_6 ("the hexagon") exists if and only if $v \equiv 1$ or $3 \pmod 6$, $v \geq 7$. (Let us note at the outset, however, that to obtain an MPT with hexagonal leave is easy; one may use for this any Steiner triple system - Section 2 below.)

There exists, however, an additional motivation for an enumeration of MPT (13)'s with hexagonal leave. First, consider a 2-chromatic S(2,4,25). In any proper 2-colouring of its elements, the induced structure on one of its colour classes is a partial triple system with 13 elements and 24 triples. The latter is either obtained by deleting two triples from a Steiner triple system of order 13, or is an MPT with hexagonal leave. Second, the derived structure of a largest maximal partial Steiner quadruple system on 14 elements (other than a Steiner quadruple system) is an MPT(13) with hexagonal leave.

It is well known that there exist exactly two nonisomorphic Steiner triple systems on 13 elements (see, e.g., [4]), each of which contains 26 triples. It is easy to determine the number of partial triple systems on 13

elements with exactly 25 triples. In this paper, we enumerate all partial triple systems of order 13 with 24 triples. There are exactly 76 nonisomorphic (non-maximal) partial triple systems of order 13 with 24 triples obtained from a Steiner triple system of order 13 by omitting two triples. There are exactly 196 MPT(13)'s with hexagonal leave, of which 57 can be obtained from a Steiner triple system; the remaining 139 MPTs with hexagonal leave (those which cannot be obtained from a Steiner triple system through a triangle replacement, cf. Section 2) are listed in the Appendix, together with some of the invariants associated with them.

2. Basic definitions and observations. A partial triple system (PTS) is a pair (V,B) where V is a v–set of elements, and B is a set of 3–subsets of V called triples such that each 2–subset of V is contained in at most one triple of B. The leave of a partial triple system (V,B) is the graph (V,E) where E contains all pairs that do not appear in B. A partial triple system (V,B) is maximal (MPT) if its leave is triangle–free. A Steiner triple system (STS) is a PTS whose leave has no edges.

A maximal partial triple system of order v with hexagonal leave (briefly C_6–MPT(v)) is one whose leave is the graph consisting of C_6, the cycle of length 6, and v–6 isolated vertices. This is a special case of what was termed in [2] an MPT with a quadratic leave. It is easy to show that a C_6–MPT(v) exists if and only if $v \equiv 1$ or 3 (mod 6), $v \geq 7$. The necessity is implied by the fact that the number of elements must be odd, and the number of pairs of elements must be a multiple of three. For sufficiency, one only has to take any STS of order v (these are well known to exist if and only if $v \equiv 1$ or 3 (mod 6), see, e.g., [6]), and in it any triangle, i.e. a set of three triples of the form axy, byz, cxz. Such a set is easily seen to exist in any STS of order $v \geq 7$. Deleting these 3 triples and adding the triple xyz results in a C_6–MPT with leave (x,a,y,b,z,c). We refer to the above transformation as a triangle replacement.

The question arises whether, conversely, one can obtain an STS from a given C_6–MPT with leave (x,a,y,b,z,c). Obviously, the answer is "yes" if this C_6–MPT contains the triple xyz or the triple abc (or both), and "no" otherwise. There indeed exist C_6–MPT(13)'s which cannot be obtained from either of the two STS(13)'s in the above way (but see Section 3 below for a different wav, through "switching").

Let us deal first with those C_6–MPT(13)'s which can be obtained from an STS(13) through a triangle replacement. In any STS(13) there are 6 triples containing a given element i, thus there are $\binom{6}{2} = 15$ pairs of triples through i, and each of these can be completed in four ways to a triangle. Thus for a given element i, there are 15.4 = 60 triangles in which i occurs in two triples, and there are a total of 13.60/3 = 260 triangles in any STS(13) (of course, this number is also obtained as $\binom{13}{3} - 26$). The two STS(13)'s have automorphism group of order 39 and 6, respectively (cf. [4]). Letting now the automorphism group act on the 60 triangles containing a given element i as a vertex in the case of the first STS (it is transitive), and on the 260 triangles of the second STS gives the following.

The number of nonisomorphic C_6–MPT(13)'s which can be obtained from the two nonisomorphic STS(13)'s through a triangle replacement, is 8 and 49, respectively, for a total of 57 such systems.

3. <u>C_6–MPTs that cannot be obtained from STS(13) through a triangle replacement.</u>

If (1,2,3,4,5,6) is the leave of a C_6–MPT which cannot be obtained from an STS through a triangle replacement (for the sake of brevity, let us call such C_6–MPTs <u>proper</u>) then neither 135 nor 246 is a triple, and each of the 9 edges 13,14,15,24,25,26,35,36,46 (the <u>diagonals</u>) belong to a different triple. Any element $x \in \{1,2,3,4,5,6\}$ is in at most 3 triples containing two elements of a diagonal, and there must be at least one such x occuring in at least two triples containing two elements of a diagonal (as there are 9 diagonals but only 7 candidates for x). This observation served as a basis for generating proper C_6–MPTs (see Figure 1). They were generated by the computer hierarchically. First, all those C_6–MPTs were generated which contained triples 14x, 25x, 36x for some x (see Fig. 1a). Next, those C_6–MPTs were generated containing no 3 triples with such x as above but containing triples 14y, 26y, 35y for some y (Fig. 1b), next those containing no 3 triples with x or y as above but containing triples 14z, 36z for some z (Fig. 1c) etc.

