On the classifications of weighing matrices of order 12

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Abstract

A weighing matrix $A = A(n, k)$ of order n and weight k is a square matrix of order n, with entries $0, \pm 1$ which satisfies $AA^T = kI_n$.

H.C.Chan, C.A.Rodger and J.Seberry "On inequivalent weighing matrices, *Ars Combinatoria*, (1986), 21-A, 299-333" showed that there were exactly 5 inequivalent weighing matrices of order 12 and weight 4 and exactly 2 inequivalent matrices of weight 5. They showed the weighing matrices of order 12 and weights 2,3 and 11 were unique. Q.M.Husain, "On the totality of the solutions for the symmetric block designs: $\lambda = 2, k = 5$ or 6, *Sankyā* 7 (1945), 204-208" had shown that the Hadamard matrix of order 12 (the weighing matrix of weight 12) is unique.

In this paper, we complete the classification of weighing matrices of order 12 by showing there are seven inequivalent matrices of weight 6, three of weight 7, six of weight 8, four of weight 9 and four of weight 10.

These results have considerable implications to inequivalence results for orders greater than 12.

1 Introduction

A *weighing* matrix $A = A(n, k)$ of order n and weight k is an orthogonal $(n \times n)$ matrix with entries 0,-1 and 1 and k non-zero entries in each row and each column.

Two weighing matrices A and B, both of order n and weight k, are said to be *equivalent* if and only if one can be transformed into the other by using the following operations:

(i) multiply any row or column by -1, and

(ii) interchange two rows or two columns.

The *intersection pattern conditions* (IPC) [1] are useful to classify the weighing matrices. We shall restate the IPC for convenience' sake. Given any row, say row j, of a specific weighing matrix, $A(n, k)$, we say that p_{2i} rows of $A(n, k)$

[1] This research was carried out while on study leave at University College, UNSW, ADFA, Canberra and The University of Sydney

intersect row j in $2i$ places if there are p_{2i} rows, each of which has exactly $2i$ non-zero elements occuring in the columns containing non-zero elements in row j. Then the IPC satisfies:

1. $\sum_{j=0} p_{2j} = n - 1$; and
2. $\sum_{j=0} jp_{2j} = k(k-1)/2$.

Let $\mathbf{r}_i, \mathbf{r}_j, \mathbf{r}_k$ and $\mathbf{r}_\ell$ be row vectors of a matrix, then a *generalized inner product* of these vectors is defined as $|\sum a_{im}a_{jm}a_{km}a_{\ell m}|$, where

$$\mathbf{r}_i = (a_{i1}, \ldots, a_{im}, \ldots), \ldots, \mathbf{r}_\ell = (a_{\ell 1}, \ldots, a_{\ell m}, \ldots).$$

A generalized inner product is invariant under equivalence operations, so it may be used to check whether two weighing matrices are equivalent or not. We shall use the term *G-Table* to tabulate the results of applying the generalized inner product.

H.C.Chan, C.A.Rodger and Jennifer Seberry [1] classified the inequivalent weighing matrices of any order with weight less than 6. In this paper, we classify the weighing matrices of order 12 and weight k, where $6 \leq k \leq 10$. We shall construct weighing matrices step by step. We shall use the following notations.

$\mathbf{I}_n$: Identity matrix of order n
$\mathbf{O}_{\ell \times m}$, $\mathbf{O}_\ell$, $\mathbf{O}_n$: Zero matrices or zero row vector
$\mathbf{G}^T$: Transpose of a matrix $\mathbf{G}$.
$\mathbf{K} \sim \mathbf{L}$: $\mathbf{K}$ and $\mathbf{L}$ are equivalent matrices.
$\pi(i, \underline{j}, \ldots, k)$: Row signed permutation of a matrix as follows: Move the i^{th} row to the first row, the j^{th} row to the second but by multiplying -1, ..., the k^{th} row to the last row.
$\rho(i, \underline{j}, \ldots, k)$: Column signed permutation.
$\mathbf{a}, \mathbf{b}, \ldots, \mathbf{r}, \mathbf{s}$: Row or Column vectors.
$\mathbf{a} \cdot \mathbf{b}$: Hadamard (element by element) product.
$|\mathbf{a}|$: Number of non-zero elements of $\mathbf{a}$.
$|\mathbf{a}_1 \cdot \mathbf{a}_2 \cdot \ldots \cdot \mathbf{a}_\ell|$: Intersection number of $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_\ell$.

2 Classifications of weighing matrices of order 12 and weight 10

Let $A = (a_{ij})$ be a weighing matrix of order 12 and weight 10. The IPC are $p_8 + p_{10} = 11$ and $4p_8 + 5p_{10} = 45$, so we have the unique solutions $p_8 = 10$ and $p_{10} = 1$. So, without loss of generality, we may assume that $\mathbf{r}_1 = (1, \ldots, 1, 0, 0)$ and $\mathbf{r}_2 = (1, \ldots, 1, -, \ldots, 0, 0)$ as the first and second rows of A respectively, and $\mathbf{c}_{11}^T = (0, 0, 1, \ldots, 1, -, \ldots, -)$ and $\mathbf{c}_{12}^T = (0, 0, 1, \ldots, 1)$ where $\mathbf{c}_{11}$ and $\mathbf{c}_{12}$ are eleventh and twelth columns of A, respectively. Let $A(\ell)$ be a submatrix of A, where $1 \leq \ell \leq 4$ and

$$\begin{aligned} A(1) &= (a_{i,j}),(i=3,\ldots,7,j=1,\ldots,5),\\ A(2) &= (a_{i,j}),(i=8,\ldots,12,j=1,\ldots,5),\\ A(3) &= (a_{i,j}),(i=3,\ldots,7,j=6,\ldots,10),\\ A(4) &= (a_{i,j}),(i=8,\ldots,12,j=6,\ldots,10) \end{aligned}$$

Lemma 1 *Each column and row of* $\mathbf{A}(\ell)$ *has to contain one 0, two 1s and two -1s where* $1 \le \ell \le 4$. *Moreover it may be assumed, without loss of generality, that*

$$\begin{aligned} a_{i+2,i} &= 0 \ \ (1 \le i \le 10), \ \ a_{i,i+3} = 0 \ \ (3 \le i \le 7) \ \textit{and}\\ a_{i+m,i} &= 0 \ \ (1 \le i \le 5). \end{aligned}$$

Proof: Let $\mathbf{a} = (\alpha_1, \ldots, \alpha_{10}, \gamma, 1)$ and $\mathbf{b} = (\beta_1, \ldots, \beta_{10}, \delta, 1)$ be vectors which are orthogonal to $\mathbf{r}_1$ and $\mathbf{r}_2$ and with weight 10, where $\alpha_i, \beta_j \in \{0, 1, -1\}$ and $\gamma, \delta \in \{1, -1\}$.

Let x_1 and x_0 be the numbers of 1s and 0s in the set $\{\alpha_1, \ldots, \alpha_5\}$, respectively, and y_1 and y_0 be the numbers of 1s and 0s in the set $\{\alpha_6, \ldots, \alpha_{10}\}$, respectively. Then we have the following equations from the orthogonality of $\mathbf{a}$ and $\mathbf{r}_1$ and $\mathbf{a}$ and $\mathbf{r}_2$:

$$x_1 + y_1 = (5 - x_1 - x_0) + (5 - y_1 - y_0) \text{ and}$$

$$x_1 + (5 - y_1 - y_0) = (5 - x_1 - x_0) + y_1.$$

So we have the unique solutions $x_1 = y_1 = 2$, $x_0 = y_0 = 1$. Therefore, each row of $A(\ell)$ has to contain one 0, two 1s and two -1s. It is similar for each column of $A(\ell)$.

Next, let $\gamma = \delta$, then it is impossible that $\mathbf{a}$ and $\mathbf{b}$ are orthogonal and $|\mathbf{a} \cdot \mathbf{b}| = 10$. So if $\mathbf{a}$ and $\mathbf{b}$ are orthogonal and $|\mathbf{a} \cdot \mathbf{b}| = 10$, then $\gamma \ne \delta$.

So we have the last half by suitable row and column permutations.

Lemma 2 *It may be assumed, without loss of generality, that* $a_{3,2} = a_{3,3} = 1$, $a_{3,4} = a_{3,5} = -1$ *and* $a_{3,7} = 1$.

Proof: If $a_{3,7} = -1$, operate with $\pi(2,1,3)$ and $\rho(1, 2, 3, 4, 5, \underline{6}, \underline{7}, \underline{8}, \underline{9}, \underline{10}, 11, 12)$ on the matrix $[\mathbf{r}_1^T, \mathbf{r}_2^T, \mathbf{r}^T]^T$ where $\mathbf{r} = (0, \alpha_2, \alpha_3, \alpha_4, \alpha_5, 0, -1, \alpha_8, \alpha_9, \alpha_{10}, 1, 1)$ is the third row of A. The other assertations of the statement of the lemma can easily be shown by operating with suitable row and column permutations on the matrix $[\mathbf{r}_1^T, \mathbf{r}_2^T, \mathbf{r}^T]^T$.

Let X be a $(j \times 12)$ matrix such that $XX^T = 10I_j$ and its entries are 0, 1 and -1, where $3 \le j \le 12$. Then X may be called *normal* of level j if it is a matrix of the following form:

$$X = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0\\ 1 & 1 & 1 & 1 & 1 & - & - & - & - & - & 0 & 0\\ 0 & 1 & 1 & - & - & 0 & 1 & * & * & * & 1 & 1\\ & \ddots & & & & & \ddots & & & & \vdots & \vdots\\ & & 0 & & & & & 0 & & & 1 & 1 \end{bmatrix} \text{ if } 3 \le j \le 7$$

and

$$X = \left[\begin{array}{cccccccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & - & - & - & - & - & 0 & 0 \\
0 & 1 & 1 & - & - & 0 & 1 & * & * & * & 1 & 1 \\
 & \ddots & & & & & \ddots & & & & \vdots & \vdots \\
 & & \ddots & & & & & \ddots & & & \vdots & \vdots \\
 & & & \ddots & & & & & \ddots & & \vdots & \vdots \\
 & & & & 0 & & & & & 0 & 1 & 1 \\
0 & & & & & 0 & & & & & - & 1 \\
 & \ddots & & & & & \ddots & & & & \vdots & \vdots \\
 & & \ddots & & & & & \ddots & & & \vdots & \vdots \\
 & & & 0 & & & & & 0 & & - & 1
\end{array}\right] \quad \text{if } 8 \le j \le 12$$

(-1 is denoted here and on all vectors and matrices with -)

If $j = 12$, X may be called a *normal weighing matrix.*

Lemma 3 *There are two normal matrices of level 3, up to equivalence.*

Proof: There are three vectors, say $\mathbf{r}_3(1), \mathbf{r}_3(2)$ and $\mathbf{r}_3(3)$, which satisfy the conditions for the third row of a normal matrix of level 3:

$$\begin{array}{lcl}
\mathbf{r}(1) & = & (\,0\ \ 1\ \ 1\ \ -\ \ -\ \ 0\ \ 1\ \ 1\ \ -\ \ -\ \ 1\ \ 1\,), \\
\mathbf{r}(2) & = & (\,0\ \ 1\ \ 1\ \ -\ \ -\ \ 0\ \ 1\ \ -\ \ 1\ \ -\ \ 1\ \ 1\,) \text{ and} \\
\mathbf{r}(3) & = & (\,0\ \ 1\ \ 1\ \ -\ \ -\ \ 0\ \ 1\ \ -\ \ -\ \ 1\ \ 1\ \ 1\,).
\end{array}$$

Then it is easy to check that

$$\begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \mathbf{r}_3(2) \end{bmatrix} \sim \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \mathbf{r}_3(3) \end{bmatrix} \text{ and } \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \mathbf{r}_3(1) \end{bmatrix} \not\sim \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \mathbf{r}_3(2) \end{bmatrix}.$$

In the following, we shall consider two cases $\begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \mathbf{r}_3(1) \end{bmatrix}$ and $\begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \mathbf{r}_3(2) \end{bmatrix}$ separately, (say Case I and II, respectively).

Case I

Lemma 4 *There are three normal matrices of level 4, up to equivalence, such that the first three rows of each matrix equals those of the matrix* $[\mathbf{r}_1^T, \mathbf{r}_2^T, \mathbf{r}_3(1)^T]^T$.

Proof: There are only seven vectors, say $\mathbf{r}_4(i)$, $(1 \le i \le 7)$, such that the matrices X_i are normal matrices of level 4, where $X_i^T = [\mathbf{r}_1^T, \mathbf{r}_2^T, \mathbf{r}_3^T, \mathbf{r}_4(i)^T]^T$ and

$$\begin{array}{lcl}
\mathbf{r}_4(1) & = & (\ 1\ 0\ -\ 1\ -\ 1\ 0\ -\ 1\ -\ 1\ 1\ ), \\
\mathbf{r}_4(2) & = & (\ 1\ 0\ -\ 1\ -\ 1\ 0\ -\ -\ 1\ 1\ 1\ ), \\
\mathbf{r}_4(3) & = & (\ -\ 0\ -\ 1\ 1\ -\ 0\ 1\ 1\ -\ 1\ 1\ ), \\
\mathbf{r}_4(4) & = & (\ -\ 0\ 1\ -\ 1\ -\ 0\ -\ 1\ 1\ 1\ 1\ ), \\
\mathbf{r}_4(5) & = & (\ -\ 0\ -\ 1\ 1\ -\ 0\ 1\ -\ 1\ 1\ 1\ ), \\
\mathbf{r}_4(6) & = & (\ 1\ 0\ -\ -\ 1\ 1\ 0\ -\ -\ 1\ 1\ 1\ ) \quad \text{and} \\
\mathbf{r}_4(7) & = & (\ 1\ 0\ -\ -\ 1\ 1\ 0\ -\ 1\ -\ 1\ 1\ ).
\end{array}$$

But it is easy to check that $X_1 \sim X_6$, $X_2 \sim X_7$ and $X_3 \sim X_5$.

Moreover $X_3 \sim X_4$, because we can obtain X_4 by operating with ρ(6, 7, 8, 10, 9, 1, 2, 3, 5, 4, 11, 12) and π(1, $\underline{2}$, 3, 4) on X_3.

Lemma 5 *It is impossible to extend X_3 to a normal weighing matrix.*

Proof: Let $\mathbf{a} = (\alpha_1, \ldots, \alpha_{10}, 1, 1,)$ be a vector such that $X_3\mathbf{a}^T = 0$ and $\alpha_4 = \alpha_9 = 0$, where $\alpha_i \in \{1, -1\}$ $(i \neq 4, 9)$. From Lemma 1, we have $\alpha_3 = -1$ and $\alpha_5 = 1$. Also, we have the equations $\alpha_2 + \alpha_7 + \alpha_8 = \alpha_{10}$ and $\alpha_{10} = \alpha_1 + \alpha_6 + \alpha_8$ from the orthogonality of $\mathbf{a}$ and $\mathbf{r}_3(1)$ and $\mathbf{a}$ and $\mathbf{r}_4(3)$ respectively. From Lemma 1 again, we may consider two cases $\alpha_1 = 1$, $\alpha_2 = -1$ or $\alpha_1 = -1$, $\alpha_2 = 1$. But it is impossible to obtain solutions which satisfy Lemma 1. This means we can't construct a normal matrix of level 6 which contains X_3 as a submatrix.

