Indecomposable Triple Systems with $\lambda = 6$

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Abstract

A triple system $B[3,\lambda;v]$ is indecomposable if it is not the union of two triple systems $B[3,\lambda_1;v]$ and $B[3,\lambda_2;v]$ with $\lambda = \lambda_1 + \lambda_2$. We prove that indecomposable triple systems with $\lambda = 6$ exist for $v = 8$, 14 and for all $v \geq 17$.

1. Introduction

A ***balanced incomplete block design***, denoted $B[k,\lambda;v]$, is a pair (V,B); V is a v-set of ***elements*** and B is a collection of k-element subsets of V called ***blocks***. Each 2-subset of V appears in precisely λ blocks. When B contains no repeated blocks we say the block design is ***simple***. When $k = 3$ block designs are called ***triple systems***.

A $B[k,\lambda;v]$ design (V,B) is ***decomposable*** if $B = B_1 \cup B_2$ and (V,B_1) is a $B[k,\lambda_1;v]$ design and (V,B_2) is a $B[k,\lambda_2;v]$ design with $\lambda_1 + \lambda_2 = \lambda$. If a balanced incomplete block design is not decomposable then it is termed ***indecomposable***.

It would be of interest to find all orders v for which there exist simple indecomposable triple systems. Kramer [5] solved this problem for $\lambda = 2$ and 3. He showed that when $\lambda = 2$, simple indecomposable $B[3,2;v]$ exist for all $v \equiv 0,1 \pmod 3$, except for $v = 7$ and when $\lambda = 3$, simple indecomposable $B[3,3;v]$ exist for all $v \equiv 1 \pmod 2$, $v \geq 5$. The case of $\lambda = 4$ was solved by Colbourn and Rosa [2] who showed that simple indecomposable $B[3,4;v]$ exist if and only if $v \equiv 0,1 \pmod 3$ and $v \geq 10$. When λ is odd and v is suffiently large, Colburn [1] proved the existence of indecomposable $B[3,\lambda;v]$. These triple systems are not necessarily simple, however. We see that the case of $\lambda = 6$ is the smallest unknown case. In this paper, we will establish that simple indecomposable B[3,6,v] exists for v = 8, 14 and for all $v \geq 17$.

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1. Main Result

In this section we will establish our main Theorem. We begin with the cases of $v = 8$ and $v = 14$.

Lemma 2.1 *There exists simple indecomposable B[3,6,8] and B[3,6,14].*

Proof. Consider the complete design consisting of all 56 triples on a set of size 8. This is a simple B[3,6;8]. Furthermore it is indecomposable since by the usual numerical conditions on v, k and λ, it is easy to see that there does not exist any design B[3,λ;8] with $\lambda < 6$. By the same numerical considerations, any simple B[3,6,14] must be indecomposable. In [3] Dehon deduces the existence of simple B[3,λ,v] for all λ and v which satisfy the necessary conditions. Thus a simple (indecomposable) B[3,6,14] exists. □

The following lemma is the main tool in constructing indecomposable designs for larger orders. It makes note of the fact that if a subdesign of a design is indecomposable, then the entire design is indecomposable. A B[k,λ;v'] design (V',B') is a ***subdesign*** of a B[k,λ;v] design (V,B) if V' $\subset$ V and B' $\subset$ B. In this case we also say that the design B[k,λ;v'] is *embedded* in the design B[k,λ;v].

Lemma 2.2 *If a B[k,λ;v] design D contains an indecomposable subdesign B[k,λ;v], then D is indecomposable.*

Proof. A decomposition of the entire system would necessitate a decomposition of the (indecomposable) subsystem. □

It is our plan to construct triple systems B[3,6,v] for $17 \leq v \leq 36$ which contain the indecomposable subsystem B[3,6;8]. In order to do so we use a slightly modified version of the Stinson hill-climbing algorithm for Steiner triple systems (i.e. triple systems with $\lambda = 1$). In [7] Stinson describes a hill-climbing algorithm which is very successful at finding Steiner triple systems even with large fixed subsystems. We modified that algorithm to work for higher λ. For a given v we construct a simple triple system on v points which contains all of the triples from the set {1,..,8}. By the previous lemma we have that these systems are indecomposable.

Lemma 2.3 *There exist simple indecomposable triple systems B[3,6;v] for v = 18,19,21,22,24,25,26,27,28,30,31,33,34, and 36.*

Proof. These designs are all given in [4]. They all contain the indecomposable subsystem B[3,6;8] on the set {1,..,8}. Lemmata 2.1 and 2.2 assure the indecomposablity of the triple systems. □

In order to proceed recursively we cite the following results of Colbourn and Rosa [2].

Theorem 2.4 *A simple B[3,λ;v] can be embedded in a simple B[3,λ;2v+1], a simple B[3,λ;2v+4], and a simple B[3,λ;2v+7].*

Lemma 2.5 *There exist simple indecomposable triple systems B[3,6;v] for v = 17,20,23,29,32 and 35 .*

Proof. Embed the simple indecomposable B[3,6;8] into simple (and thus indecomposable) designs B[3,6;17], B[3,6;20], and B[3,6;23]. Embed the simple indecomposable B[3,6;14] into simple indecomposable designs B[3,6;29], B[3,6;32], and B[3,6;35]. □

Just to reiterate what was shown in Lemmata 2.3 and 2.5, we now have that there exist simple indecomposable triple systems B[3,6;v] for all $17 \leq v \leq 36$. We are now in a position to prove the main theorem.

Theorem 2.6 *There exist simple indecomposable B[3,6,v] exists for v = 8,14 and for all $v \geq 17$.*

Proof. If $v \leq 36$ then the theorem holds by Lemmata 2.1, 2.3 or 2.5. Now consider an arbitrary $v \geq 37$. By way of induction assume that there exists simple indecomposable triple systems B[3,6;v'] for all $v' < v$. If v is even, then $17 \leq (v-4)/2 < v$ and thus there exists a simple indecomposable B[3,6;(v-4)/2]. By Theorem 2.4 this design can be embedded into a simple B[3,6;v] which is therefore indecomposable. If v is odd, then $17 \leq (v-1)/2 < v$ and thus again there exists a simple indecomposable B[3,6;$(v-1)/2$] which embeds into a simple (indecomposable) B[3,6;v]. □

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Addendum. We have learned that this same result has recently been proven independently by S. Milici [6]. In his proof he derives recursive constructions which embed simple B[3,6;v] into simple B[3,6;w] for w = 2v+1, 2v+2 and 2v+4. He then constructs roughly the same set of "small" orders as were constructed in this paper but without the use of a computer.

References

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