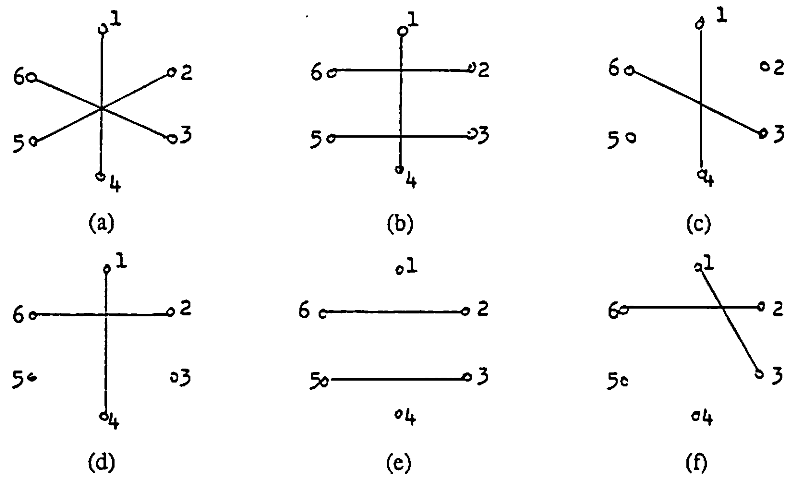

Figure 1

Isomorph rejection was performed simultaneously during the generation. The number of nonisomorphic solutions obtained is given in the following table:

Case	a	b	c	d	e	f	Total
Number of designs	3	7	60	52	12	5	139

(Independently, ten thousand C_6–MPT(13)'s were generated by hill–climbing (cf., e.g., [5]). These were then tested for isomorphism, giving 139 nonisomorphic proper C_6–MPTs and 52 nonisomorphic C_6–MPTs obtainable from STS by triangle replacement. Subsequent verification showed that the set of 139 proper C_6–MPTs and the set of 139 designs obtained earlier by orderly generation are the same.)

For each of the 139 proper C_6–MPT(13)'s, we computed the order of the automorphism group as well as three other invariants whose description is given below. The only orders of the automorphism group that occur are 1,2,3 and 6. There are 116, 15, and 7 designs, whose group has order 1,2, and 3, respectively; there is a unique design whose automorphism group has order 6 (design No. 3).

Any set of 4 disjoint triples in an MPT(13) is an almost parallel class. An MPT(13) whose triples can be partitioned into almost parallel classes is almost resolvable. The number of almost parallel classes in the 139 proper C_6–MPTs ranges from 0 to a maximum of 13. There is a unique design without an almost parallel class (No. 3), and a unique design which is almost resolvable (No. 75).

If (1,2,3,4,5,6) is the leave of an C_6–MPT(13) and 13x is a triple, then deleting this triple and adding the triple 123 produces another C_6–MPT(13) whose leave is (1,x,3,4,5,6). We refer to this operation as switching on the pair 13. Clearly, we can perform switching on any of the 6 pairs 13, 24, 35, 46, 15, 26, and thus obtain 6 new C_6–MPT(13)'s. Further switching can be performed on the latter, etc. The deficiency of a proper C_6–MPT(13) is the minimum number of switchings (if it exists) that one has to perform in order to arrive at a C_6–MPT obtainable from an STS by triangle replacement. The deficiency of our 139 proper designs turns out to be 1,2,3, or 5, with a unique design (No.3) having deficiency 5.

Finally, a fragment (sometimes also called a Pasch configuration) in a PTS is a set of four triples involving only six elements (cf., e.g., [3]). The number of fragments in our 139 proper designs ranges from 0 to 13, with only two designs (Nos. 3 and 106) having no fragments at all.

We list all 139 proper C_6–MPT(13)'s at the end of this paper. Preceding an explicit listing of the 24 triples of each design is its (1) number, (2) order of the automorphism group, (3) the number of distinct almost parallel classes, (4) the deficiency, and (5) the number of fragments. The set of these 4 invariants, while not distinguishing between several pairs of designs, is nevertheless a reasonably good approximation to a complete set of invariants.

For the sake of completeness let us note at the conclusion that the non-maximal PTSs of order 13 with 24 triples can all be obtained from the two nonisomorphic STS(13)'s by omitting two triples from the latter. It is then easily obtained (by letting the automorphism group of the respective design act on the set of pairs of triples) that there are exactly 5, and exactly 4 nonisomorphic PTSs with 24 triples obtained by omitting two intersecting triples, and two disjoint triples, respectively, from the transitive STS(13). The corresponding numbers for the second STS(13) are 39, and 28, respectively. Thus the total number of nonmaximal PTS(13)'s with 24 triples is $4 + 5 + 39 + 28 = 76$.