Lemma 6 *There is a unique normal matrix of level 7, say $\bar{X}_1$, which may be constructed from X_1 and such that the first four rows of $\bar{X}_1$ equal the first four rows of X_1.*

Proof: Let $\bar{X}_1$ be a (7×12) matrix as follows:

$$\bar{X}_1 = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & - & - & - & - & - & 0 & 0 \\
0 & 1 & 1 & - & - & 0 & 1 & 1 & - & - & 1 & 1 \\
1 & 0 & - & 1 & - & 1 & 0 & - & 1 & - & 1 & 1 \\
1 & - & 0 & - & 1 & 1 & - & 0 & - & 1 & 1 & 1 \\
- & 1 & - & 0 & 1 & - & 1 & - & 0 & 1 & 1 & 1 \\
- & - & 1 & 1 & 0 & - & - & 1 & 1 & 0 & 1 & 1
\end{bmatrix}.$$

Then $\bar{X}_1$ is a normal matrix of level 7 and it is easy to prove the uniqueness.

Lemma 7 *There are three normal matrices of level 8, up to equivalence, such that the first seven rows of each matrix equal those of $\bar{X}_1$. Moreover, for each matrix, it is possible to construct two normal weighing matrices.*

Proof: Let $\mathbf{a} = (\alpha_1, \alpha_2, \ldots, \alpha_{10}, -1, 1)$ be a vector such that the matrix $[\bar{X}_1{}^T, \mathbf{a}^T]^T$ is a normal matrix of level 8. So, $\alpha_1 = \alpha_6 = 0$. We may assume that $\alpha_2 = 1$, because if $\alpha_2 = -1$, we may multiply the kth row $(8 \leq k \leq 12)$ by -1 and exchange the 11th for the 12th columns. Also, it is easily shown that there are three vectors, say $\mathbf{r}_8(1)$, $\mathbf{r}_8(2)$ and $\mathbf{r}_8(3)$, satisfying the conditions, where

$$\mathbf{r}_8(1) \ = \ (\ 0\ 1\ 1\ -\ -\ 0\ -\ -\ 1\ 1\ -\ 1\ ),$$

$$\mathbf{r}_8(2) = (\,0\ 1\ -\ 1\ -\ 0\ -\ 1\ -\ 1\ -\ 1\,) \text{ and}$$
$$\mathbf{r}_8(3) = (\,0\ 1\ -\ -\ 1\ 0\ -\ 1\ 1\ -\ -\ 1\,).$$

Put $X_1^{(i)} = \begin{bmatrix} \bar{X}_1 \\ \mathbf{r}_8(i) \end{bmatrix}$ $(i = 1, 2, 3)$. Then we can construct two normal weighing matrices, say $A_{(i-1)\times 2+1}$ and $A_{(i-1)\times 2+2}$, from each $X_1^{(i)}$ $i = 1, 2, 3$, where

$$A_1 = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & & \\
1 & 1 & 1 & 1 & 1 & - & - & - & - & - & & \\
 & 1 & 1 & - & - & & 1 & 1 & - & - & 1 & 1 \\
1 & & - & 1 & - & 1 & & - & 1 & - & 1 & 1 \\
1 & - & & - & 1 & 1 & - & & - & 1 & 1 & 1 \\
- & 1 & - & & 1 & - & 1 & - & & 1 & 1 & 1 \\
- & - & 1 & 1 & & - & - & 1 & 1 & & 1 & 1 \\
 & 1 & 1 & - & - & & - & - & 1 & 1 & - & 1 \\
1 & & - & 1 & - & - & & 1 & - & 1 & - & 1 \\
1 & - & & - & 1 & - & 1 & & 1 & - & - & 1 \\
- & 1 & - & & 1 & 1 & - & 1 & & - & - & 1 \\
- & - & 1 & 1 & & 1 & 1 & - & - & & - & 1
\end{bmatrix},$$

$$A_2 = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & & \\
1 & 1 & 1 & 1 & 1 & - & - & - & - & - & & \\
 & 1 & 1 & - & - & & 1 & 1 & - & - & 1 & 1 \\
1 & & - & 1 & - & 1 & & - & 1 & - & 1 & 1 \\
1 & - & & - & 1 & 1 & - & & - & 1 & 1 & 1 \\
- & 1 & - & & 1 & - & 1 & - & & 1 & 1 & 1 \\
- & - & 1 & 1 & & - & - & 1 & 1 & & 1 & 1 \\
 & 1 & 1 & - & - & & - & - & 1 & 1 & - & 1 \\
1 & & - & - & 1 & - & & 1 & 1 & - & - & 1 \\
1 & - & & 1 & - & - & 1 & & - & 1 & - & 1 \\
- & - & 1 & & 1 & 1 & 1 & - & & - & - & 1 \\
- & 1 & - & 1 & & 1 & - & 1 & - & & - & 1
\end{bmatrix},$$

$$A_3 = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & & \\
1 & 1 & 1 & 1 & 1 & - & - & - & - & - & & \\
 & 1 & 1 & - & - & & 1 & 1 & - & - & 1 & 1 \\
1 & & - & 1 & - & 1 & & - & 1 & - & 1 & 1 \\
1 & - & & - & 1 & 1 & - & & - & 1 & 1 & 1 \\
- & 1 & - & & 1 & - & 1 & - & & 1 & 1 & 1 \\
- & - & 1 & 1 & & - & - & 1 & 1 & & 1 & 1 \\
 & 1 & - & 1 & - & & - & 1 & - & 1 & - & 1 \\
1 & & 1 & - & - & - & & - & 1 & 1 & - & 1 \\
- & 1 & & - & 1 & 1 & - & & 1 & - & - & 1 \\
1 & - & - & & 1 & - & 1 & 1 & & - & - & 1 \\
- & - & 1 & 1 & & 1 & 1 & - & - & & - & 1
\end{bmatrix},$$

$$
A_4 = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & & \\
1 & 1 & 1 & 1 & 1 & - & - & - & - & - & & \\
 & 1 & 1 & - & - & & 1 & 1 & - & - & 1 & 1 \\
1 & & - & 1 & - & 1 & & - & 1 & - & 1 & 1 \\
1 & - & & - & 1 & 1 & - & & - & 1 & 1 & 1 \\
- & 1 & - & & 1 & - & 1 & - & & 1 & 1 & 1 \\
- & - & 1 & 1 & & - & - & 1 & 1 & & 1 & 1 \\
 & 1 & - & 1 & - & & - & 1 & - & 1 & - & 1 \\
1 & & - & - & 1 & - & & 1 & 1 & - & - & 1 \\
- & - & & 1 & 1 & 1 & 1 & & - & - & - & 1 \\
1 & - & 1 & & - & - & 1 & - & & 1 & - & 1 \\
- & 1 & 1 & - & & 1 & - & - & 1 & & - & 1
\end{bmatrix},
$$

$$
A_5 = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & & \\
1 & 1 & 1 & 1 & 1 & - & - & - & - & - & & \\
 & 1 & 1 & - & - & & 1 & 1 & - & - & 1 & 1 \\
1 & & - & 1 & - & 1 & & - & 1 & - & 1 & 1 \\
1 & - & & - & 1 & 1 & - & & - & 1 & 1 & 1 \\
- & 1 & - & & 1 & - & 1 & - & & 1 & 1 & 1 \\
- & - & 1 & 1 & & - & - & 1 & 1 & & 1 & 1 \\
 & 1 & - & - & 1 & & - & 1 & 1 & - & - & 1 \\
1 & & 1 & - & - & - & & - & 1 & 1 & - & 1 \\
- & 1 & & 1 & - & 1 & - & & - & 1 & - & 1 \\
- & - & 1 & & 1 & 1 & 1 & - & & - & - & 1 \\
1 & - & - & 1 & & - & 1 & 1 & - & & - & 1
\end{bmatrix} \text{ and}
$$

$$
A_6 = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & & \\
1 & 1 & 1 & 1 & 1 & - & - & - & - & - & & \\
 & 1 & 1 & - & - & & 1 & 1 & - & - & 1 & 1 \\
1 & & - & 1 & - & 1 & & - & 1 & - & 1 & 1 \\
1 & - & & - & 1 & 1 & - & & - & 1 & 1 & 1 \\
- & 1 & - & & 1 & - & 1 & - & & 1 & 1 & 1 \\
- & - & 1 & 1 & & - & - & 1 & 1 & & 1 & 1 \\
 & 1 & - & - & 1 & & - & 1 & 1 & - & - & 1 \\
1 & & - & 1 & - & - & & 1 & - & 1 & - & 1 \\
- & - & & 1 & 1 & 1 & 1 & & - & - & - & 1 \\
- & 1 & 1 & & - & 1 & - & - & & 1 & - & 1 \\
1 & - & 1 & - & & - & 1 & - & 1 & & - & 1
\end{bmatrix}.
$$

(all unspecified positions are 0 here and in all matrices)

Lemma 8 *There are two normal matrices of level 5, up to equivalence, such that the first four rows of each matrix equal those of X_2.*

Proof: There are three vectors, say $\mathbf{r}_5(i)$, $i = 1, 2, 3$, such that the matrices $X_2^{(i)} = [X_2^T, \mathbf{r}_5(i)^T]^T$ are normal matrices of level 5, where

$$
\begin{array}{lcl}
\mathbf{r}_5(1) & = & (\ 1\ -\ 0\ -\ 1\ 1\ -\ 0\ 1\ -\ 1\ 1\ ), \\
\mathbf{r}_5(2) & = & (\ -\ 1\ 0\ -\ 1\ -\ -\ 0\ 1\ 1\ 1\ 1\ ) \quad \text{and} \\
\mathbf{r}_5(3) & = & (\ -\ -\ 0\ 1\ 1\ -\ 1\ 0\ 1\ -\ 1\ 1\ ).
\end{array}
$$

But it is easy to check that $X_2^{(2)} \sim X_2^{(3)}$ by operating with $\rho(6, 7, 8, 10, 9, 1, 2, 3, 5, 4, 11, 12)$ and $\pi(1, \underline{2}, 3, 4, 5)$ on $X_2^{(2)}$.

Lemma 9 *It is impossible to extend $X_2^{(2)}$ to a normal weighing matrix.*

Proof: We can prove it similarly to Lemma 6.

Lemma 10 *There are two normal matrices of level 7 such that the first five rows of each matrix equal those of $X_2^{(1)}$. Moreover, for each matrix, it is uniquely possible to construct a normal weighing matrix.*

Proof: It is easily checked that there are two vectors, say $\mathbf{r}_6(1)$ and $\mathbf{r}_6(2)$, such that the matrices $[X_2^{(1)T}, \mathbf{r}_6(1)^T]^T$ and $X_2^{(1)T}, \mathbf{r}_6(2)^T]^T$ are normal of level 6, where

$$
\begin{array}{lcl}
\mathbf{r}_6(1) & = & (\ -\ 1\ -\ 0\ 1\ -\ -\ 1\ 0\ 1\ 1\ 1\ ) \text{ and} \\
\mathbf{r}_6(2) & = & (\ -\ -\ 1\ 0\ 1\ -\ 1\ -\ 0\ 1\ 1\ 1\ ).
\end{array}
$$

But each matrix is extended uniquely to a matrix of level 7, say $\bar{X}_2^{(1)}$ and $\bar{\bar{X}}_2^{(1)}$, where

$$
\bar{X}_2^{(1)} = \left[\begin{array}{cccccccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & & \\
1 & 1 & 1 & 1 & 1 & - & - & - & - & - & & \\
 & 1 & 1 & - & - & & 1 & 1 & - & - & 1 & 1 \\
1 & & - & 1 & - & 1 & & - & - & 1 & 1 & 1 \\
1 & - & & - & 1 & 1 & - & & 1 & - & 1 & 1 \\
- & 1 & - & & 1 & - & - & 1 & & 1 & 1 & 1 \\
- & - & 1 & 1 & & - & 1 & - & 1 & & 1 & 1
\end{array}\right] \text{ and}
$$

$$
\bar{\bar{X}}_2^{(1)} = \left[\begin{array}{cccccccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & & \\
1 & 1 & 1 & 1 & 1 & - & - & - & - & - & & \\
 & 1 & 1 & - & - & & 1 & 1 & - & - & 1 & 1 \\
1 & & - & 1 & - & 1 & & - & - & 1 & 1 & 1 \\
1 & - & & - & 1 & 1 & - & & 1 & - & 1 & 1 \\
- & - & 1 & & 1 & - & 1 & - & & 1 & 1 & 1 \\
- & 1 & - & 1 & & - & - & 1 & 1 & & 1 & 1
\end{array}\right].
$$

Also, we can construct, uniquely, normal weighing matrices, say A_7 and A_8, from $\bar{X}_2^{(1)}$ and $\bar{\bar{X}}_2^{(1)}$, where

$$
A_7 = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & & \\
1 & 1 & 1 & 1 & 1 & - & - & - & - & - & & \\
 & 1 & 1 & - & - & & 1 & 1 & - & - & 1 & 1 \\
1 & & - & 1 & - & 1 & & - & - & 1 & 1 & 1 \\
1 & - & & - & 1 & 1 & - & & 1 & - & 1 & 1 \\
- & 1 & - & & 1 & - & - & 1 & & 1 & 1 & 1 \\
- & - & 1 & 1 & & - & 1 & - & 1 & & 1 & 1 \\
 & 1 & 1 & - & - & & - & - & 1 & 1 & - & 1 \\
1 & & - & 1 & - & - & & 1 & 1 & - & - & 1 \\
1 & - & & - & 1 & - & 1 & & - & 1 & - & 1 \\
- & 1 & - & & 1 & 1 & 1 & - & & - & - & 1 \\
- & - & 1 & 1 & & 1 & - & 1 & - & & - & 1
\end{bmatrix} \text{ and }
$$

$$
A_8 = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & & \\
1 & 1 & 1 & 1 & 1 & - & - & - & - & - & & \\
 & 1 & 1 & - & - & & 1 & 1 & - & - & 1 & 1 \\
1 & & - & 1 & - & 1 & & - & - & 1 & 1 & 1 \\
1 & - & & - & 1 & 1 & - & & 1 & - & 1 & 1 \\
- & - & 1 & & 1 & - & 1 & - & & 1 & 1 & 1 \\
- & 1 & - & 1 & & - & - & 1 & 1 & & 1 & 1 \\
 & 1 & 1 & - & - & & - & - & 1 & 1 & - & 1 \\
1 & & - & - & 1 & - & & 1 & - & 1 & - & 1 \\
1 & - & & 1 & - & - & 1 & & 1 & - & - & 1 \\
- & 1 & - & & 1 & 1 & 1 & - & & - & - & 1 \\
- & - & 1 & 1 & & 1 & - & 1 & - & & - & 1
\end{bmatrix}.
$$

Remark 1. Operate with $\rho(2, 1, 4, 5, 3, 7, 6, 9, 10, 8, 11, 12)$ and $\pi(1, 2, 4, 3)$ on X_2. Then we obtain a matrix $\tilde{X}_2$ which belongs to Case II, where $\tilde{X}_2 = [\mathbf{r}_1^T, \mathbf{r}_2^T, \mathbf{r}_3(2)^T, \mathbf{r}(X_2)^T]^T$ and

$$\mathbf{r}(X_2) = (\ 1\ \ 0\ \ -\ \ -\ \ 1\ \ 1\ \ 0\ \ -\ \ -\ \ 1\ \ 1\ \ 1\ ).$$

Remark 2. Operate with $\rho(\underline{2}, \underline{1}, \underline{3}, \underline{5}, \underline{4}, \underline{7}, \underline{6}, \underline{8}, \underline{10}, \underline{9}, 11, 12)$ and $\pi(\underline{1}, \underline{2}, 4, 3)$ on X_4. Then we obtain a matrix $\tilde{X}_4$ which belongs to Case II, where $\tilde{X}_4 = [\mathbf{r}_1^T, \mathbf{r}_2^T, \mathbf{r}_3(2)^T, \mathbf{r}(X_4)^T]^T$ and

$$\mathbf{r}(X_4) = (\ -\ \ 0\ \ -\ \ 1\ \ 1\ \ -\ \ 0\ \ -\ \ 1\ \ 1\ \ 1\ \ 1\ ).$$

Case II

Lemma 11 *There are three normal matrices of level 4, up to equivalence, such that the first three rows of each matrix equal those of the matrix* $[\mathbf{r}_1^T, \mathbf{r}_2^T, \mathbf{r}_3(2)^T]^T$.