1		3	3	2	3							
	ACH	ADG	AEI	AJM	AKL	BDJ	BEG	BFK	BHI	BLM	CEL	CFG
	CIJ	CKM	DFM	DHL	DIK	EHM	EJK	FHJ	FIL	GHK	GIM	GJL
2		2	2	2	2							
	ACH	ADG	AEI	AJM	AKL	BDJ	BEG	BFK	BHI	BLM	CEL	CFG
	CIM	CJK	DFM	DHK	DIL	EHJ	EKM	FHL	FIJ	GHM	GIK	GJL
3		6	0	5	0							
	ACH	ADG	AEI	AJL	AKM	BDJ	BEG	BFK	BHI	BLM	CEL	CFG
	CIM	CJK	DFM	DHL	DIK	EHK	EJM	FHJ	FIL	GHM	GIJ	GKL
4		1	5	1	3							
	ACG	ADH	AEJ	AIL	AKM	BDK	BEG	BFL	BHI	BJM	CEM	CFI
	CHJ	CKL	DFG	DIJ	DLM	EHL	EIK	FHM	FJK	GHK	GIM	GJL
5		1	6	1	4							
	ACG	ADH	AEJ	AIK	ALM	BDK	BEG	BFL	BHI	BJM	CEM	CFI
	CHJ	CKL	DFG	DIM	DJL	EHK	EIL	FHM	FJK	GHL	GIJ	GKM
6		1	7	1	5							
	ACG	ADH	AEJ	AIL	AKM	BDK	BEG	BFL	BHI	BJM	CEM	CFI
	CHJ	CKL	DFG	DIM	DJL	EHL	EIK	FHM	FJK	GHK	GIJ	GLM
7		2	6	1	4							
	ACG	ADH	AEJ	AIL	AKM	BDK	BEG	BFL	BHI	BJM	CEM	CFI
	CHK	CJL	DFG	DIJ	DLM	EHL	EIK	FHM	FJK	GHJ	GIM	GKL
8		2	8	1	6							
	ACG	ADH	AEJ	AIK	ALM	BDK	BEG	BFL	BHI	BJM	CEM	CFI
	CHL	CJK	DFG	DIM	DJL	EHK	EIL	FHJ	FKM	GHM	GIJ	GKL
9		2	6	1	4							
	ACG	ADH	AEJ	AIL	AKM	BDK	BEG	BFL	BHJ	BIM	CEM	CFI
	CHK	CJL	DFG	DIJ	DLM	EHL	EIK	FHM	FJK	GHI	GJM	GKL
10		2	8	1	6							
	ACG	ADH	AEJ	AIL	AKM	BDK	BEG	BFL	BHM	BIJ	CEM	CFI
	CHJ	CKL	DFG	DIM	DJL	EHL	EIK	FHK	FJM	GHI	GJK	GLM
11		1	7	2	7							
	ACH	ADG	AEI	AJK	ALM	BDH	BEJ	BFI	BGL	BKM	CEM	CFG
	CIJ	CKL	DFK	DIL	DJM	EGK	EHL	FHM	FJL	GHJ	GIM	HIK
12		1	10	2	10							
	ACH	ADG	AEI	AJK	ALM	BDH	BEJ	BFI	BGL	BKM	CEK	CFG
	CIL	CJM	DFM	DIJ	DKL	EGM	EHL	FHK	FJL	GHJ	GIK	HIM
13		1	4	3	4							
	ACH	ADG	AEI	AJK	ALM	BDH	BEJ	BFI	BGL	BKM	CEL	CFG
	CIK	CJM	DFM	DIJ	DKL	EGK	EHM	FHK	FJL	GHJ	GIM	HIL
14		1	10	3	10							
	ACH	ADG	AEI	AJK	ALM	BDH	BEJ	BFI	BGL	BKM	CEK	CFG
	CIL	CJM	DFM	DIK	DJL	EGM	EHL	FHJ	FKL	GHK	GIJ	HIM
15		1	7	2	6							
	ACH	ADG	AEI	AJK	ALM	BDH	BEJ	BFL	BGI	BKM	CEK	CFG
	CIL	CJM	DFJ	DIM	DKL	EGM	EHL	FHM	FIK	GHK	GJL	HIJ
16		1	13	2	13							
	ACH	ADG	AEI	AJK	ALM	BDH	BEJ	BFL	BGI	BKM	CEK	CFG
	CIL	CJM	DFK	DIM	DJL	EGM	EHL	FHM	FIJ	GHJ	GKL	HIK
17		1	3	2	3							
	ACH	ADG	AEI	AJK	ALM	BDH	BEJ	BFL	BGI	BKM	CEK	CFG
	CIM	CJL	DFK	DIL	DJM	EGM	EHL	FHM	FIJ	GHJ	GKL	HIK
18		1	7	1	7							
	ACH	ADG	AEI	AJK	ALM	BDH	BEJ	BFL	BGI	BKM	CEL	CFG
	CIK	CJM	DFK	DIM	DJL	EGM	EHK	FHM	FIJ	GHJ	GKL	HIL
19		1	3	2	3							
	ACH	ADG	AEI	AJK	ALM	BDH	BEJ	BFK	BGL	BIM	CEL	CFG
	CIJ	CKM	DFI	DJM	DKL	EGM	EHK	FHM	FJL	GHJ	GIK	HIL
20		1	9	2	9							
	ACH	ADG	AEI	AJK	ALM	BDH	BEJ	BFL	BGM	BIK	CEM	CFG
	CIJ	CKL	DFI	DJL	DKM	EGK	EHL	FHK	FJM	GHJ	GIL	HIM
21		1	10	1	10							
	ACH	ADG	AEI	AJK	ALM	BDH	BEJ	BFL	BGM	BIK	CEL	CFG
	CIJ	CKM	DFK	DIM	DJL	EGK	EHM	FHI	FJM	GHJ	GIL	HKL