Proof: There are only seven vectors, say $\mathbf{s}_4(i)$ $(1 \le i \le 7)$, such that the matrices Y_i are normal matrices of level 4, where $Y_i^T = [\mathbf{r}_1^T, \mathbf{r}_2^T, \mathbf{r}_3(2)^T, \mathbf{s}_4(i)^T]^T$ and

$$
\begin{array}{lcl}
s_4(1) & = & (\ 1\ \ 0\ \ -\ \ 1\ \ -\ \ 1\ \ 0\ \ -\ \ -\ \ 1\ \ 1\ \ 1\ ), \\
s_4(2) & = & (\ 1\ \ 0\ \ -\ \ 1\ \ -\ \ 1\ \ 0\ \ 1\ \ -\ \ -\ \ 1\ \ 1\ ), \\
s_4(3) & = & (\ -\ \ 0\ \ 1\ \ 1\ \ -\ \ -\ \ 0\ \ 1\ \ -\ \ 1\ \ 1\ \ 1\ ), \\
s_4(4) & = & (\ 1\ \ 0\ \ -\ \ -\ \ 1\ \ 1\ \ 0\ \ -\ \ -\ \ 1\ \ 1\ \ 1\ ), \\
s_4(5) & = & (\ 1\ \ 0\ \ -\ \ -\ \ 1\ \ 1\ \ 0\ \ 1\ \ -\ \ -\ \ 1\ \ 1\ ), \\
s_4(6) & = & (\ -\ \ 0\ \ -\ \ 1\ \ 1\ \ -\ \ 0\ \ 1\ \ 1\ \ -\ \ 1\ \ 1\ ) \quad \text{and} \\
s_4(7) & = & (\ -\ \ 0\ \ -\ \ 1\ \ 1\ \ -\ \ 0\ \ -\ \ 1\ \ 1\ \ 1\ \ 1\ ).
\end{array}
$$

But it is easily shown that $Y_1 \sim Y_5$ by operating with $\rho($ 2, 1, 4, 5, 3, 7, 6, 9, 10, 8, 11, 12$)$ and $\pi(1, 2, 4, 3)$ on Y_1 and $Y_3 \sim Y_6$ by operating with $\rho($ 6, 7, 9, 8, 10, 12, 4, 3, 5, 11, 12$)$ and $\pi($ 1, $\underline{2}$, 3, 4$)$ on Y_3.

Moreover, $Y_4 \sim X_2$ from Remark 1 and $Y_7 \sim X_4$ from Remark 2.

Lemma 12 *For each Y_i, there is a unique normal matrix, say $\bar{Y}_i$, of level 7 such that the first four rows of $\bar{Y}_i$ equal the first four rows of Y_i, where $i = 1, 2, 3$.*

Proof: It is obvious that $\bar{Y}_i$ is normal of level 7, where $i = 1, 2, 3$ and

$$
\bar{Y}_1 = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & & \\
1 & 1 & 1 & 1 & 1 & - & - & - & - & - & & \\
 & 1 & 1 & - & - & & 1 & - & 1 & - & 1 & 1 \\
1 & & - & 1 & - & 1 & & - & - & 1 & 1 & 1 \\
1 & - & & - & 1 & - & - & & 1 & 1 & 1 & 1 \\
- & 1 & - & & 1 & 1 & - & 1 & & - & 1 & 1 \\
- & - & 1 & 1 & & - & 1 & 1 & - & & 1 & 1
\end{bmatrix},
$$

$$
\bar{Y}_2 = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & & \\
1 & 1 & 1 & 1 & 1 & - & - & - & - & - & & \\
 & 1 & 1 & - & - & & 1 & - & 1 & - & 1 & 1 \\
1 & & - & 1 & - & 1 & & 1 & - & - & 1 & 1 \\
1 & - & & - & 1 & - & 1 & & - & 1 & 1 & 1 \\
- & 1 & - & & 1 & 1 & - & - & & 1 & 1 & 1 \\
- & - & 1 & 1 & & - & - & 1 & 1 & & 1 & 1
\end{bmatrix} \quad \text{and}
$$

$$
\bar{Y}_3 = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & & \\
1 & 1 & 1 & 1 & 1 & - & - & - & - & - & & \\
 & 1 & 1 & - & - & & 1 & - & 1 & - & 1 & 1 \\
- & & 1 & 1 & - & - & & 1 & - & 1 & 1 & 1 \\
1 & - & & - & 1 & - & - & & 1 & 1 & 1 & 1 \\
- & 1 & - & & 1 & 1 & - & 1 & & - & 1 & 1 \\
1 & - & - & 1 & & 1 & 1 & - & - & & 1 & 1
\end{bmatrix}.
$$

And it is easy to check the uniqueness for each matrix $\bar{Y}_i$.

Remark 3: Operate with $\pi($ $\underline{2}$, 1, 6, 3, 5, 4, 7$)$ and $\rho($ 9, 6, 8, 7, 10, $\underline{4}$, $\underline{1}$, $\underline{3}$, $\underline{2}$, $\underline{5}$, 11, 12$)$ on $\bar{Y}_1$. Then we obtain $X_2^{(1)}$.

Operate with $\pi($ $\underline{1}$, 2, 7, 3, 4, 5, 6$)$ and $\rho($ $\underline{10}$, $\underline{6}$, $\underline{7}$, $\underline{8}$, $\underline{9}$, $\underline{5}$, $\underline{1}$, $\underline{2}$, $\underline{3}$, $\underline{4}$, 11, 12$)$ on $\bar{Y}_2$. Then we obtain $X_2^{(1)}$. —

Operate with $\pi(2, 1, 6, 7, 4, 3, 5)$ and $\rho(4, 5, 2, 1, 3, \underline{9}, \underline{10}, \underline{7}, \underline{6}, \underline{8}, 11, 12)$ on $\bar{Y}_3$. Then we obtain $X_2^{(1)}$.

Theorem 1 *There are four weighing matrices, up to equivalence, of order 12 and weight 10.*

Proof: By the above lemmas and remarks, it is sufficient to check whether the eight matrices A_i $(1 \le i \le 8)$ are equivalent or not. It is easy to check $A_4 \sim A_2$, $A_4 \sim A_3$, $A_4 \sim A_6$, $A_5 \sim A_6$, $A_7^T \sim A_4$, $A_8^T \sim A_8$ and $A_1^T \sim A_1$. We shall list an equivalence table in the following:

Old	π-operations	ρ-operations	New
A_4	$5\ 10\ 3\ \underline{6}\ 1\ \underline{4}\ 7\ 8\ \underline{11}\ \underline{2}\ 9\ \underline{12}$	$6\ \underline{9}\ 12\ \underline{2}\ 5\ 1\ \underline{4}\ 11\ \underline{7}\ 10\ 3\ 8$	A_2
A_4	$6\ 11\ \underline{3}\ \underline{5}\ 7\ 4\ 1\ \underline{8}\ \underline{10}\ 12\ 9\ \underline{2}$	$\underline{6}\ \underline{8}\ 10\ 7\ 12\ \underline{1}\ \underline{3}\ 5\ 2\ 11\ 4\ 9$	A_3
A_4	$4\ 9\ 3\ \underline{7}\ 1\ \underline{5}\ 6\ \underline{8}\ \underline{12}\ \underline{2}\ 10\ 11$	$1\ \underline{10}\ 12\ \underline{3}\ 9\ 6\ \underline{5}\ 11\ \underline{8}\ 4\ 2\ 7$	A_6
A_5	$3\ 8\ 4\ \underline{7}\ 1\ \underline{6}\ 5\ \underline{9}\ \underline{12}\ \underline{2}\ 11\ 10$	$2\ \underline{10}\ 12\ \underline{4}\ 8\ 7\ \underline{5}\ 11\ \underline{9}\ 3\ 1\ 6$	A_6
A_7^T	$1\ 6\ 12\ 2\ 3\ \underline{4}\ \underline{5}\ 11\ 8\ 7\ \underline{9}\ \underline{10}$	$1\ 5\ 4\ \underline{6}\ \underline{7}\ 2\ 10\ 9\ \underline{11}\ \underline{12}\ 8\ 3$	A_2
A_8^T	$1\ 6\ 12\ 2\ 3\ \underline{4}\ \underline{5}\ 11\ 7\ 8\ \underline{9}\ \underline{10}$	$1\ 4\ 5\ \underline{6}\ \underline{7}\ 2\ 9\ 10\ \underline{11}\ \underline{12}\ 8\ 3$	A_8
A_1	Identity	12 11 1 2 3 4 5 6 7 8 9 10	Sym

(where Sym: symmetrical matrix)

In the above equivalence table, it means that, for example, we obtain A_2 by operating with $\pi(5\ 10\ 3\ \underline{6}\ 1\ \underline{4}\ 7\ 8\ \underline{11}\ \underline{2}\ 9\ \underline{12})$ and $\rho(6\ \underline{9}\ 12\ \underline{2}\ 5\ 1\ \underline{4}\ 11\ \underline{7}\ 10\ 3\ 8)$ on A_4.

Next, in order to check whether A_1, A_4, A_7 and A_8 are equivalent or not, we count the numbers of generalized inner products when fixing each row of each matrix. We list these in a table called the G-table in the following:

Matrices	Mult.	0	1	2	3	4	5	6	7	8	9	10
A_1	12	80	0	40	0	40	0	0	0	5	0	0
A_4	12	100	0	24	0	24	0	16	0	1	0	0
A_7	12	80	0	40	0	40	0	0	0	5	0	0
A_8	12	80	0	32	0	32	0	8	0	1	0	0

When fixing a row of a weighing matrix of order 12 and weight 10, there are ${}_{11}C_3$ generalized inner products, each of which is one of 0, 1, ..., 10. In the above G-table, it means that, for example, there are 12 rows of A_1, each of which has eighty 0s, forty 2s, forty 4s and five 8s.

Clearly, $A_1 \not\sim A_4$ and A_8, $A_4 \not\sim A_7$ and A_8, and $A_7 \not\sim A_8$. But $A_1 \not\sim A_7$, because $A_1 \sim A_1^T$ but $A_7 \not\sim A_7^T$. This completes our proof.

3 Classifications of weighing matrices of order 12 and weight 9

Let $A = (a_{ij})$ be a weighing matrix of order 12 and weight 9. The *IPC* are $p_6 + p_8 = 11$ and $3p_6 + 4p_8 = 36$, so we have the unique solutions $p_6 = 8$ and $p_8 = 3$. So, without loss of generality, we may assume that $\mathbf{r}_1$ = (1, 1, 1, 1, 1, 1, 1, 1, 1, 0, 0, 0) and $\mathbf{r}_2$ = (1, 1, 1, 1, -, -, -, -, 0, 1, 0, 0) as the first and second rows of A, respectively, and $\mathbf{c}_{11}^T$ = (0, 0, 1, 0, 1, 1, 1, 1, -, -, -, -) and $\mathbf{c}_{12}^T$ = (0, 0, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1), where $\mathbf{c}_{11}$ and $\mathbf{c}_{12}$ are the 11th and 12th columns of A respectively.

Let $\mathbf{a} = (\alpha_1, \ldots, \alpha_8, \gamma, \delta, \xi, \eta)$ be a vector of weight 9 which is orthogonal to $\mathbf{r}_1$ and $\mathbf{r}_2$, where $\alpha_i, \gamma, \delta, \xi, \eta \in \{0, -1, 1\}$ and $1 \le i \le 8$. Let x_1, x_0 and x_- be the numbers of 1, 0 and -1 in the set $\{\alpha_1, \ldots, \alpha_4\}$ and y_1, y_0 and y_- be the numbers of 1, 0 and -1 in the set $\{\alpha_5, \ldots, \alpha_8\}$, respectively. Then we obtain the equations $4x_1 = 8 - 2x_0 - \gamma - \delta$ and $4y_1 = 8 - 2y_0 - \gamma + \delta$ by the orthogonality of $\mathbf{a}$ and $\mathbf{r}_1$ and $\mathbf{a}$ and $\mathbf{r}_1$, respectively. Note that $\gamma + \delta$ is even. So, we obtain a set of solutions as described in Table 2.

By the *IPC*, there are two other rows, say $\mathbf{r}_3$ and $\mathbf{r}_4$, in A such that each row has the common weight 8 with V_1. So, $\mathbf{r}_3$ and $\mathbf{r}_4$ have to belong to one of (1), (4), (7) and (13) in Table 2. If $\mathbf{r}_3$ and $\mathbf{r}_4$ do not belong to (1), then $|\mathbf{r}_3| \ge 10$ and $|\mathbf{r}_4| \ge 10$ because of the choice of $\mathbf{c}_{11}$ and $\mathbf{c}_{12}$. So, we may assume, up to equivalence, $\mathbf{r}_3$ = (1, 1, -, -, 1, 1, -, -, 0, 0, 1, 0). But for $\mathbf{r}_4$, there are two cases, say $\mathbf{r}_4(1)$ and $\mathbf{r}_4(2)$ where

$$\begin{aligned} \mathbf{r}_4(1) &= (1,1,-,-,-,-,1,1,0,0,0,1) \text{ and} \\ \mathbf{r}_4(2) &= (1,-,1,-,1,-,1,-,0,0,0,1). \end{aligned}$$

In the following, we shall consider two cases such that the first four rows of the weighing matrix are

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & & & \\ 1 & 1 & 1 & 1 & & - & - & - & - & 1 & & \\ 1 & 1 & - & - & 1 & 1 & - & - & & & 1 & \\ 1 & 1 & - & - & - & - & 1 & 1 & & & & 1 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & & & \\ 1 & 1 & 1 & 1 & - & - & - & - & & 1 & & \\ 1 & 1 & - & - & 1 & 1 & - & - & & & 1 & \\ 1 & - & 1 & - & 1 & - & 1 & - & & & & 1 \end{bmatrix},$$

(say Case I and Case II, respectively) separately.

	x_1	x_0	x_-	y_1	y_0	y_-	γ	δ	$\lvert \mathbf{r}_1 \cdot \mathbf{a} \rvert$
(1)	2	0	2	2	0	2	0	0	8
(2)		"		1	2	1	"		6
(3)	1	2	1	2	0	2	"		6
(4)	1	1	2	2	0	2	1	1	8
(5)		"		1	2	1	"		6
(6)	0	3	1	2	0	2	"		6
(7)	2	1	1	2	0	2	-1	-1	8
(8)		"		1	2	1	"		6
(9)	1	3	0	2	0	2	"		6
(10)	2	0	2	1	1	2	1	-1	8
(11)		"		0	3	1	"		6
(12)	1	2	1	1	1	2	"		6
(13)	2	0	2	2	1	1	-1	1	8
(14)		"		1	3	0	"		6
(15)	1	2	1	2	1	1	"		6

Table 2

Case I

Let A be a weighing matrix for Case I. Let $\mathbf{a} = (\alpha_1, \ldots, \alpha_9, \gamma, \delta, \xi, \eta)$ be a vector of weight 9 which is orthogonal to $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$, and $\mathbf{r}_4(1)$. Let w_1 and w_0, x_1 and x_0, y_1 and y_0 and z_1 and z_0 be the numbers of 1s and 0s in the sets $\{\alpha_1, \alpha_2\}$, $\{\alpha_3, \alpha_4\}$, $\{\alpha_5, \alpha_6\}$ and $\{\alpha_7, \alpha_8\}$, respectively. Then we have the equations:

$$\begin{aligned} 8w_1 &= 8 - 4w_0 - (\gamma + \delta + \xi + \eta), \\ 8x_1 &= 8 - 4x_0 - (\gamma + \delta - \xi - \eta), \\ 8y_1 &= 8 - 4y_0 - (\gamma - \delta + \xi - \eta) \text{ and} \\ 8z_1 &= 8 - 4z_0 - (\gamma - \delta - \xi + \eta). \end{aligned}$$

by the orthogonalities of $\mathbf{a}$ and $\mathbf{r}_1$, $\mathbf{a}$ and $\mathbf{r}_2$, $\mathbf{a}$ and $\mathbf{r}_3$ and $\mathbf{a}$ and $\mathbf{r}_4(1)$.

Note that $\gamma + \delta + \xi + \gamma \equiv 0 \pmod 4$.

So, without loss of generality, we may assume that

$$A = \begin{bmatrix} A(1) & A(2) \\ A(3) & A(4) \end{bmatrix}, \text{where } A(2) = I_4,$$

$$A(1) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & - & - & - & - \\ 1 & 1 & - & - & 1 & 1 & - & - \\ 1 & 1 & - & - & - & - & 1 & 1 \end{bmatrix} \text{ and}$$

$$A(4)^T = \begin{bmatrix} 1 & 1 & - & - & - & - & 1 & 1 \\ 1 & 1 & - & - & 1 & 1 & - & - \\ 1 & 1 & 1 & 1 & - & - & - & - \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

Let $X = \begin{bmatrix} X(1) & X(2) \\ X(3) & X(4) \end{bmatrix}$ be a $(j \times 12)$ matrix such that its entries are 0, -1 and 1 and $XX^T = 9I_j$, where $X(1), X(2), X(3)$ and $X(4)$ are (4×8), (4×4), $((j-4) \times 8)$ and $((j-4) \times 4)$submatrices of X, respectively.

Then X is called normal of type I and level j if $[X(1), X(2)] = [A(1), A(2)]$ and $X(4) = \bar{A}(4)$, where $\bar{A}(4)$ is the first $((j-4) \times 4)$ submatrix of $A(4)$.

Lemma 13 *There are two normal matrices of type I and level 5, up to equivalence.*

Proof: Let $\mathbf{a} = (\alpha_1, \ldots, \alpha_8, 1, 1, 1, 1)$ be a vector which is the fifth row of a normal matrix of type I and level 5. Then $|\mathbf{r}_1 \cdot \mathbf{a}| = 6$. So, $\mathbf{a}$ belongs to either (5) or (6) in Table 2. If $\mathbf{a}$ belongs to (5), we may assume, without loss of generality, $\mathbf{a} = \mathbf{r}_5(1)$, where $\mathbf{r}_5(1) =$ (0, -, 1, -, 1, -, 0, 0, 1, 1, 1, 1) and if $\mathbf{a}$ belongs to (6), we may assume $\mathbf{a} = \mathbf{r}_5(2)$, where $\mathbf{r}_5(2) =$ (-, 0, 0, 0, 1, -, 1, -, 1, 1, 1, 1).

$$\text{Put } X_1 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & & & \\ 1 & 1 & 1 & 1 & - & - & - & - & & 1 & & \\ 1 & 1 & - & - & 1 & 1 & - & - & & & 1 & \\ 1 & 1 & - & - & - & - & 1 & 1 & & & & 1 \\ & - & 1 & - & 1 & - & & & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$\text{and } X_2 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & & & \\ 1 & 1 & 1 & 1 & - & - & - & - & & 1 & & \\ 1 & 1 & - & - & 1 & 1 & - & - & & & 1 & \\ 1 & 1 & - & - & - & - & 1 & 1 & & & & 1 \\ - & & & & 1 & - & 1 & - & 1 & 1 & 1 & 1 \end{bmatrix}$$

Clearly, $X_1 \not\sim X_2$.

Lemma 14 *For X_1, it is uniquely possible, up to equivalence, to construct an weighing matrix such that the first five rows equal those of X_1.*

Proof: It is easy to show that there is the unique vector, say $\mathbf{r}_6$, such that a matrix $[X_1^T, \mathbf{r}_6^T]^T$ is normal of type I and level 6, where $\mathbf{r}_6 = (\,-, 0, -, 1, -, 1, 0, 0, 1, 1, 1, 1)$.

Moreover, there are four normal matrices, up to equivalence, of type I and level 8 such that the first six rows equal those of $[X_1^T, \mathbf{r}_6^T]^T$, i.e.

$$X_1^{(1)} = \left[\begin{array}{cccccccccccc} \multicolumn{12}{c}{X_1} \\ - & & - & 1 & - & 1 & & & 1 & 1 & 1 & 1 \\ 1 & - & & 1 & & & 1 & - & - & - & 1 & 1 \\ - & 1 & 1 & & & & - & 1 & - & - & 1 & 1 \end{array}\right],$$

$$X_1^{(2)} = \left[\begin{array}{cccccccccccc} \multicolumn{12}{c}{X_1} \\ - & & - & 1 & - & 1 & & & 1 & 1 & 1 & 1 \\ - & 1 & 1 & & & & 1 & - & - & - & 1 & 1 \\ 1 & - & & 1 & & & - & 1 & - & - & 1 & 1 \end{array}\right],$$

$$X_1^{(3)} = \left[\begin{array}{cccccccccccc} \multicolumn{12}{c}{X_1} \\ - & & - & 1 & - & 1 & & & 1 & 1 & 1 & 1 \\ 1 & - & 1 & & - & 1 & & & - & - & 1 & 1 \\ - & 1 & & 1 & 1 & - & & & - & - & 1 & 1 \end{array}\right] \text{ and}$$

$$X_1^{(4)} = \left[\begin{array}{cccccccccccc} \multicolumn{12}{c}{X_1} \\ - & & - & 1 & - & 1 & & & 1 & 1 & 1 & 1 \\ - & 1 & & 1 & 1 & - & & & - & - & 1 & 1 \\ 1 & - & 1 & & - & 1 & & & - & - & 1 & 1 \end{array}\right].$$

But clearly, we can't extend $X_1^{(3)}$ and $X_1^{(4)}$ to a weighing matrix of weight 9. Moreover $X_1^{(1)} \sim X_1^{(2)}$, because we obtain $X_1^{(1)}$ by operating with $\pi(\underline{1}, \underline{2}, 3, 4, 7, 8, 5, 6)$ and $\rho(\underline{4}\ \underline{3}\ \underline{1}\ \underline{2}\ \underline{8}\ \underline{7}\ \underline{6}\ \underline{5}\ \underline{9}\ \underline{10}\ 11\ 12)$ on $X_1^{(2)}$. From $X_1^{(1)}$, we can construct uniquely a weighing matrix, say A_1, where

$$A_1 = \left[\begin{array}{cccccccccccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & & & \\ 1 & 1 & 1 & 1 & - & - & - & - & & 1 & & \\ 1 & 1 & - & - & 1 & 1 & - & - & & & 1 & \\ 1 & 1 & - & - & - & - & 1 & 1 & & & & 1 \\ & - & 1 & - & 1 & - & & & 1 & 1 & 1 & 1 \\ - & & - & 1 & - & 1 & & & 1 & 1 & 1 & 1 \\ 1 & - & & 1 & & & 1 & - & - & - & 1 & 1 \\ - & 1 & 1 & & & & - & 1 & - & - & 1 & 1 \\ 1 & - & & & & 1 & - & 1 & - & 1 & - & 1 \\ - & 1 & & & 1 & & 1 & - & - & 1 & - & 1 \\ & & 1 & - & - & 1 & & - & 1 & - & - & 1 \\ & & - & 1 & 1 & - & - & & 1 & - & - & 1 \end{array}\right].$$

Lemma 15 *For X_2, it is uniquely possible, up to equivalence, to construct a weighing matrix such that the first five rows equal those of X_2.*

Proof: We can construct such a weighing matrix, say A_2, by step by step, moreover uniquely, where

$$A_2 = \left[\begin{array}{cccccccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & & & \\
1 & 1 & 1 & 1 & - & - & - & - & & 1 & & \\
1 & 1 & - & - & 1 & 1 & - & - & & & 1 & \\
1 & 1 & - & - & - & - & 1 & 1 & & & & 1 \\
- & & & & 1 & - & 1 & - & 1 & 1 & 1 & 1 \\
 & - & & & - & 1 & - & 1 & 1 & 1 & 1 & 1 \\
 & & 1 & & 1 & - & - & 1 & - & - & 1 & 1 \\
 & & & 1 & - & 1 & 1 & - & - & - & 1 & 1 \\
1 & - & - & 1 & 1 & & & & - & 1 & - & 1 \\
- & 1 & 1 & - & & 1 & & & - & 1 & - & 1 \\
- & 1 & - & 1 & & & - & & 1 & - & - & 1 \\
1 & - & 1 & - & & & & - & 1 & - & - & 1
\end{array}\right].$$

Case II

Let A be an weighing matrix for Case II. Then we can assume, without loss of generality, that

$$A = \begin{bmatrix} A(5) & A(6) \\ A(7) & A(8) \end{bmatrix}, \text{ where } A_6 = I_4,$$

$$A(5) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & - & - & - & - \\ 1 & 1 & - & - & 1 & 1 & - & - \\ 1 & - & 1 & - & 1 & - & 1 & - \end{bmatrix} \text{ and}$$

$$A(8)^T = \begin{bmatrix} 1 & - & 1 & - & 1 & - & 1 & - \\ 1 & 1 & - & - & 1 & 1 & - & - \\ 1 & 1 & 1 & 1 & - & - & - & - \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

Let $Y = \begin{bmatrix} Y(1) & Y(2) \\ Y(3) & Y(4) \end{bmatrix}$ be a $(j \times 12)$ matrix such that its entries are 0, -1 and 1 and $YY^T = 9I_j$, where $4 \le j \le 12$ and $Y(1), Y(2), Y(3)$ and $Y(4)$ are (4×12), (4×4), $((j-4) \times 8)$ and $((j-4) \times 4)$ submatrices of Y, respectively. Then Y is called normal of type II and level j if $[Y(1), Y(2)] = [A(5), I_4]$ and $Y(4) = \bar{A}(8)$, where $\bar{A}(8)$ is the first $((j-4) \times 4)$ submatrix of $A(8)$.

Lemma 16 *There are three normal matrices of type II and level 5, up to equivalence.*

Proof: Let **a** be a vector which is the fifth row of a normal matrix of type II and level 5. So, **a** has to belong to either (5) or (6) in Table 2. If **a** belongs to (6), we may assume, up to equivalence,

$$\mathbf{a} = \mathbf{a}_1 = (-, 0, 0, 0, 1, -, -, 1, 1, 1, 1, 1).$$

If **a** belongs to (5), there are eleven cases, say $\mathbf{a}_\ell$, $(2 \le \ell \le 12)$, where

$$\mathbf{a}_2 \;=\; (\; - \;\; 1 \;\; 0 \;\; - \;\; 0 \;\; - \;\; 0 \;\; 1 \;\; 1 \;\; 1 \;\; 1 \;\; 1 \;),$$

$$
\begin{array}{lcl}
\mathbf{a}_3 & = & (\,1\ -\ -\ 0\ -\ 0\ 0\ 1\ 1\ 1\ 1\ 1\,),\\
\mathbf{a}_4 & = & (\,-\ 0\ 1\ -\ -\ 1\ 0\ 0\ 1\ 1\ 1\ 1\,),\\
\mathbf{a}_5 & = & (\,-\ 0\ -\ 1\ 1\ -\ 0\ 0\ 1\ 1\ 1\ 1\,),\\
\mathbf{a}_6 & = & (\,-\ 1\ 0\ -\ -\ 0\ 1\ 0\ 1\ 1\ 1\ 1\,),\\
\mathbf{a}_7 & = & (\,-\ -\ 0\ 1\ 1\ 0\ -\ 0\ 1\ 1\ 1\ 1\,),\\
\mathbf{a}_8 & = & (\,-\ 1\ -\ 0\ 0\ -\ 1\ 0\ 1\ 1\ 1\ 1\,),\\
\mathbf{a}_9 & = & (\,-\ -\ 1\ 0\ 0\ 1\ -\ 0\ 1\ 1\ 1\ 1\,),\\
\mathbf{a}_{10} & = & (\,-\ 0\ 1\ -\ 0\ 0\ -\ 1\ 1\ 1\ 1\ 1\,),\\
\mathbf{a}_{11} & = & (\,-\ 0\ -\ 1\ 0\ 0\ 1\ -\ 1\ 1\ 1\ 1\,),\\
\mathbf{a}_{12} & = & (\,-\ -\ 0\ 1\ 0\ 1\ 0\ -\ 1\ 1\ 1\ 1\,).
\end{array}
$$

Let $Y_i = [\mathbf{r}_1^T, \mathbf{r}_2^T, \mathbf{r}_3^T, \mathbf{r}_4(2)^T, \mathbf{a}_i^T]^T$, $i = 1, 2, \ldots, 12$. Then we obtain an equivalence table as follows:

Old	π-operations	ρ-operations	New
Y_2	2 1 3 4 5	1 2 3 4 $\underline{8}\ \underline{7}\ \underline{6}\ \underline{5}$ 10 9 11 12	Y_6
Y_4	"	"	Y_{10}
Y_5	"	"	Y_{11}
Y_7	"	"	Y_{12}
Y_4	1 3 2 4 5	1 2 5 6 3 4 7 8 9 11 10 12	Y_5
Y_4	1 2 4 3 5	1 3 2 4 5 7 6 8 9 10 12 11	Y_6
Y_4	1 4 2 3 5	1 3 5 7 2 4 6 8 9 12 10 11	Y_8
Y_5	1 2 4 3 5	1 3 2 4 5 7 6 8 9 10 12 11	Y_7
Y_8	1 2 4 3 5	1 3 2 4 5 7 6 8 9 10 12 11	Y_9

Clearly, $Y_1 \not\sim Y_2, Y_3$ and $Y_2 \not\sim Y_3$.