22		1	4	2	3							
	ACH	ADG	AEI	AJK	ALM	BDH	BEJ	BFL	BGM	BIK	CEK	CFG
	CIL	CJM	DFJ	DIM	DKL	EGL	EHM	FHI	FKM	GHK	GIJ	HJL
23		1	7	1	7							
	ACH	ADG	AEI	AJK	ALM	BDH	BEJ	BFK	BGL	BIM	CEK	CFG
	CIL	CJM	DFM	DIJ	DKL	EGM	EHL	FHI	FJL	GHJ	GIK	HKM
24		1	7	2	6							
	ACH	ADG	AEI	AJK	ALM	BDH	BEJ	BFL	BGK	BIM	CEM	CFG
	CIJ	CKL	DFJ	DIL	DKM	EGL	EHK	FHM	FIK	GHI	GJM	HJL
25		1	4	2	3							
	ACH	ADG	AEI	AJK	ALM	BDH	BEJ	BFL	BGM	BIK	CEM	CFG
	CIJ	CKL	DFJ	DIL	DKM	EGK	EHL	FHK	FIM	GHI	GJL	HJM
26		1	10	1	10							
	ACH	ADG	AEI	AJK	ALM	BDH	BEJ	BFL	BGM	BIK	CEL	CFG
	CIJ	CKM	DFK	DIL	DJM	EGK	EHM	FHJ	FIM	GHI	GJL	HKL
27		1	5	1	4							
	ACH	ADG	AEI	AJK	ALM	BDI	BEH	BFJ	BGL	BKM	CEM	CFG
	CIJ	CKL	DFK	DHL	DJM	EGK	EJL	FHM	FIL	GHJ	GIM	HIK
28		1	4	1	3							
	ACH	ADG	AEI	AJK	ALM	BDI	BEH	BFJ	BGL	BKM	CEM	CFG
	CIJ	CKL	DFL	DHK	DJM	EGK	EJL	FHM	FIK	GHJ	GIM	HIL
29		1	6	1	5							
	ACH	ADG	AEI	AJK	ALM	BDI	BEH	BFJ	BGL	BKM	CEK	CFG
	CIL	CJM	DFM	DHJ	DKL	EGM	EJL	FHL	FIK	GHK	GIJ	HIM
30		1	5	1	4							
	ACH	ADG	AEI	AJK	ALM	BDI	BEH	BFJ	BGL	BKM	CEM	CFG
	CIK	CJL	DFK	DHL	DJM	EGJ	EKL	FHM	FIL	GHK	GIM	HIJ
31		1	7	2	6							
	ACH	ADG	AEI	AJK	ALM	BDJ	BEH	BFI	BGL	BKM	CEM	CFG
	CIJ	CKL	DFK	DHM	DIL	EGK	EJL	FHL	FJM	GHJ	GIM	HIK
32		1	5	3	4							
	ACH	ADG	AEI	AJK	ALM	BDJ	BEH	BFI	BGL	BKM	CEM	CFG
	CIJ	CKL	DFL	DHM	DIK	EGK	EJL	FHK	FJM	GHJ	GIM	HIL
33		1	6	2	5							
	ACH	ADG	AEI	AJK	ALM	BDJ	BEH	BFI	BGL	BKM	CEM	CFG
	CIK	CJL	DFK	DHM	DIL	EGJ	EKL	FHL	FJM	GHK	GIM	HIJ
34		1	7	3	6							
	ACH	ADG	AEI	AJK	ALM	BDJ	BEH	BFI	BGL	BKM	CEK	CFG
	CIL	CJM	DFM	DHL	DIK	EGM	EJL	FHJ	FKL	GHK	GIJ	HIM
35		1	7	1	7							
	ACH	ADG	AEI	AJK	ALM	BDI	BEJ	BFL	BGH	BKM	CEL	CFG
	CIK	CJM	DFM	DHJ	DKL	EGK	EHM	FHK	FIJ	GIM	GJL	HIL
36		1	6	2	6							
	ACH	ADG	AEI	AJK	ALM	BDJ	BEL	BFI	BGH	BKM	CEJ	CFG
	CIM	CKL	DFK	DHM	DIL	EGM	EHK	FHL	FJM	GIK	GJL	HIJ
37		1	10	1	10							
	ACH	ADG	AEI	AJK	ALM	BDI	BEJ	BFK	BGL	BHM	CEM	CFG
	CIJ	CKL	DFH	DJL	DKM	EGK	EHL	FIL	FJM	GHJ	GIM	HIK
38		1	7	1	7							
	ACH	ADG	AEI	AJK	ALM	BDI	BEJ	BFK	BGL	BHM	CEL	CFG
	CIJ	CKM	DFH	DJM	DKL	EGM	EHK	FIM	FJL	GHJ	GIK	HIL
39		1	8	1	8							
	ACH	ADG	AEI	AJK	ALM	BDI	BEJ	BFL	BGK	BHM	CEM	CFG
	CIJ	CKL	DFH	DJL	DKM	EGL	EHK	FIK	FJM	GHJ	GIM	HIL
40		1	10	1	10							
	ACH	ADG	AEI	AJK	ALM	BDI	BEJ	BFL	BGM	BHK	CEM	CFG
	CIJ	CKL	DFH	DJL	DKM	EGK	EHL	FIK	FJM	GHJ	GIL	HIM
41		1	10	1	10							
	ACH	ADG	AEI	AJK	ALM	BDJ	BEK	BFI	BGL	BHM	CEM	CFG
	CIK	CJL	DFH	DIL	DKM	EGJ	EHL	FJM	FKL	GHK	GIM	HIJ
42		1	13	1	13							
	ACH	ADG	AEI	AJK	ALM	BDJ	BEL	BFI	BGK	BHM	CEM	CFG
	CIK	CJL	DFH	DIL	DKM	EGJ	EHK	FJM	FKL	GHL	GIM	HIJ