Lemma 17 *For Y_1, there are two normal matrices of type II and level 6, up to equivalence, such that the first five rows equal those of Y_1. Moreover, for each matrix, it is uniquely possible to construct a weighing matrix.*

Proof: A vector to be added to Y_1 has to belong to (14) or (15) in Table 2. There is not such a vector which belongs to (14). But there are three vectors, say $\mathbf{r}_6(1)$, $\mathbf{r}_6(2)$ and $\mathbf{r}_6(3)$, which belong to (15), where

$$\mathbf{r}_6(1) = (\,-\ 0\ 0\ 1\ 0\ 1\ 1\ -\ -\ 1\ 1\ 1\,),$$

$$\begin{array}{rcl}\mathbf{r}_6(2) & = & (\ 1\ \ -\ \ 0\ \ 0\ \ -\ \ 1\ \ 0\ \ 1\ \ -\ \ 1\ \ 1\ \ 1\ ) \text{ and} \\ \mathbf{r}_6(3) & = & (\ 1\ \ 0\ \ -\ \ 0\ \ -\ \ 0\ \ 1\ \ 1\ \ -\ \ 1\ \ 1\ \ 1\ ).\end{array}$$

Put $Y_1^{(i)} = [Y_1^T, \mathbf{r}_6(i)^T]^T$, $(i = 1, 2, 3)$. But $Y_1^{(2)} \sim Y_1^{(3)}$ because operating with $\pi(1, 2, 4, 3, 5, 6)$ and $\rho(1, 3, 2, 4, 5, 7, 6, 8, 9, 10, 12, 11)$ on $Y_1^{(2)}$ gives $Y_1^{(3)}$. Next, it is easily shown that we can uniquely construct a weighing matrix, say A_{i+2}, from each $Y_1^{(i)}$, $(i = 1, 2)$, where

$$A_3 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & & & \\ 1 & 1 & 1 & 1 & - & - & - & - & & 1 & & \\ 1 & 1 & - & - & 1 & 1 & - & - & & & 1 & \\ 1 & - & 1 & - & 1 & - & 1 & - & & & & 1 \\ - & & & & 1 & - & - & 1 & 1 & 1 & 1 & 1 \\ - & & & 1 & & 1 & 1 & - & - & 1 & 1 & 1 \\ 1 & - & - & 1 & - & & & & 1 & - & 1 & 1 \\ & 1 & 1 & - & - & & & 1 & - & - & 1 & 1 \\ & 1 & - & - & - & & 1 & & 1 & 1 & - & 1 \\ 1 & - & & & & 1 & - & 1 & - & 1 & - & 1 \\ - & & 1 & & & 1 & - & - & 1 & - & - & 1 \\ & 1 & - & 1 & 1 & - & & & - & - & - & 1 \end{bmatrix} \text{ and}$$

$$A_4 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & & & \\ 1 & 1 & 1 & 1 & - & - & - & - & & 1 & & \\ 1 & 1 & - & - & 1 & 1 & - & - & & & 1 & \\ 1 & - & 1 & - & 1 & - & 1 & - & & & & 1 \\ - & & & & 1 & - & - & 1 & 1 & 1 & 1 & 1 \\ 1 & - & & & - & 1 & & 1 & - & 1 & 1 & 1 \\ - & & 1 & & - & 1 & & - & 1 & - & 1 & 1 \\ & 1 & - & 1 & & - & 1 & & - & - & 1 & 1 \\ & 1 & - & - & - & & 1 & & 1 & 1 & - & 1 \\ - & & & 1 & 1 & 1 & & - & - & 1 & - & 1 \\ 1 & - & - & 1 & & & - & & 1 & - & - & 1 \\ & 1 & 1 & - & & & - & 1 & - & - & - & 1 \end{bmatrix}.$$

Lemma 18 *For Y_2, there are three normal matrices of type II and level 6 such that the first five rows equal those of Y_2. Moreover, for each matrix, it is uniquely possible to construct a weighing matrix.*

Proof: Let **a** be a vector to be added to Y_2. Then **a** has to be chosen from (14) or (15) in Table 2. If **a** belongs to (14), put $\mathbf{a} = \bar{\mathbf{r}}_6(1)$, where $\bar{\mathbf{r}}_6(1) = (1, -, -, 1, 0, 0, 0, -, 1, 1, 1)$. If **a** belongs to (15), there are two vectors, say $\bar{\mathbf{r}}_6(2)$ and $\bar{\mathbf{r}}_6(3)$, satisfying the conditions, where

$$\begin{array}{rcl}\bar{\mathbf{r}}_6(2) & = & (-,0,0,1,0,1,1,-,-,1,1,1) \text{ and} \\ \bar{\mathbf{r}}_6(3) & = & (1,-,0,0,-,1,0,1,-,1,1,1).\end{array}$$

Put $Y_2^{(i)} = [Y_2^T, \bar{\mathbf{r}}_6(i)^T]^T$, $(i = 1,2,3)$. Then, from each $Y_2^{(i)}$, we can uniquely construct a weighing matrix, say A_{i+4}, where $i = 1,2,3$ and

$$A_5 = \left[\begin{array}{cccccccccccc}
1&1&1&1&1&1&1&1&1&&& \\
1&1&1&1&-&-&-&-&&1&& \\
1&1&-&-&1&1&-&-&&&1& \\
1&-&1&-&1&-&1&-&&&&1 \\
-&1&&-&&-&&1&1&1&1&1 \\
1&-&-&1&&&&1&-&1&1&1 \\
&-&1&&-&1&-&&1&-&1&1 \\
-&1&&1&&&1&-&-&-&1&1 \\
&&-&&-&1&1&-&1&1&-&1 \\
-&&1&&1&1&-&&-&1&-&1 \\
&&-&1&1&-&-&&1&-&-&1 \\
1&1&&-&-&&&1&-&-&-&1
\end{array}\right],$$

$$A_6 = \left[\begin{array}{cccccccccccc}
1&1&1&1&1&1&1&1&1&&& \\
1&1&1&1&-&-&-&-&&1&& \\
1&1&-&-&1&1&-&-&&&1& \\
1&-&1&-&1&-&1&-&&&&1 \\
-&1&&-&&-&&1&1&1&1&1 \\
-&&&1&&1&1&-&-&1&1&1 \\
&-&1&&-&1&-&&1&-&1&1 \\
1&&-&1&&-&&1&-&-&1&1 \\
&-&-&1&1&&-&&1&1&-&1 \\
1&&&-&-&1&&1&-&1&-&1 \\
&1&-&&-&&1&-&1&-&-&1 \\
-&1&1&&1&&-&&-&-&-&1
\end{array}\right] \text{ and}$$

$$A_7 = \left[\begin{array}{cccccccccccc}
1&1&1&1&1&1&1&1&1&&& \\
1&1&1&1&-&-&-&-&&1&& \\
1&1&-&-&1&1&-&-&&&1& \\
1&-&1&-&1&-&1&-&&&&1 \\
-&1&&-&&-&&1&1&1&1&1 \\
1&-&&&-&1&&1&-&1&1&1 \\
&&-&1&-&&1&-&1&-&1&1 \\
-&&1&1&1&&-&&-&-&1&1 \\
&-&-&1&1&&-&&1&1&-&1 \\
-&1&&&&1&1&-&-&1&-&1 \\
&&1&-&-&1&-&&1&-&-&1 \\
1&1&-&&&-&&1&-&-&-&1
\end{array}\right].$$

Lemma 19 *For Y_3, there are two normal matrices of type II and level 8 such that the first five rows equal those of Y_3. Moreover, it is uniquely possible to construct a weighing matrix from each matrix of level 8.*

Proof: There are five normal matrices, say $Y_3^{(i)}$ of type II and level 8 such that the first five rows equal those of Y_3, where $i \le i \le 5$, and

$$Y_3^{(i)} = \begin{bmatrix} A(5) & I_4 \\ Z^{(i)} & A(9) \end{bmatrix}, \quad A(9) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ - & 1 & 1 & 1 \\ 1 & - & 1 & 1 \\ - & - & 1 & 1 \end{bmatrix},$$

$$\begin{aligned}
Z^{(1)} &= \begin{bmatrix} 1 & - & - & & - & & & 1 \\ - & 1 & & & 1 & - & & 1 \\ - & 1 & & & - & & 1 & - \\ & - & 1 & 1 & & 1 & - & \end{bmatrix}, \\
Z^{(2)} &= \begin{bmatrix} 1 & - & - & & - & & & 1 \\ - & 1 & 1 & - & & & & 1 \\ - & & & 1 & 1 & - & - & \\ & & & 1 & - & 1 & 1 & - \end{bmatrix}, \\
Z^{(3)} &= \begin{bmatrix} 1 & - & - & & - & & & 1 \\ - & & & 1 & & 1 & 1 & - \\ - & 1 & 1 & - & - & & & \\ & & & 1 & 1 & - & - & 1 \end{bmatrix}, \\
Z^{(4)} &= \begin{bmatrix} 1 & - & - & & - & & & 1 \\ - & & 1 & & 1 & & - & 1 \\ - & & 1 & & - & 1 & & - \\ & 1 & - & 1 & & - & 1 & \end{bmatrix} \text{ and} \\
Z^{(5)} &= \begin{bmatrix} 1 & - & - & & - & & & 1 \\ - & & & 1 & & 1 & 1 & - \\ - & & & 1 & 1 & - & - & \\ & 1 & 1 & - & - & & & 1 \end{bmatrix}.
\end{aligned}$$

Clearly, we can't construct a weighing matrix from $Y_3^{(5)}$. But $Y_3^{(1)} \sim Y_3^{(4)}$ and $Y_3^{(2)} \sim Y_3^{(3)}$. We shall give an equivalence table in the following:

Old	π-operations	ρ-operations	New
$Y_3^{(1)}$	1 2 4 3 5 6 7 8	1 3 2 4 5 7 6 8 9 10 12 11	$Y_3^{(4)}$
$Y_3^{(2)}$	2 1 3 4 5 7 6 8	1 2 3 4 $\underline{8}$ $\underline{7}$ $\underline{6}$ $\underline{5}$ 10 9 11 12	$Y_3^{(3)}$

Next, we can uniquely construct a weighing matrix, say A_{i+7}, from each $Y_3^{(i)}$ $(i = 1, 2)$, where

$$A_8 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & & & \\ 1 & 1 & 1 & 1 & - & - & - & - & & 1 & & \\ 1 & 1 & - & - & 1 & 1 & - & - & & & 1 & \\ 1 & - & 1 & - & 1 & - & 1 & - & & & & 1 \\ 1 & - & - & & - & & & 1 & 1 & 1 & 1 & 1 \\ - & 1 & 1 & - & & & & 1 & - & 1 & 1 & 1 \\ - & & & 1 & 1 & - & - & & 1 & - & 1 & 1 \\ & & & 1 & - & 1 & 1 & - & - & - & 1 & 1 \\ - & 1 & - & & & & 1 & - & 1 & 1 & - & 1 \\ & - & & 1 & 1 & 1 & - & & - & 1 & - & 1 \\ & & 1 & - & - & 1 & - & & 1 & - & - & 1 \\ 1 & 1 & - & & & - & & 1 & - & - & - & 1 \end{bmatrix} \text{ and}$$

$$A_9 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & & & \\ 1 & 1 & 1 & 1 & - & - & - & - & & 1 & & \\ 1 & 1 & - & - & 1 & 1 & - & - & & & 1 & \\ 1 & - & 1 & - & 1 & - & 1 & - & & & & 1 \\ 1 & - & - & & - & & & 1 & 1 & 1 & 1 & 1 \\ - & & & 1 & & 1 & 1 & - & - & 1 & 1 & 1 \\ - & & & 1 & 1 & - & - & & 1 & - & 1 & 1 \\ & 1 & 1 & - & - & & & 1 & - & - & 1 & 1 \\ - & & 1 & - & & 1 & - & & 1 & 1 & - & 1 \\ & 1 & - & & 1 & - & & 1 & - & 1 & - & 1 \\ & 1 & - & & - & & 1 & - & 1 & - & - & 1 \\ 1 & - & & 1 & & 1 & - & & - & - & - & 1 \end{bmatrix}.$$

Theorem 2 *There are four weighing matrices up to equivalence of order 12 and weight 9.*

Proof: It is sufficient to check whether matrices A_i, $(i \leq i \leq 9)$ are equivalent or not. We give an equivalence table in the following:

Old	π-operations	ρ-operations	New
A_1	Identity	9 10 11 12 1 2 3 4 5 6 7 8	Sym
A_1	2 3 1 4 5 6 11 12 8 7 9 10	2 1 $\underline{8}$ $\underline{7}$ 3 4 $\underline{6}$ $\underline{5}$ 10 11 9 12	A_2
A_3	9 7 12 8 $\underline{11}$ $\underline{6}$ $\underline{3}$ 5 1 10 $\underline{2}$ $\underline{4}$	12 $\underline{3}$ $\underline{5}$ 9 2 $\underline{11}$ $\underline{4}$ 10 7 1 $\underline{6}$ 8	A_6
A_4^T	2 3 4 7 $\underline{8}$ 6 $\underline{11}$ $\underline{9}$ $\underline{10}$ $\underline{5}$ $\underline{12}$ 1	1 2 $\underline{11}$ 12 8 $\underline{4}$ 9 3 $\underline{6}$ 7 10 $\underline{5}$	A_7
A_5	2 $\underline{1}$ 3 4 6 8 5 7 10 12 9 11	$\underline{8}$ $\underline{7}$ $\underline{6}$ $\underline{5}$ 1 2 3 4 10 $\underline{9}$ 11 12	A_4
A_6^T	1 4 6 8 $\underline{2}$ $\underline{7}$ 11 3 10 5 $\underline{12}$ $\underline{9}$	1 $\underline{5}$ 8 2 10 3 $\underline{6}$ 4 $\underline{12}$ 9 7 $\underline{11}$	A_6
A_7	Identity	12 11 10 9 1 2 3 4 5 6 7 8	Sym
A_8	12 9 6 5 $\underline{10}$ $\underline{7}$ $\underline{1}$ 11 $\underline{3}$ 8 2 $\underline{4}$	12 2 $\underline{3}$ $\underline{11}$ 8 $\underline{9}$ 1 $\underline{10}$ $\underline{6}$ 7 $\underline{4}$ $\underline{5}$	A_6
A_9	Identity	12 11 10 9 1 2 3 4 5 6 7 8	Sym

But A_1, A_3, A_4 and A_9 are not equivalent to each other, as we can see from the following G-table:

Matrices	Mult.	0	1	2	3	4	5	6	7	8	9
A_1	12	48	32	64	0	20	0	0	0	1	0
A_3	12	55	12	72	14	6	6	0	0	0	0
A_4	12	55	16	72	8	6	8	0	0	0	0
A_9	12	73	0	48	32	12	0	0	0	0	0

This completes our proof.

4 Classifications of weighing matrices of order 12 and weight 8

Let A be an weighing matrix of order 12 and weight 8. The IPC are $p_4+p_6+p_8 = 11$ and $2p_4 + 3p_6 + 4p_8 = 28$. So we get four kinds of solutions:

I $p_8 = 3$, $p_6 = 0$, $p_4 = 8$.

II $p_8 = 2$, $p_6 = 2$, $p_4 = 7$.

III $p_8 = 1$, $p_6 = 4$, $p_4 = 6$.

IV $p_8 = 0$, $p_6 = 6$, $p_4 = 5$.

Lemma 20 *There is no weighing matrix containing a row which has Case II for the IPC.*

Proof: Let A be a weighing matrix for Case II. Then we can assume, without loss of generality,

$$A = \begin{bmatrix} A(1) & A(2) \\ A(3) & A(4) \end{bmatrix}, \text{ where } A(1) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & - & - & - & - \\ 1 & 1 & - & - & 1 & 1 & - & - \end{bmatrix},$$

$A(2) = 0_{3\times4}$, and $A(3)$ and $A(4)$ are (9×8) and (9×4) submatrices of A. But $A(4)^T A(4) = 8I_4$ and each column of $A(4)$ has weight 8. So, there is a row, say the i_0th row, in $A(4)$ which is a zero vector. Then the i_0th row of $A(3)$ has the weight 8. This contradicts to our Case II.

Lemma 21 *Any weighing matrix containing a row which has Case IV for the IPC is non-existent or equivalent to the transpose of an weighing matrix which was constructed from Cases I or III.*

Proof: Let A be a weighing matrix for Case IV. Put $A = [\bar{A}, \bar{B}]$, where the first rows' components of $\bar{A}$ and $\bar{B}$ are all ones and all zeros, respectively. Then we may assume that $|\mathbf{b}' \cdot \mathbf{b}''| = 6$, where $\mathbf{b}'$ and $\mathbf{b}''$ are distinct column vectors of $\bar{B}$, because if there is a pair of column vectors, say $\mathbf{b}'$ and $\mathbf{b}''$, of $\bar{B}$ such as $|\mathbf{b}' \cdot \mathbf{b}''| = 8$, then A^T will be equivalent to an weighing matrix which was constructed from Cases I or III.

So, we may assume without loss of generality, that

$$\bar{B}^T = \left[\begin{array}{cc|ccccc|cc|cc|c} 0 & 1 & \multicolumn{5}{c|}{B(1)^T} & \multicolumn{2}{c|}{B(2)^T} & \multicolumn{2}{c|}{B(3)^T} & b \\ 0 & 1 & & & & & & & & & & 1 \\ \hline 0 & 1 & 1 & 1 & - & - & - & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \end{array}\right], \text{ where } b \in \{1, -1\}$$

.

But $B(2)$ and $B(3)$ must be forms of matrices such as $\begin{bmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{bmatrix}$ or $\begin{bmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{bmatrix}$ and there is one zero row vector in $B(1)$. So, we may assume that $B(2)$ and $B(3)$ are forms of matrices such as $\begin{bmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{bmatrix}$, by row permutations of $\bar{B}$.

Then it is easily shown that there are three matrices, say $\bar{B}_1, \bar{B}_2$, and $\bar{B}_3$, satisfying the above conditions, where

$$\bar{B}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ - & - & 1 & 1 \\ - & 1 & - & 1 \\ 1 & - & - & 1 \\ - & - & - & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & - & 1 & 0 \\ - & 0 & 1 & 0 \\ - & 1 & 0 & 0 \end{bmatrix}, \quad \bar{B}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ - & 1 & 1 & 1 \\ 1 & - & 1 & 1 \\ 0 & 0 & - & 1 \\ 1 & 1 & - & 1 \\ - & - & - & 1 \\ 0 & - & 0 & 1 \\ - & 0 & 0 & 1 \\ 0 & - & 1 & 0 \\ - & 0 & 1 & 0 \\ - & 1 & 0 & 0 \end{bmatrix}, \quad \bar{B}_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & - & 1 & 1 \\ - & - & 1 & 1 \\ 0 & 0 & - & 1 \\ - & 1 & - & 1 \\ 1 & - & - & 1 \\ 0 & 1 & 0 & 1 \\ - & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ - & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

But $\bar{B}_1 \sim \bar{B}_2$, because we get $\bar{B}_1$ by operating with $\pi($ 1, 6, 5, 7, $\underline{4}$, $\underline{3}$, $\underline{2}$, $\underline{10}$, $\underline{11}$, 8, 9, 12$)$ and $\rho($ 1, 2, 4, $\underline{3})$ on $\bar{B}_2$. $\bar{B}_2 \sim \bar{B}_3$, because operating with $\pi($ 1, 4, 2, 3, 5, 7, 6, 8, 9, 10, 11, $\underline{12})$ and $\rho($ 1, $\underline{2}$, 3, 4$)$ on $\bar{B}_2$ gives $\bar{B}_3$. So, we shall use $\bar{B}_1$ for $\bar{B}$. Then we may assume without a loss of generality that $A = \begin{bmatrix} X & J \\ Y & K \end{bmatrix}$, where $[J^T, K^T]^T$ is a matrix which was obtained by operating with $\pi($ 1, 2, 4, 5, 6, 7, 3, 8, 9, 10, 11, 12$)$ on $\bar{B}_1$:

$$J = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ - & - & 1 & 1 \\ - & 1 & - & 1 \\ 1 & - & - & 1 \\ - & - & - & 1 \end{bmatrix} \text{ and } K = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & - & 1 & 0 \\ - & 0 & 1 & 0 \\ - & 1 & 0 & 0 \end{bmatrix}.$$

We want to find a matrix X such that $[\mathbf{r} \cdot \mathbf{r}'] \neq 8$, where $\mathbf{r}$ and $\mathbf{r}'$ are any two different row vectors of the matrix $[X, J]$.

But we can obtain two such matrices, say X_1 and X_2 without loss of generality and up to equivalence, where

$$X_1 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & & & & \\ 1 & 1 & - & - & & & & & 1 & 1 & 1 & 1 \\ - & & - & & 1 & & 1 & & - & - & 1 & 1 \\ & - & & - & & 1 & & 1 & - & 1 & - & 1 \\ & & & & 1 & 1 & - & - & 1 & - & - & 1 \\ 1 & 1 & & & - & - & & & - & - & 1 & 1 \end{bmatrix} \text{ and}$$

$$X_2 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & & & & \\ 1 & 1 & - & - & & & & & 1 & 1 & 1 & 1 \\ & & & & - & 1 & 1 & - & - & - & 1 & 1 \\ & 1 & 1 & & - & & - & & - & 1 & - & 1 \\ - & & & - & & 1 & & 1 & 1 & - & - & 1 \\ 1 & & - & & 1 & - & & & - & - & - & 1 \end{bmatrix}.$$

Let $\mathbf{a} = (\alpha_1, \ldots, \alpha_8, 0, 0, 1, 1)$ be a vector such that $[X, J]\mathbf{a}^T = \mathbf{0}$ or $[X_2, J]\mathbf{a}^T = \mathbf{0}$, where $|\mathbf{a}| = 8$. But no vector $\mathbf{a}$ can exist such that $\sum_{i=1}^{8} |\alpha_i| = 6$.

Lemma 22 *For Case I, there are six weighing matrices which could be inequivalent.*

Proof: Let A be a weighing matrix for Case I. We may assume, without loss of generality, that $A = \begin{bmatrix} A(5) & A(6) \\ A(7) & A(8) \end{bmatrix}$, where $A(6) = 0_{4\times4}$. Moreover, without loss of generality, the $A(5)$ part is divided into two cases, say (i) and (ii), and the $A(8)$ part is divided into two cases, say (iii) and (iv), where

$$\text{(i)}: \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & - & - & - & - \\ 1 & 1 & - & - & 1 & 1 & - & - \\ 1 & 1 & - & - & - & - & 1 & 1 \end{bmatrix}, \quad \text{(iii)} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ - & - & 1 & 1 \\ - & - & 1 & 1 \\ - & 1 & - & 1 \\ - & 1 & - & 1 \\ 1 & - & - & 1 \\ 1 & - & - & 1 \end{bmatrix},$$

$$\text{(ii)} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & - & - & - & - \\ 1 & 1 & - & - & 1 & 1 & - & - \\ 1 & - & 1 & - & 1 & - & 1 & - \end{bmatrix},$$

$$
\text{(iv)}\quad \begin{bmatrix}
1 & 1 & 1 & 1 \\
- & 1 & 1 & 1 \\
1 & - & 1 & 1 \\
- & - & 1 & 1 \\
1 & 1 & - & 1 \\
- & 1 & - & 1 \\
1 & - & - & 1 \\
- & - & - & 1
\end{bmatrix}.
$$

We construct weighing matrices from three cases (i,iii), (i,iv), (ii,iv) only, because matrices which are constructed from case (ii,iii) will be equivalent to the transpose matrices which are constructed from case (i,iv).

For cases (i,iii) and (i,iv), we can assume that the first row vector of $A(7)$ is $(1, -, 1, -, 0, 0, 0, 0)$. On the other hand, for case (ii,iv), there are 24 possibilities for the first row vector of $A(7)$ as follows:

(1)	(1 - - 1 0 0 0 0)	(7)	(0 1 - 0 0 - 1 0)
(2)	(1 - 0 0 - 1 0 0)	(8)	(0 1 0 - - 0 1 0)
(3)	(1 - 0 0 0 0 - 1)	(9)	(0 1 0 - 0 - 0 1)
(4)	(1 0 - 0 0 - 0 1)	(10)	(0 0 1 - - 1 0 0)
(5)	(1 0 - 0 - 0 1 0)	(11)	(0 0 1 - 0 0 - 1)
(6)	(1 0 0 - - 0 0 1)	(12)	(0 0 0 0 1 - - 1)

$(n)^*$: Sign changed vector of (n), where $n = 1, 2, \ldots, 12$.

But we can easily show that all cases are equivalent to each other. So, for case (ii,iv), we assume that the first row of vector $A(7)$ is $(1, -, -, 1, 0, 0, 0, 0)$.

From case (i,iii), we can construct two weighing matrices, say A_1 and A_2, where

$$
A_1 = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & & & & \\
1 & 1 & 1 & 1 & - & - & - & - & & & & \\
1 & 1 & - & - & 1 & 1 & - & - & & & & \\
1 & 1 & - & - & - & - & 1 & 1 & & & & \\
1 & - & 1 & - & & & & & 1 & 1 & 1 & 1 \\
- & 1 & - & 1 & & & & & 1 & 1 & 1 & 1 \\
1 & - & - & 1 & & & & & - & - & 1 & 1 \\
- & 1 & 1 & - & & & & & - & - & 1 & 1 \\
 & & & & 1 & - & 1 & - & - & 1 & - & 1 \\
 & & & & - & 1 & - & 1 & - & 1 & - & 1 \\
 & & & & 1 & - & - & 1 & 1 & - & - & 1 \\
 & & & & - & 1 & 1 & - & 1 & - & - & 1
\end{bmatrix} \quad \text{and}
$$

$$
A_2 = \left[\begin{array}{cccccccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & & & & \\
1 & 1 & 1 & 1 & - & - & - & - & & & & \\
1 & 1 & - & - & 1 & 1 & - & - & & & & \\
1 & 1 & - & - & - & - & 1 & 1 & & & & \\
1 & - & 1 & - & & & & & 1 & 1 & 1 & 1 \\
- & 1 & - & 1 & & & & & 1 & 1 & 1 & 1 \\
 & & & & 1 & - & 1 & - & - & - & 1 & 1 \\
 & & & & - & 1 & - & 1 & - & - & 1 & 1 \\
1 & - & - & 1 & & & & & - & 1 & - & 1 \\
- & 1 & 1 & - & & & & & - & 1 & - & 1 \\
 & & & & 1 & - & - & 1 & 1 & - & - & 1 \\
 & & & & - & 1 & 1 & - & 1 & - & - & 1
\end{array}\right].
$$

From case (i,iv), we can construct three weighing matrices, say A_3, A_4 and A_5, and from case (ii,iv) an weighing matrices, say A_6, where

$$
A_3 = \left[\begin{array}{cccccccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & & & & \\
1 & 1 & 1 & 1 & - & - & - & - & & & & \\
1 & 1 & - & - & 1 & 1 & - & - & & & & \\
1 & 1 & - & - & - & - & 1 & 1 & & & & \\
1 & - & 1 & - & & & & & 1 & 1 & 1 & 1 \\
- & 1 & & & 1 & - & & & - & 1 & 1 & 1 \\
 & & - & 1 & & & - & 1 & 1 & - & 1 & 1 \\
 & & & & - & 1 & 1 & - & - & - & 1 & 1 \\
 & & - & 1 & & & 1 & - & 1 & 1 & - & 1 \\
 & & & & - & 1 & - & 1 & - & 1 & - & 1 \\
- & 1 & 1 & - & & & & & 1 & - & - & 1 \\
1 & - & & & 1 & - & & & - & - & - & 1
\end{array}\right],
$$

$$
A_4 = \left[\begin{array}{cccccccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & & & & \\
1 & 1 & 1 & 1 & - & - & - & - & & & & \\
1 & 1 & - & - & 1 & 1 & - & - & & & & \\
1 & 1 & - & - & - & - & 1 & 1 & & & & \\
1 & - & 1 & - & & & & & 1 & 1 & 1 & 1 \\
- & 1 & & & 1 & - & & & - & 1 & 1 & 1 \\
 & & - & 1 & & & - & 1 & 1 & - & 1 & 1 \\
 & & & & - & 1 & 1 & - & - & - & 1 & 1 \\
- & 1 & & & - & 1 & & & 1 & 1 & - & 1 \\
1 & - & - & 1 & & & & & - & 1 & - & 1 \\
 & & & & 1 & - & 1 & - & 1 & - & - & 1 \\
 & & 1 & - & & & - & 1 & - & - & - & 1
\end{array}\right],
$$

$$
A_5 = \left[\begin{array}{cccccccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & & & & \\
1 & 1 & 1 & 1 & - & - & - & - & & & & \\
1 & 1 & - & - & 1 & 1 & - & - & & & & \\
1 & 1 & - & - & - & - & 1 & 1 & & & & \\
1 & - & 1 & - & & & & & 1 & 1 & 1 & 1 \\
- & 1 & & & 1 & - & & & - & 1 & 1 & 1 \\
- & 1 & & & - & 1 & & & 1 & - & 1 & 1 \\
1 & - & - & 1 & & & & & - & - & 1 & 1 \\
 & & - & 1 & & & 1 & - & 1 & 1 & - & 1 \\
 & & & & - & 1 & - & 1 & - & 1 & - & 1 \\
 & & & & 1 & - & - & 1 & 1 & - & - & 1 \\
 & & 1 & - & & & 1 & - & - & - & - & 1
\end{array}\right] \text{ and}
$$

$$
A_6 = \left[\begin{array}{cccccccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & & & & \\
1 & 1 & 1 & 1 & - & - & - & - & & & & \\
1 & 1 & - & - & 1 & 1 & - & - & & & & \\
1 & - & 1 & - & 1 & - & 1 & - & & & & \\
1 & - & - & 1 & & & & & 1 & 1 & 1 & 1 \\
- & 1 & & & & & 1 & - & - & 1 & 1 & 1 \\
 & & 1 & - & - & 1 & & & 1 & - & 1 & 1 \\
 & & & & 1 & - & - & 1 & - & - & 1 & 1 \\
- & & 1 & & 1 & & - & & 1 & 1 & - & 1 \\
1 & & & - & - & & & 1 & - & 1 & - & 1 \\
 & 1 & - & & & - & 1 & & 1 & - & - & 1 \\
 & - & & 1 & & 1 & & - & - & - & - & 1
\end{array}\right].
$$

Lemma 23 *For Case III, there are six weighing matrices which could be inequivalent.*

Proof: Let A be a weighing matrix for Case III. We may assume, without loss of generality, that $A = \begin{bmatrix} A(9) & A(10) \\ A(11) & A(12) \end{bmatrix}$, where

$$
A(9) = \left[\begin{array}{cccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & - & - & - & - \\
\hline
\multicolumn{8}{c}{} \\
\multicolumn{8}{c}{E} \\
\multicolumn{8}{c}{}
\end{array}\right],
$$

$$
A(10) = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
e & f & 0 & 0 \\
0 & 0 & 1 & 1 \\
0 & 0 & g & h
\end{bmatrix} \text{ and } A(12) = \left[\begin{array}{c|cc}
 & 1 & 1 \\
 & 1 & 1 \\
F & - & 1 \\
 & - & 1 \\
 & - & 1 \\
 & - & 1
\end{array}\right],
$$

also where $|\mathbf{e}| = 6$ (e is any row vector of E), $|\mathbf{r}| = 4$ (r is any row vector of $A(11)$), and $e, f, g, h \in \{1, -1\}$.