43.		1	6	1	6							
	ACH	ADG	AEI	AJK	ALM	BDJ	BEL	BFI	BGM	BHK	CEK	CFG
	CIL	CJM	DFH	DIM	DKL	EGJ	EHM	FJL	FKM	GHL	GIK	HIJ
44		1	7	1	7							
	ACH	ADG	AEI	AJK	ALM	BDJ	BEL	BFI	BGM	BHK	CEK	CFG
	CIM	CJL	DFH	DIL	DKM	EGJ	EHM	FJM	FKL	GHL	GIK	HIJ
45		1	4	1	4							
	ACH	ADG	AEI	AJK	ALM	BDI	BEJ	BFL	BGK	BHM	CEL	CFG
	CIK	CJM	DFM	DHJ	DKL	EGH	EKM	FHK	FIJ	GIM	GJL	HIL
46		1	3	2	3							
	ACH	ADG	AEI	AJK	ALM	BDJ	BEL	BFI	BGK	BHM	CEJ	CFG
	CIM	CKL	DFM	DHL	DIK	EGH	EKM	FHK	FJL	GIL	GJM	HIJ
47		1	4	3	2							
	ACH	ADG	AEI	AJK	ALM	BDH	BEJ	BFG	BIL	BKM	CEM	CFI
	CGJ	CKL	DFK	DIM	DJL	EGL	EHK	FHL	FJM	GHM	GIK	HIJ
48		1	8	2	6							
	ACH	ADG	AEI	AJK	ALM	BDH	BEJ	BFG	BIL	BKM	CEK	CFI
	CGL	CJM	DFM	DIJ	DKL	EGM	EHL	FHK	FJL	GHJ	GIK	HIM
49		1	10	3	8							
	ACH	ADG	AEI	AJK	ALM	BDH	BEJ	BFG	BIL	BKM	CEK	CFI
	CGL	CJM	DFM	DIK	DJL	EGM	EHL	FHJ	FKL	GHK	GIJ	HIM
50		1	6	2	4							
	ACH	ADG	AEI	AJK	ALM	BDH	BEJ	BFG	BIL	BKM	CEL	CFI
	CGK	CJM	DFM	DIK	DJL	EGM	EHK	FHJ	FKL	GHL	GIJ	HIM
51		1	11	3	9							
	ACH	ADG	AEI	AJK	ALM	BDH	BEJ	BFG	BIL	BKM	CEK	CFL
	CGI	CJM	DFJ	DIM	DKL	EGM	EHL	FHM	FIK	GHK	GJL	HIJ
52		1	4	3	2							
	ACH	ADG	AEI	AJK	ALM	BDH	BEJ	BFG	BIL	BKM	CEK	CFM
	CGI	CJL	DFJ	DIM	DKL	EGL	EHM	FHL	FIK	GHK	GJM	HIJ
53		1	11	3	10							
	ACH	ADG	AEI	AJK	ALM	BDH	BEJ	BFG	BIL	BKM	CEK	CFL
	CGI	CJM	DFK	DIM	DJL	EGM	EHL	FHM	FIJ	GHJ	GKL	HIK
54		1	5	2	4							
	ACH	ADG	AEI	AJK	ALM	BDH	BEJ	BFG	BIL	BKM	CEL	CFK
	CGJ	CIM	DFI	DJM	DKL	EGM	EHK	FHM	FJL	GHL	GIK	HIJ
55		1	5	2	4							
	ACH	ADG	AEI	AJK	ALM	BDH	BEJ	BFG	BIL	BKM	CEL	CFM
	CGJ	CIK	DFI	DJM	DKL	EGK	EHM	FHK	FJL	GHL	GIM	HIJ
56		1	2	3	1							
	ACH	ADG	AEI	AJK	ALM	BDH	BEJ	BFG	BIL	BKM	CEK	CFL
	CGM	CIJ	DFK	DIM	DJL	EGL	EHM	FHI	FJM	GHJ	GIK	HKL
57		1	3	2	2							
	ACH	ADG	AEI	AJK	ALM	BDH	BEJ	BFG	BIL	BKM	CEL	CFK
	CGJ	CIM	DFL	DIK	DJM	EGM	EHK	FHM	FIJ	GHI	GKL	HJL
58		1	11	2	9							
	ACH	ADG	AEI	AJK	ALM	BDH	BEJ	BFG	BIL	BKM	CEL	CFM
	CGK	CIJ	DFJ	DIM	DKL	EGM	EHK	FHL	FIK	GHI	GJL	HJM
59		1	4	2	2							
	ACH	ADG	AEI	AJK	ALM	BDH	BEJ	BFG	BIL	BKM	CEM	CFL
	CGK	CIJ	DFJ	DIM	DKL	EGL	EHK	FHM	FIK	GHI	GJM	HJL
60		1	9	2	8							
	ACH	ADG	AEI	AJK	ALM	BDH	BEJ	BFG	BIL	BKM	CEL	CFK
	CGM	CIJ	DFL	DIK	DJM	EGK	EHM	FHJ	FIM	GHI	GJL	HKL
61		1	8	1	6							
	ACH	ADG	AEI	AJK	ALM	BDI	BEH	BFG	BJL	BKM	CEJ	CFM
	CGK	CIL	DFL	DHK	DJM	EGM	EKL	FHJ	FIK	GHL	GIJ	HIM
62		1	7	1	5							
	ACH	ADG	AEI	AJK	ALM	BDI	BEH	BFG	BJL	BKM	CEJ	CFM
	CGL	CIK	DFK	DHL	DJM	EGM	EKL	FHJ	FIL	GHK	GIJ	HIM
63		1	10	3	7							
	ACH	ADG	AEI	AJK	ALM	BDJ	BEH	BFG	BIL	BKM	CEM	CFI
	CGJ	CKL	DFL	DHM	DIK	EGK	EJL	FHK	FJM	GHL	GIM	HIJ

64		1	5	2	2							
	ACH	ADG	AEI	AJK	ALM	BDJ	BEH	BFG	BIL	BKM	CEM	CFI
	CGK	CJL	DFL	DHM	DIK	EGJ	EKL	FHK	FJM	GHL	GIM	HIJ
65		1	9	2	7							
	ACH	ADG	AEI	AJK	ALM	BDJ	BEH	BFG	BIL	BKM	CEK	CFL
	CGI	CJM	DFK	DHL	DIM	EGM	EJL	FHM	FIJ	GHJ	GKL	HIK
66		1	5	2	3							
	ACH	ADG	AEI	AJK	ALM	BDJ	BEH	BFG	BIL	BKM	CEM	CFL
	CGJ	CIK	DFM	DHI	DKL	EGK	EJL	FHK	FIJ	GHL	GIM	HJM
67		1	6	1	5							
	ACH	ADG	AEI	AJK	ALM	BDI	BEJ	BFG	BHL	BKM	CEK	CFM
	CGJ	CIL	DFH	DJM	DKL	EGL	EHM	FIK	FJL	GHK	GIM	HIJ
68		1	7	1	6							
	ACH	ADG	AEI	AJK	ALM	BDI	BEJ	BFG	BHL	BKM	CEM	CFK
	CGJ	CIL	DFH	DJM	DKL	EGL	EHK	FIM	FJL	GHM	GIK	HIJ
69		1	5	1	4							
	ACH	ADG	AEI	AJK	ALM	BDJ	BEL	BFG	BHI	BKM	CEJ	CFM
	CGK	CIL	DFH	DIM	DKL	EGM	EHK	FIK	FJL	GHL	GIJ	HJM
70		1	9	1	8							
	ACH	ADG	AEI	AJK	ALM	BDJ	BEK	BFG	BHL	BIM	CEJ	CFM
	CGI	CKL	DFH	DIL	DKM	EGL	EHM	FIK	FJL	GHK	GJM	HIJ
71		1	7	1	6							
	ACH	ADG	AEI	AJK	ALM	BDJ	BEL	BFG	BHK	BIM	CEJ	CFM
	CGI	CKL	DFH	DIL	DKM	EGK	EHM	FIK	FJL	GHL	GJM	HIJ
72		1	7	1	6							
	ACH	ADG	AEI	AJK	ALM	BDJ	BEL	BFG	BHK	BIM	CEJ	CFM
	CGL	CIK	DFH	DIL	DKM	EGK	EHM	FIJ	FKL	GHI	GJM	HJL
73		3	6	4	3							
	ACH	ADG	AEI	AJK	ALM	BDJ	BEL	BFG	BHI	BKM	CEK	CFJ
	CGM	CIL	DFL	DHK	DIM	EGH	EJM	FHM	FIK	GIJ	GKL	HJL
74		1	10	2	7							
	ACH	ADG	AEI	AJK	ALM	BDJ	BEL	BFG	BHI	BKM	CEK	CFJ
	CGM	CIL	DFL	DHM	DIK	EGH	EJM	FHK	FIM	GIJ	GKL	HJL
75		3	12	2	9							
	ACH	ADG	AEI	AJK	ALM	BDJ	BEL	BFG	BHM	BIK	CEK	CFJ
	CGM	CIL	DFL	DHI	DKM	EGH	EJM	FHK	FIM	GIJ	GKL	HJL
76		3	12	3	9							
	ACH	ADG	AEI	AJK	ALM	BDJ	BEL	BFG	BHK	BIM	CEK	CFJ
	CGM	CIL	DFL	DHM	DIK	EGH	EJM	FHI	FKM	GIJ	GKL	HJL
77		1	6	1	6							
	ACH	ADI	AEJ	AGK	ALM	BDH	BEL	BFG	BIK	BJM	CEG	CFK
	CIM	CJL	DFJ	DGM	DKL	EHI	EKM	FHM	FIL	GHL	GIJ	HJK
78		1	3	1	3							
	ACH	ADI	AEJ	AGK	ALM	BDH	BEL	BFG	BIM	BJK	CEG	CFM
	CIK	CJL	DFJ	DGM	DKL	EHI	EKM	FHK	FIL	GHL	GIJ	HJM
79		1	6	1	6							
	ACH	ADI	AEJ	AGK	ALM	BDJ	BEL	BFG	BHI	BKM	CEG	CFK
	CIL	CJM	DFH	DGM	DKL	EHM	EIK	FIM	FJL	GHL	GIJ	HJK
80		3	6	1	6							
	ACH	ADI	AEJ	AGK	ALM	BDJ	BEL	BFG	BHI	BKM	CEG	CFM
	CIL	CJK	DFH	DGM	DKL	EHK	EIM	FIK	FJL	GHL	GIJ	HJM
81		3	3	4	3							
	ACH	ADI	AEG	AJK	ALM	BDJ	BEK	BFH	BGM	BIL	CEJ	CFL
	CGI	CKM	DFG	DHM	DKL	EHL	EIM	FIK	FJM	GHK	GJL	HIJ
82		3	9	3	9							
	ACH	ADI	AEG	AJK	ALM	BDJ	BEL	BFH	BGM	BIK	CEJ	CFK
	CGL	CIM	DFG	DHM	DKL	EHI	EKM	FIL	FJM	GHK	GIJ	HJL
83		1	6	2	5							
	ACH	ADG	AEI	AJK	ALM	BDH	BEJ	BFL	BGI	BKM	CEM	CFG
	CIJ	CKL	DFK	DIL	DJM	EGK	EHL	FHJ	FIM	GHM	GJL	HIK
84		1	11	2	10							
	ACH	ADG	AEI	AJK	ALM	BDH	BEJ	BFL	BGI	BKM	CEK	CFG
	CIL	CJM	DFM	DIJ	DKL	EGM	EHL	FHJ	FIK	GHK	GJL	HIM

85		1	5	3	4							
	ACH	ADG	AEI	AJK	ALM	BDH	BEJ	BFL	BGI	BKM	CEK	CFG
	CIM	CJL	DFM	DIJ	DKL	EGL	EHM	FHJ	FIK	GHK	GJM	HIL
86		1	11	2	10							
	ACH	ADG	AEI	AJK	ALM	BDH	BEJ	BFL	BGI	BKM	CEK	CFG
	CIL	CJM	DFM	DIK	DJL	EGM	EHL	FHK	FIJ	GHJ	GKL	HIM
87		1	11	2	10							
	ACH	ADG	AEI	AJK	ALM	BDH	BEJ	BFL	BGM	BIK	CEM	CFG
	CIJ	CKL	DFK	DIM	DJL	EGK	EHL	FHI	FJM	GHJ	GIL	HKM
88		1	8	2	7							
	ACH	ADG	AEI	AJK	ALM	BDH	BEJ	BFK	BGL	BIM	CEL	CFG
	CIJ	CKM	DFM	DIK	DJL	EGK	EHM	FHJ	FIL	GHI	GJM	HKL
89		1	11	2	10							
	ACH	ADG	AEI	AJK	ALM	BDH	BEJ	BFL	BGM	BIK	CEM	CFG
	CIJ	CKL	DFK	DIL	DJM	EGK	EHL	FHJ	FIM	GHI	GJL	HKM
90		1	6	2	5							
	ACH	ADG	AEI	AJK	ALM	BDH	BEJ	BFL	BGK	BIM	CEK	CFG
	CIL	CJM	DFM	DIJ	DKL	EGM	EHL	FHJ	FIK	GHI	GJL	HKM
91		1	8	2	6							
	ACH	ADG	AEI	AJK	ALM	BDJ	BEH	BFL	BGI	BKM	CEM	CFG
	CIJ	CKL	DFK	DHM	DIL	EGK	EJL	FHJ	FIM	GHL	GJM	HIK
92		1	7	2	5							
	ACH	ADG	AEI	AJK	ALM	BDJ	BEH	BFL	BGI	BKM	CEK	CFG
	CIL	CJM	DFM	DHL	DIK	EGM	EJL	FHK	FIJ	GHJ	GKL	HIM
93		1	11	1	10							
	ACH	ADG	AEI	AJK	ALM	BDI	BEJ	BFL	BGH	BKM	CEM	CFG
	CIJ	CKL	DFK	DHM	DJL	EGK	EHL	FHJ	FIM	GIL	GJM	HIK
94		1	5	1	4							
	ACH	ADG	AEI	AJK	ALM	BDI	BEJ	BFL	BGH	BKM	CEM	CFG
	CIJ	CKL	DFK	DHM	DJL	EGL	EHK	FHJ	FIM	GIK	GJM	HIL
95		1	7	1	6							
	ACH	ADG	AEI	AJK	ALM	BDI	BEJ	BFL	BGH	BKM	CEK	CFG
	CIM	CJL	DFM	DHJ	DKL	EGL	EHM	FHK	FIJ	GIK	GJM	HIL
96		2	11	1	10							
	ACH	ADG	AEI	AJK	ALM	BDI	BEJ	BFL	BGH	BKM	CEK	CFG
	CIL	CJM	DFM	DHK	DJL	EGM	EHL	FHJ	FIK	GIJ	GKL	HIM
97		2	11	2	10							
	ACH	ADG	AEI	AJK	ALM	BDJ	BEL	BFI	BGH	BKM	CEM	CFG
	CIJ	CKL	DFK	DHL	DIM	EGK	EHJ	FHM	FJL	GIL	GJM	HIK
98		1	11	2	10							
	ACH	ADG	AEI	AJK	ALM	BDJ	BEL	BFI	BGH	BKM	CEM	CFG
	CIJ	CKL	DFK	DHM	DIL	EGK	EHJ	FHL	FJM	GIM	GJL	HIK
99		1	3	2	2							
	ACH	ADG	AEI	AJK	ALM	BDJ	BEL	BFI	BGH	BKM	CEK	CFG
	CIL	CJM	DFM	DHL	DIK	EGJ	EHM	FHK	FJL	GIM	GKL	HIJ
100		1	8	2	7							
	ACH	ADG	AEI	AJK	ALM	BDJ	BEL	BFI	BGH	BKM	CEM	CFG
	CIK	CJL	DFK	DHM	DIL	EGJ	EHK	FHL	FJM	GIM	GKL	HIJ
101		2	10	1	9							
	ACH	ADG	AEI	AJK	ALM	BDJ	BEL	BFK	BGI	BHM	CEM	CFG
	CIJ	CKL	DFH	DIL	DKM	EGK	EHJ	FIM	FJL	GHL	GJM	HIK
102		2	11	1	10							
	ACH	ADG	AEI	AJK	ALM	BDJ	BEL	BFK	BGI	BHM	CEM	CFG
	CIK	CJL	DFH	DIM	DKL	EGJ	EHK	FIL	FJM	GHL	GKM	HIJ
103		2	5	1	4							
	ACH	ADG	AEI	AJK	ALM	BDJ	BEL	BFM	BGI	BHK	CEK	CFG
	CIL	CJM	DFH	DIM	DKL	EGJ	EHM	FIK	FJL	GHL	GKM	HIJ
104		2	4	1	3							
	ACH	ADG	AEI	AJK	ALM	BDJ	BEK	BFL	BGI	BHM	CEM	CFG
	CIK	CJL	DFH	DIM	DKL	EGL	EHJ	FIJ	FKM	GHK	GJM	HIL
105		2	3	1	2							
	ACH	ADG	AEI	AJK	ALM	BDI	BEJ	BFK	BGL	BHM	CEM	CFG
	CIK	CJL	DFL	DHK	DJM	EGH	EKL	FHJ	FIM	GIJ	GKM	HIL