Then, up to equivalence, there are four cases for the matrix E, say E_i ($1 \leq i \leq 4$), and for each case $e = f = g = h = 1$, where

$$E_1 = \begin{bmatrix} 1 & - & 0 & 0 & 1 & 1 & - & - \\ - & 1 & 0 & 0 & 1 & - & 1 & - \\ 0 & 0 & 1 & - & 1 & - & - & 1 \\ 0 & 0 & 1 & - & - & 1 & 1 & - \end{bmatrix},$$

$$E_2 = \begin{bmatrix} 1 & - & 0 & 0 & 1 & 1 & - & - \\ - & 1 & 0 & 0 & 1 & - & 1 & - \\ - & 1 & 1 & - & 0 & 1 & - & 0 \\ 1 & - & 1 & - & 0 & - & 1 & 0 \end{bmatrix},$$

$$E_3 = \begin{bmatrix} 1 & - & 0 & 0 & 1 & 1 & - & - \\ 1 & - & 0 & 0 & - & - & 1 & 1 \\ 0 & 0 & 1 & - & 1 & - & 1 & - \\ 0 & 0 & 1 & - & - & 1 & - & 1 \end{bmatrix}, \text{and}$$

$$E_4 = \begin{bmatrix} 1 & - & 0 & 0 & 1 & 1 & - & - \\ 1 & - & 0 & 0 & - & - & 1 & 1 \\ 1 & 1 & - & - & 1 & - & 0 & 0 \\ - & - & 1 & 1 & 1 & - & 0 & 0 \end{bmatrix}.$$

Similarly, up to equivalence, there are two cases for the matrix F, say F_1 and F_2, where

$$F_1 = \begin{bmatrix} 1 & 1 \\ - & - \\ - & 1 \\ - & 1 \\ 1 & - \\ 1 & - \end{bmatrix} \text{ and } F_2 = \begin{bmatrix} - & - \\ - & 1 \\ 1 & - \\ 1 & 1 \\ - & 1 \\ 1 & - \end{bmatrix}$$

For each pattern (E_i, F_j), we shall find the $A(11)$–parts in order to construct weighing matrices, where $1 \leq i \leq 4$, $1 \leq j \leq 2$. But it is easy to check that we can uniquely construct weighing matrices from each pattern (E_i, F_1), say A_{i+6}, and two weighing matrices from (E_4, F_2), say A_{11} and A_{12}, where $1 \leq i \leq 4$. We list the matrices A_i, where $7 \leq i \leq 12$.

$$A_7 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & & & & \\ 1 & 1 & 1 & 1 & - & - & - & - & & & & \\ 1 & - & & & 1 & 1 & - & - & 1 & 1 & & \\ - & 1 & & & 1 & - & 1 & - & 1 & 1 & & \\ & & 1 & - & 1 & - & - & 1 & & & 1 & 1 \\ & & 1 & - & - & 1 & 1 & - & & & 1 & 1 \\ & & - & 1 & - & & & 1 & 1 & 1 & 1 & 1 \\ & & - & 1 & 1 & & & - & - & - & 1 & 1 \\ - & 1 & & & & 1 & - & & - & 1 & - & 1 \\ 1 & - & & & & - & 1 & & - & 1 & - & 1 \\ 1 & 1 & - & - & & & & & 1 & - & - & 1 \\ - & - & 1 & 1 & & & & & 1 & - & - & 1 \end{bmatrix}$$

$$A_8 = \left[\begin{array}{cccccccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & & & & \\
1 & 1 & 1 & 1 & - & - & - & - & & & & \\
1 & - & & & 1 & 1 & - & - & 1 & 1 & & \\
- & 1 & & & 1 & - & 1 & - & 1 & 1 & & \\
- & 1 & 1 & - & & 1 & - & & & & 1 & 1 \\
1 & - & 1 & - & & - & 1 & & & & 1 & 1 \\
 & & - & 1 & - & & & 1 & 1 & 1 & 1 & 1 \\
 & & - & 1 & 1 & & & - & - & - & 1 & 1 \\
1 & 1 & - & - & & & & & - & 1 & - & 1 \\
- & - & 1 & 1 & & & & & - & 1 & - & 1 \\
 & & & & 1 & - & - & 1 & 1 & - & - & 1 \\
 & & & & - & 1 & 1 & - & 1 & - & - & 1
\end{array}\right]$$

$$A_9 = \left[\begin{array}{cccccccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & & & & \\
1 & 1 & 1 & 1 & - & - & - & - & & & & \\
1 & - & & & 1 & 1 & - & - & 1 & 1 & & \\
1 & - & & & - & - & 1 & 1 & 1 & 1 & & \\
 & & 1 & - & 1 & - & 1 & - & & & 1 & 1 \\
 & & 1 & - & - & 1 & - & 1 & & & 1 & 1 \\
- & 1 & - & 1 & & & & & 1 & 1 & 1 & 1 \\
1 & - & - & 1 & & & & & - & - & 1 & 1 \\
 & & & & 1 & - & - & 1 & - & 1 & - & 1 \\
 & & & & - & 1 & 1 & - & - & 1 & - & 1 \\
1 & 1 & - & - & & & & & 1 & - & - & 1 \\
- & - & 1 & 1 & & & & & 1 & - & - & 1
\end{array}\right]$$

$$A_{10} = \left[\begin{array}{cccccccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & & & & \\
1 & 1 & 1 & 1 & - & - & - & - & & & & \\
1 & - & & & 1 & 1 & - & - & 1 & 1 & & \\
1 & - & & & - & - & 1 & 1 & 1 & 1 & & \\
1 & 1 & - & - & 1 & - & & & & & 1 & 1 \\
- & - & 1 & 1 & 1 & - & & & & & 1 & 1 \\
- & 1 & & & - & 1 & & & 1 & 1 & 1 & 1 \\
1 & - & & & - & 1 & & & - & - & 1 & 1 \\
 & & 1 & - & & & 1 & - & - & 1 & - & 1 \\
 & & - & 1 & & & - & 1 & - & 1 & - & 1 \\
 & & 1 & - & & & - & 1 & 1 & - & - & 1 \\
 & & - & 1 & & & 1 & - & 1 & - & - & 1
\end{array}\right]$$

$$A_{11} = \left[\begin{array}{cccccccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & & & & \\
1 & 1 & 1 & 1 & - & - & - & - & & & & \\
1 & - & & & 1 & 1 & - & - & 1 & 1 & & \\
1 & - & & & - & - & 1 & 1 & 1 & 1 & & \\
1 & 1 & - & - & 1 & - & & & & & 1 & 1 \\
- & - & 1 & 1 & 1 & - & & & & & 1 & 1 \\
 & & - & 1 & - & 1 & & & - & 1 & 1 & 1 \\
 & & 1 & - & - & 1 & & & 1 & - & 1 & 1 \\
- & 1 & & & & & - & 1 & 1 & 1 & - & 1 \\
 & & 1 & - & & & 1 & - & - & 1 & - & 1 \\
 & & - & 1 & & & 1 & - & 1 & - & - & 1 \\
1 & - & & & & & - & 1 & - & - & - & 1
\end{array}\right]$$

$$A_{12} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & & & & \\ 1 & 1 & 1 & 1 & - & - & - & - & & & & \\ 1 & - & & & 1 & 1 & - & - & 1 & 1 & & \\ 1 & - & & & - & - & 1 & 1 & 1 & 1 & & \\ 1 & 1 & - & - & 1 & - & & & & & 1 & 1 \\ - & - & 1 & 1 & 1 & - & & & & & 1 & 1 \\ & & & & - & 1 & 1 & - & - & 1 & 1 & 1 \\ & & & & - & 1 & - & 1 & 1 & - & 1 & 1 \\ - & 1 & - & 1 & & & & & 1 & 1 & - & 1 \\ & & 1 & - & & & - & 1 & - & 1 & - & 1 \\ & & 1 & - & & & 1 & - & 1 & - & - & 1 \\ 1 & - & - & 1 & & & & & - & - & - & 1 \end{bmatrix}$$

Theorem 3 *There are six weighing matrices of order 12 and weight 8, up to equivalence.*

Proof: It is sufficient to check whether A_i are equivalent or not, where $1 \leq i \leq 12$. We list an equivalence table:

Old	π-operations	ρ-operations	New
A_1	$1\ 2\ 3\ 4\ 5\ \underline{6}\ 7\ \underline{8}\ 9\ \underline{10}\ \underline{12}\ 11$	Identity	Sym
A_2	$1\ 2\ 3\ 4\ 5\ 6\ 9\ 10\ 7\ 8\ 11\ 12$	$1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 11\ 10\ 12$	A_1
A_4	$1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ \underline{12}\ \underline{11}\ \underline{10}\ \underline{9}$	$1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ 12\ 11$	A_3
$(A_6)^T$	$12\ 10\ 11\ 9\ 1\ 2\ 5\ 6\ 3\ 4\ 7\ 8$	$5\ 6\ 9\ 10\ 7\ 8\ 11\ 12\ 4\ 2\ 3\ 1$	A_6
$(A_7)^T$	$1\ 2\ 9\ 10\ 6\ 7\ \underline{5}\ \underline{8}\ 11\ 12\ \underline{4}\ 3$	$11\ \underline{12}\ 1\ 2\ 3\ \underline{9}\ 10\ \underline{4}\ \underline{8}\ 7\ \underline{5}\ 6$	A_7
A_7	$5\ 6\ 1\ 2\ \underline{8}\ 7\ 4\ 3\ \underline{12}\ 9\ 10\ \underline{11}$	$3\ \underline{4}\ 12\ 11\ 8\ 5\ \underline{6}\ \underline{7}\ 1\ 2\ 9\ 10$	A_{12}
$(A_8)^T$	$1\ \underline{2}\ 3\ \underline{4}\ 7\ 6\ \underline{5}\ \underline{8}\ \underline{11}\ 9\ \underline{10}\ 12$	$6\ \underline{5}\ 3\ \underline{4}\ 1\ 2\ \underline{10}\ 9\ \underline{7}\ \underline{8}\ \underline{11}\ 12$	A_{11}
$(A_9)^T$	$5\ 6\ 7\ 8\ 1\ \underline{4}\ \underline{10}\ \underline{11}\ \underline{9}\ \underline{12}\ \underline{2}\ 3$	$1\ \underline{2}\ 3\ \underline{4}\ 5\ \underline{6}\ 9\ \underline{10}\ 8\ \underline{12}\ 11\ \underline{7}$	A_3
$(A_9)^T$	$5\ 6\ 7\ 8\ 1\ \underline{4}\ \underline{3}\ 2\ \underline{10}\ \underline{11}\ 12\ 9$	$1\ \underline{2}\ 3\ \underline{4}\ 5\ \underline{6}\ \underline{10}\ 9\ 8\ \underline{7}\ \underline{12}\ 11$	A_5
A_{10}	$\underline{10}\ 9\ 12\ \underline{11}\ 1\ \underline{4}\ 3\ \underline{2}\ \underline{6}\ 7\ 8\ \underline{5}$	$7\ \underline{8}\ 3\ \underline{4}\ \underline{10}\ 9\ \underline{12}\ 11\ 1\ 2\ 5\ 6$	A_5

Next, we shall check by the G-table whether A_1, A_3, A_6, A_7 and A_8 are equivalent. We list the G-table:

Matrices	Mult	0	1	2	3	4	5	6	7	8
A_1	12	144	0	0	0	20	0	0	0	1
A_3	4	108	0	48	0	8	0	0	0	1
	8	114	0	40	0	9	0	2	0	0
A_3^T	4	111	0	40	0	14	0	0	0	0
	8	123	0	28	0	12	0	2	0	0
A_6	4	81	32	0	0	12	0	0	0	0
	8	72	48	36	0	5	4	0	0	0
A_7	4	102	0	56	0	5	0	2	0	0
	8	103	0	53	0	8	0	1	0	0
A_8	12	104	0	50	0	11	0	0	0	0

So, A_1, A_3, A_6, A_7, A_8 and A_9 are inequivalent to each other.
This completes our proof.

5 Classifications of weighing matrices of order 12 and weight 7

Let A be a weighing matrix of order 12 and weight 7. The IPC are $p_2+p_4+p_6 = 11$ and $p_2 + 2p_4 + 3p_6 = 21$. So, we can divide the solutions into two cases:

I $p_6 \geq 1$; and

II $p_6 = 0$.

Lemma 24 *There are two weighing matrices which could be inequivalent, for Case I.*

Proof: Let A be a weighing matrix for Case I. Without a loss of generality, we may assume that $\mathbf{r}_1 = (1, 1, 1, 1, 1, 1, 1, 0, 0, 0, 0, 0)$ and $\mathbf{r}_2 = (1, 1, 1, -, -, -, 0, 1, 0, 0, 0, 0)$ as the first and second rows of A, respectively.

Let $\mathbf{a} = (\alpha_1, \ldots, \alpha_6, , \gamma, \delta, \xi_1, \ldots, \xi_4)$ be a vector which is orthogonal to $\mathbf{r}_1$ and $\mathbf{r}_2$ and $|\mathbf{a}| = 7$, where $\alpha_i, \gamma, \delta, \xi_j \in \{0, 1, -1\}$. Let x_1, x_0 and x_- be the numbers of 1, 0 and -1 in the set $\{\alpha_1, \alpha_2, \alpha_3\}$ and y_1, y_0 and y_- be the numbers of 1, 0 and -1 in the set $\{\alpha_4, \alpha_5, \alpha_6\}$, respectively.

Then we obtain the equations

$$4x_1 = 6 - 2x_0 - \gamma - \delta \text{ and } 4y_1 = 6 - 2y_0 - \gamma + \delta$$

by the orthogonality of $\mathbf{a}$ and $\mathbf{r}_1$ and $\mathbf{a}$ and $\mathbf{r}_2$, respectively.

Note that $\gamma + \delta$ is even, so we obtain the set of solutions In Table 4.

Note that we can't find a vector $\mathbf{a}$ such that $|\mathbf{a}| = 7$ and which belongs to one of (18), (19) and (20) in Table 4.

Now, $p_2 \geq 2$, because $p_2 = 1 + p_6$ and $p_6 \geq 1$. So, we have to find at least two vectors, each of which has the intersection number 2 with $\mathbf{r}_1$. Therefore, they have to belong to one of (5), (9), (13) and (17) in the Table 4. We shall put vectors the third and fourth rows of A.

Then, up to equivalence, there are two cases, say Case (i) and Case (ii), where

$$(i): \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & - & - & - & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & - & - & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & - & - & 1 & - & - & - \end{bmatrix}$$

$$(ii): \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & - & - & - & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & - & - & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & - & 1 & 1 & 1 & - & - \end{bmatrix}$$

	x_1	x_0	x_-	y_1	y_0	y_-	γ	δ	$\|\mathbf{r}_1 \cdot \mathbf{a}\|$
(1)	1	1	1	1	1	1	0	0	4
(2)	1	0	2	1	1	1	1	1	6
(3)	1	0	2	0	3	0	1	1	4
(4)	0	2	1	1	1	1	1	1	4
(5)	0	2	1	0	3	0	1	1	2
(6)	2	0	1	1	1	1	-1	-1	6
(7)	2	0	1	0	3	0	-1	-1	4
(8)	1	2	0	1	1	1	-1	-1	4
(9)	1	2	0	0	3	0	-1	-1	2
(10)	1	1	1	1	0	2	1	-1	6
(11)	1	1	1	0	2	1	1	-1	4
(12)	0	3	0	1	0	2	1	-1	4
(13)	0	3	0	0	2	1	1	-1	2
(14)	1	1	1	2	0	1	-1	1	6
(15)	1	1	1	1	2	0	-1	1	4
(16)	0	3	0	2	0	1	-1	1	4
(17)	0	3	0	1	2	0	-1	1	2
(18)	1	1	1	0	3	0	0	0	2
(19)	0	3	0	1	1	1	0	0	2
(20)	0	3	0	0	3	0	0	0	0

Table 4.

But for both cases, we can't find another vector which has the intersection number 2 with $\mathbf{r}_1$.

This means that if there is a weighing matrix of order 12 and weight 7 which has a row of Case I, then the solution of the IPC is unique: $p_2 = 2$, $p_4 = 8$ and $p_6 = 1$.

For Case (i), without loss of generality, we may assume that the first three column vectors of A are $(1, 1, 1, 0, 1, 1, 1, 1, 0, 0, 0, 0)^T$, $(1, 1, 0, 1, -, -, 0, 0, 1, 1, 0, 0)^T$ and $(1, 1, 0, 0, 0, 0, -, -, -, -, 1, 0)^T$ in order. So, we have to find eight row vectors, six of which belong to one of (10), (11), (14) and (15), one of (8) and one of (16) in Table 4 respectively.

But we can't find such a vector from (8).

For Case (ii), there are two cases for the first three column vectors of A:

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & - & - & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & - & - & - & - & 0 & 1 \end{bmatrix}^T \text{ and}$$

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & - & - & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & - & 0 & 0 & - & - & - & - & 0 \end{bmatrix}^T.$$

But we can't construct a weighing matrix from the latter, because we can't find the fifth row vector which belongs to (3) in Table 4.