106		2	1	4	0							
	ACH	ADG	AEI	AJK	ALM	BDJ	BEK	BFI	BGL	BHM	CEM	CFG
	CIJ	CKL	DFL	DHK	DIM	EGH	EJL	FHJ	FKM	GIK	GJM	HIL
107		1	7	3	5							
	ACH	ADG	AEI	AJK	ALM	BDH	BEJ	BFG	BIL	BKM	CEK	CFL
	CGI	CJM	DFM	DIJ	DKL	EGM	EHL	FHJ	FIK	GHK	GJL	HIM
108		1	13	2	11							
	ACH	ADG	AEI	AJK	ALM	BDH	BEJ	BFG	BIL	BKM	CEK	CFL
	CGI	CJM	DFM	DIK	DJL	EGM	EHL	FHK	FIJ	GHJ	GKL	HIM
109		1	5	2	3							
	ACH	ADG	AEI	AJK	ALM	BDH	BEJ	BFG	BIL	BKM	CEK	CFM
	CGI	CJL	DFL	DIK	DJM	EGM	EHL	FHK	FIJ	GHJ	GKL	HIM
110		1	6	2	4							
	ACH	ADG	AEI	AJK	ALM	BDH	BEJ	BFG	BIL	BKM	CEL	CFK
	CGI	CJM	DFM	DIK	DJL	EGM	EHK	FHL	FIJ	GHJ	GKL	HIM
111		1	7	2	5							
	ACH	ADG	AEI	AJK	ALM	BDH	BEJ	BFG	BIL	BKM	CEL	CFK
	CGJ	CIM	DFM	DIJ	DKL	EGM	EHK	FHI	FJL	GHL	GIK	HJM
112		1	7	2	5							
	ACH	ADG	AEI	AJK	ALM	BDH	BEJ	BFG	BIL	BKM	CEK	CFL
	CGJ	CIM	DFM	DIK	DJL	EGM	EHL	FHK	FIJ	GHI	GKL	HJM
113		1	10	2	8							
	ACH	ADG	AEI	AJK	ALM	BDH	BEJ	BFG	BIL	BKM	CEL	CFK
	CGJ	CIM	DFM	DIK	DJL	EGM	EHK	FHL	FIJ	GHI	GKL	HJM
114		1	9	2	7							
	ACH	ADG	AEI	AJK	ALM	BDH	BEJ	BFG	BIL	BKM	CEL	CFM
	CGJ	CIK	DFK	DIM	DJL	EGM	EHK	FHL	FIJ	GHI	GKL	HJM
115		1	7	2	4							
	ACH	ADG	AEI	AJK	ALM	BDJ	BEH	BFG	BIL	BKM	CEK	CFL
	CGI	CJM	DFM	DHL	DIK	EGM	EJL	FHK	FIJ	GHJ	GKL	HIM
116		1	6	2	3							
	ACH	ADG	AEI	AJK	ALM	BDJ	BEH	BFG	BIL	BKM	CEK	CFL
	CGJ	CIM	DFM	DHI	DKL	EGL	EJM	FHK	FIJ	GHM	GIK	HJL
117		1	7	2	4							
	ACH	ADG	AEI	AJK	ALM	BDJ	BEH	BFG	BIL	BKM	CEK	CFL
	CGJ	CIM	DFM	DHI	DKL	EGM	EJL	FHK	FIJ	GHL	GIK	HJM
118		1	7	2	4							
	ACH	ADG	AEI	AJK	ALM	BDJ	BEH	BFG	BIL	BKM	CEM	CFK
	CGL	CIJ	DFL	DHM	DIK	EGK	EJL	FHI	FJM	GHJ	GIM	HKL
119		1	6	1	4							
	ACH	ADG	AEI	AJK	ALM	BDJ	BEL	BFG	BHI	BKM	CEM	CFK
	CGL	CIJ	DFH	DIM	DKL	EGJ	EHK	FIL	FJM	GHM	GIK	HJL
120		1	9	1	7							
	ACH	ADG	AEI	AJK	ALM	BDJ	BEL	BFG	BHM	BIK	CEM	CFK
	CGI	CJL	DFH	DIM	DKL	EGJ	EHK	FIL	FJM	GHL	GKM	HIJ
121		1	6	1	4							
	ACH	ADG	AEI	AJK	ALM	BDJ	BEL	BFG	BHM	BIK	CEM	CFK
	CGL	CIJ	DFH	DIM	DKL	EGJ	EHK	FIL	FJM	GHI	GKM	HJL
122		1	5	1	3							
	ACH	ADG	AEI	AJK	ALM	BDJ	BEL	BFG	BHM	BIK	CEM	CFK
	CGL	CIJ	DFH	DIL	DKM	EGK	EHJ	FIM	FJL	GHI	GJM	HKL
123		1	5	2	2							
	ACH	ADG	AEI	AJK	ALM	BDJ	BEL	BFG	BHI	BKM	CEK	CFM
	CGJ	CIL	DFL	DHM	DIK	EGH	EJM	FHK	FIJ	GIM	GKL	HJL
124		1	10	2	8							
	ACH	ADG	AEI	AJK	ALM	BDJ	BEL	BFG	BHK	BIM	CEJ	CFK
	CGM	CIL	DFM	DHI	DKL	EGH	EKM	FHL	FIJ	GIK	GJL	HJM
125		1	10	2	7							
	ACH	ADG	AEI	AJK	ALM	BDJ	BEL	BFG	BHM	BIK	CEK	CFJ
	CGL	CIM	DFM	DHI	DKL	EGH	EJM	FHK	FIL	GIJ	GKM	HJL
126		1	12	2	10							
	ACH	ADG	AEI	AJK	ALM	BDJ	BEK	BFG	BHL	BIM	CEM	CFL
	CGJ	CIK	DFM	DHI	DKL	EGH	EJL	FHK	FIJ	GIL	GKM	HJM

127		1	8	2	6							
	ACH	ADG	AEI	AJK	ALM	BDJ	BEK	BFG	BHL	BIM	CEM	CFL
	CGK	CIJ	DFM	DHI	DKL	EGH	EJL	FHJ	FIK	GIL	GJM	HKM
128		1	7	2	5							
	ACH	ADG	AEI	AJK	ALM	BDJ	BEL	BFG	BHK	BIM	CEJ	CFK
	CGM	CIL	DFM	DHL	DIK	EGH	EKM	FHI	FJL	GIJ	GKL	HJM
129		1	8	1	7							
	ACH	ADI	AEJ	AGK	ALM	BDH	BEL	BFG	BIK	BJM	CEG	CFK
	CIM	CJL	DFM	DGJ	DKL	EHI	EKM	FHL	FIJ	GHM	GIL	HJK
130		1	11	1	10							
	ACH	ADI	AEJ	AGK	ALM	BDH	BEL	BFG	BIK	BJM	CEG	CFK
	CIM	CJL	DFM	DGL	DJK	EHI	EKM	FHJ	FIL	GHM	GIJ	HKL
131		1	3	1	2							
	ACH	ADI	AEJ	AGK	ALM	BDH	BEL	BFG	BIM	BJK	CEG	CFM
	CIK	CJL	DFK	DGL	DJM	EHI	EKM	FHJ	FIL	GHM	GIJ	HKL
132		2	7	1	6							
	ACH	ADI	AEJ	AGK	ALM	BDH	BEL	BFG	BIJ	BKM	CEG	CFK
	CIM	CJL	DFM	DGL	DJK	EHM	EIK	FHJ	FIL	GHI	GJM	HKL
133		2	3	1	2							
	ACH	ADI	AEJ	AGK	ALM	BDH	BEK	BFG	BIL	BJM	CEG	CFM
	CIJ	CKL	DFL	DGM	DJK	EHL	EIM	FHJ	FIK	GHI	GJL	HKM
134		1	5	1	4							
	ACH	ADI	AEJ	AGK	ALM	BDL	BEK	BFG	BHI	BJM	CEG	CFM
	CIJ	CKL	DFH	DGJ	DKM	EHL	EIM	FIK	FJL	GHM	GIL	HJK
135		1	9	1	8							
	ACH	ADI	AEJ	AGK	ALM	BDL	BEK	BFG	BHI	BJM	CEG	CFM
	CIJ	CKL	DFH	DGJ	DKM	EHM	EIL	FIK	FJL	GHL	GIM	HJK
136		1	7	1	6							
	ACH	ADI	AEJ	AGK	ALM	BDL	BEK	BFG	BHI	BJM	CEG	CFM
	CIJ	CKL	DFH	DGM	DJK	EHL	EIM	FIK	FJL	GHJ	GIL	HKM
137		1	4	2	3							
	ACH	ADI	AEG	AJK	ALM	BDH	BEJ	BFK	BGM	BIL	CEL	CFM
	CGJ	CIK	DFG	DJL	DKM	EHK	EIM	FHL	FIJ	GHI	GKL	HJM
138		1	9	2	8							
	ACH	ADI	AEG	AJK	ALM	BDJ	BEL	BFH	BGM	BIK	CEM	CFK
	CGI	CJL	DFG	DHL	DKM	EHK	EIJ	FIL	FJM	GHJ	GKL	HIM
139		1	7	2	6							
	ACH	ADI	AEG	AJK	ALM	BDJ	BEL	BFH	BGM	BIK	CEM	CFK
	CGJ	CIL	DFG	DHL	DKM	EHK	EIJ	FIM	FJL	GHI	GKL	HJM

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