So, by rearranging rows of the first, we may assume that

$$A = \left[\begin{array}{c|c} A(1) & A(2) \\ \hline A(3) & A(4) \end{array}\right], \text{ where}$$

$$A(1) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A(2) = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ - & - & - & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & - & - & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & - & 1 & 1 & 1 & - & - \end{bmatrix} \text{ and}$$

$$A(3)^T = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & - & - & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & - & - & - & - \end{bmatrix}$$

So, we have a total of eight vectors: the 5th and 6th row vectors belong to (8); the ith row vectors to either (11) or (15); and the 11th and 12th row vectors to (1) in Table 4, where $7 \le i \le 10$.

Now we can construct two weighing matrices, up to equivalence, say A_1 and A_2, where

$$A_1 = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & & & & & \\
1 & 1 & 1 & - & - & - & & 1 & & & & \\
1 & & & & & & - & - & 1 & 1 & 1 & 1 \\
 & & & & & 1 & - & 1 & 1 & 1 & - & - \\
 & 1 & & 1 & - & & - & - & - & & - & \\
 & & 1 & - & 1 & & - & - & & - & & - \\
1 & - & & & - & 1 & & & - & & 1 & - \\
1 & - & & & 1 & - & & & - & 1 & - & \\
1 & & - & - & & 1 & & & & - & - & 1 \\
1 & & - & 1 & & - & & & 1 & - & & - \\
 & 1 & - & - & & & 1 & - & & 1 & & - \\
 & 1 & - & & 1 & & - & 1 & - & & 1 &
\end{bmatrix} \text{ and}$$

$$A_2 = \left[\begin{array}{cccccccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & & & & & \\
1 & 1 & 1 & - & - & - & & 1 & & & & \\
1 & & & & & & - & - & 1 & 1 & 1 & 1 \\
& & & & & 1 & - & 1 & 1 & 1 & - & - \\
& 1 & & 1 & - & & - & - & - & & - & \\
& & 1 & - & 1 & & - & - & & - & & - \\
1 & - & & - & & 1 & & & - & & - & 1 \\
1 & - & & 1 & & - & & & 1 & - & - & \\
1 & & - & & - & 1 & & & & - & 1 & - \\
1 & & - & & 1 & - & & & - & 1 & & - \\
& 1 & - & - & & & 1 & - & 1 & & - & \\
& 1 & - & & 1 & & - & 1 & & - & & 1
\end{array}\right].$$

Case II

Let A be a weighing matrix for Case II. The IPC solutions are $p_2 = 1$, $p_4 = 10$ and $p_6 = 0$ for Case II. So, without loss of generality, we may assume that $\mathbf{r}_1 = (1, 1, 1, 1, 1, 1, 1, 0, 0, 0, 0, 0)$, $\mathbf{r}_2 = (1, -, 0, 0, 0, 0, 0, 1, 1, 1, 1, 1)$, $\mathbf{c}_1^T = (1, 1, 1, 1, 1, 1, 1, 0, 0, 0, 0, 0)$ and $\mathbf{c}_2^T = (1, -, 0, 0, 0, 0, 0, 1, 1, 1, 1, 1)$, where $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{c}_1$ and $\mathbf{c}_2$ are the first and second row vectors and column vectors of A respectively. Moreover we may assume that $\mathbf{r}_3 = (1, 0, 1, -, -, 0, 0, 1, -, -, 0, 0)$ where $\mathbf{r}_3$ is the third row vector of A.

Let X be an $(i \times 12)$-matrix such that the entries are 0 and ± 1 and $XX^T = 7I_i$, where $3 \le i \le 12$. Then X is said normal of level i if the first three row vectors of X equal $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_3$ in order, and $X_1 = C_1$, where $X \equiv [X_1, X_2]$, $C \equiv [\mathbf{c}_1, \mathbf{c}_2] \equiv \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$ and X_1 is the $(i \times 2)$-submatrix of X.

Let A be a normal weighing matrix. Then, we may assume that $|\mathbf{c}_1 \cdot \mathbf{c}_i| = |\mathbf{c}_2 \cdot \mathbf{c}_i| = 4$, where $\mathbf{c}_i$ is the ith column vector of A ($3 \le i \le 12$), because if there is a column (say i_0th) such that $|\mathbf{c}_1 \cdot \mathbf{c}_{i_0}| = 6$ (so $|\mathbf{c}_2 \cdot \mathbf{c}_{i_0}| = 2$ or $|\mathbf{c}_1 \cdot \mathbf{c}_{i_0}| = 2$ (so $|\mathbf{c}_2 \cdot \mathbf{c}_{i_0}| = 6$), then A will be equivalent to the transpose of a matrix which was constructed from Case I.

Lemma 25 *There are four normal matrices of level 4, up to equivalence.*

Proof: Let $\mathbf{a}$ be a vector which is the fourth row of a normal matrix of level 4. By the above paragraph, we may assume that the third component of $\mathbf{a}$ equals -1. Then there are eight possibilities for it, say $\mathbf{a}_i$, where

$$\begin{aligned}
\mathbf{a}_1 &= (1, 0, -, 1, 0, -, 0, 0, -, 0, 1, -), \\
\mathbf{a}_2 &= (1, 0, -, -, 0, 1, 0, -, 0, 0, 1, -), \\
\mathbf{a}_3 &= (1, 0, -, -, 0, 1, 0, 0, 1, 0, -, -), \\
\mathbf{a}_4 &= (1, 0, -, 1, -, 0, 0, -, -, 0, 1, 0), \\
\mathbf{a}_5 &= (1, 0, -, 1, -, 0, 0, 0, 1, -, -, 0), \\
\mathbf{a}_6 &= (1, 0, -, 0, 0, 1, -, -, -, 0, 1, 0),
\end{aligned}$$

$$\begin{aligned} \mathbf{a}_7 &= (1,0,-,0,0,1,-,0,1,-,-,0) \text{ and} \\ \mathbf{a}_8 &= (1,0,-,-,0,1,0,-,1,-,0,0). \end{aligned}$$

Put $Y_i = [\mathbf{r}_1^T, \mathbf{r}_2^T, \mathbf{r}_3^T, \mathbf{a}_i^T]^T$, where $1 \le i \le 7$. Then we can show easily that $Y_2 \sim Y_6$, $Y_3 \sim Y_7$, $Y_4 \sim Y_5$, $Y_4 \sim Y_8$.

The next lemma is obvious.

Lemma 26 *A weighing matrix which is constructed from Y_4 is equivalent to the transpose to a matrix which is constructed from Case I.*

Lemma 27 *There are three weighing matrices for which the first four row vectors equal those of Y_1 up to equivalence.*

Proof: Let $\mathbf{r}$ be a row vector such that a matrix $[Y_1^T, \mathbf{r}^T]^T$ is a normal matrix of level 5. Then, without loss of generality, we may assume that the third component of $\mathbf{r}$ equals -1. So, we have three possibilities, say $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_3$, for $\mathbf{r}$, where

$$\begin{aligned} \mathbf{r}_1 &= (1,0,-,0,1,0,-,0,0,-,-,1), \\ \mathbf{r}_2 &= (1,0,-,0,-,0,1,-,0,0,-,1) \text{ and} \\ \mathbf{r}_3 &= (1,0,-,0,0,1,-,-,0,-,0,1). \end{aligned}$$

Put $\bar{Y}_i = [Y_1^T, \mathbf{r}_i^T]^T$, where $1 \le i \le 3$. Then we can uniquely construct a normal weighing matrix, say A_{i+2} for which the first five row vectors equal those of $\bar{Y}_i$ $(1 \le i \le 3)$, up to equivalence, namely:

$$A_3 = \left[\begin{array}{cccccccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & & & & & \\
1 & - & & & & & & 1 & 1 & 1 & 1 & 1 \\
1 & & 1 & - & - & & & 1 & - & - & & \\
1 & & - & 1 & & - & & & - & & 1 & - \\
1 & & - & & 1 & & - & & & - & - & 1 \\
1 & & & - & & - & 1 & - & & 1 & - & \\
1 & & & & - & 1 & - & - & 1 & & & - \\
 & 1 & 1 & & & - & - & - & & & 1 & 1 \\
 & 1 & - & - & & & 1 & & 1 & - & 1 & \\
 & 1 & - & & - & 1 & & & - & 1 & & 1 \\
 & 1 & & 1 & - & - & & 1 & 1 & & - & \\
 & 1 & & - & 1 & & - & 1 & & 1 & & -
\end{array}\right],$$

$$A_4 = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & & & & & \\
1 & - & & & & & & 1 & 1 & 1 & 1 & 1 \\
1 & & 1 & - & - & & & 1 & - & - & & \\
1 & & - & 1 & & - & & & - & & 1 & - \\
1 & & - & & - & & 1 & - & & & - & 1 \\
1 & & & - & & 1 & - & - & & 1 & & - \\
1 & & & & 1 & - & - & & 1 & - & - & \\
 & 1 & 1 & & - & - & & - & 1 & & 1 & \\
 & 1 & - & - & & & 1 & 1 & 1 & & & - \\
 & 1 & - & & & 1 & - & & & - & 1 & 1 \\
 & 1 & & 1 & - & & - & 1 & & 1 & - & \\
 & 1 & & - & 1 & - & & & - & 1 & & 1
\end{bmatrix},$$

$$A_5 = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & & & & & \\
1 & - & & & & & & 1 & 1 & 1 & 1 & 1 \\
1 & & 1 & - & - & & & 1 & - & - & & \\
1 & & - & 1 & & - & & & - & & 1 & - \\
1 & & - & & & 1 & - & - & & - & & 1 \\
1 & & & - & 1 & & - & & & 1 & - & - \\
1 & & & & - & - & 1 & - & 1 & & - & \\
 & 1 & 1 & & - & & - & - & & 1 & 1 & \\
 & 1 & - & - & & & 1 & & - & 1 & & 1 \\
 & 1 & - & & - & 1 & & 1 & 1 & & & - \\
 & 1 & & 1 & & - & - & 1 & & & - & 1 \\
 & 1 & & - & 1 & - & & & 1 & - & 1 &
\end{bmatrix}.$$

Lemma 28 *There are three weighing matrices for which the first four row vectors equal those of Y_2 up to equivalence.*

Proof: We can construct three weighing matrices, say A_{i+5}, for which the first four row vectors equal those of Y_2, similarly to Lemma 27, where $1 \le i \le 3$, and

$$A_6 = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & & & & & \\
1 & - & & & & & & 1 & 1 & 1 & 1 & 1 \\
1 & & 1 & - & - & & & 1 & - & - & & \\
1 & & - & - & & 1 & & - & & & 1 & - \\
1 & & - & & 1 & & - & & - & & - & 1 \\
1 & & & 1 & & - & - & & 1 & - & & - \\
1 & & & & - & - & 1 & - & & 1 & - & \\
 & 1 & 1 & - & & & - & - & 1 & & & 1 \\
 & 1 & - & & - & 1 & & 1 & 1 & & - & \\
 & 1 & - & & & - & 1 & & & - & 1 & 1 \\
 & 1 & & 1 & - & & - & & - & 1 & 1 & \\
 & 1 & & - & 1 & - & & 1 & & 1 & & -
\end{bmatrix},$$

$$A_7 = \begin{bmatrix}
1&1&1&1&1&1&1&&&&& \\
1&-&&&&&&1&1&1&1&1 \\
1&&1&-&-&&&1&-&-&& \\
1&&-&-&&1&&-&&&1&- \\
1&&-&&1&&-&&&-&-&1 \\
1&&&1&&-&-&&-&1&&- \\
1&&&&-&-&1&-&1&&-& \\
&1&1&-&&&-&-&&1&&1 \\
&1&-&&-&1&&1&&1&-& \\
&1&-&&&-&1&&-&&1&1 \\
&1&&1&-&&-&&1&-&1& \\
&1&&-&1&-&&1&1&&&-
\end{bmatrix} \text{ and}$$

$$A_8 = \begin{bmatrix}
1&1&1&1&1&1&1&&&&& \\
1&-&&&&&&1&1&1&1&1 \\
1&&1&-&-&&&1&-&-&& \\
1&&-&-&&1&&-&&&1&- \\
1&&-&&&-&1&&1&-&-& \\
1&&&1&-&&-&&&1&-&- \\
1&&&&1&-&-&-&-&&&1 \\
&1&1&&-&-&&-&1&&1& \\
&1&-&1&&&-&1&&-&1& \\
&1&-&&-&&1&&-&1&&1 \\
&1&&-&1&-&&1&&1&&- \\
&1&&-&&1&-&&1&&-&1
\end{bmatrix}.$$

Lemma 29 *It is uniquely possible to construct a weighing matrix from Y_3, up to equivalence.*

Proof: It is obvious. We form the matrix A_9, where

$$A_9 = \begin{bmatrix}
1&1&1&1&1&1&1&&&&& \\
1&-&&&&&&1&1&1&1&1 \\
1&&1&-&-&&&1&-&-&& \\
1&&-&-&&1&&&1&&-&- \\
1&&-&&&-&1&-&&-&1& \\
1&&&1&-&&-&-&&&-&1 \\
1&&&&1&-&-&&-&1&&- \\
&1&1&-&&&-&-&1&&1& \\
&1&-&&1&&-&1&&-&&1 \\
&1&-&&-&1&&&-&1&1& \\
&1&&1&-&-&&1&1&&&- \\
&1&&-&&-&1&&&1&-&1
\end{bmatrix}.$$

Theorem 4 *There are three weighing matrices of order 12 and weight 7, up to equivalence.*

Proof: It is sufficient to check whether A_i are equivalent or not, where $1 \leq i \leq 9$. We list an equivalence table:

Old	π-operations	ρ-operations	New
A_1^T	$6\ 1\ 7\ \underline{8}\ \underline{9}\ \underline{10}\ 5\ 2\ 4\ 3\ \underline{12}\ 11$	$1\ 7\ 9\ \underline{10}\ \underline{8}\ \underline{2}\ 4\ 3\ \underline{12}\ 11\ \underline{5}\ \underline{6}$	A_1
A_2	$2\ 1\ 3\ \underline{4}\ 6\ 5\ 10\ 9\ 8\ 7\ \underline{11}\ \underline{12}$	$1\ 3\ 2\ \underline{4}\ \underline{5}\ \underline{6}\ 8\ 7\ 12\ 11\ 10\ 9$	A_1
A_3	Identity	Identity	Sym
A_4	$\underline{10}\ 3\ 1\ \underline{9}\ \underline{5}\ 8\ \underline{4}\ \underline{7}\ 11\ 6\ 2\ 12$	$3\ 10\ 7\ \underline{2}\ \underline{6}\ \underline{12}\ \underline{11}\ 1\ \underline{4}\ \underline{5}\ \underline{9}\ 8$	A_5
A_6	$1\ 2\ 5\ 7\ 3\ 6\ 4\ 12\ 11\ 9\ 10\ 8$	$1\ 2\ 5\ 7\ 3\ 6\ 4\ 12\ 11\ 9\ 10\ 8$	A_3
A_6	$8\ 7\ \underline{9}\ \underline{3}\ 4\ \underline{2}\ \underline{12}\ \underline{10}\ 5\ \underline{1}\ 6\ 11$	$\underline{8}\ \underline{7}\ 3\ 9\ 2\ \underline{4}\ 12\ \underline{6}\ \underline{5}\ \underline{11}\ 10\ 1$	A_5
A_7	$7\ 8\ \underline{2}\ 4\ \underline{9}\ \underline{3}\ \underline{12}\ \underline{11}\ \underline{5}\ 10\ 1\ \underline{6}$	$\underline{8}\ 7\ \underline{11}\ 1\ 9\ \underline{6}\ \underline{5}\ 2\ 12\ 10\ \underline{4}\ 3$	A_5
A_8	$12\ 3\ \underline{1}\ \underline{9}\ \underline{6}\ 11\ 4\ \underline{10}\ \underline{7}\ 5\ 8\ 2$	$\underline{4}\ 9\ \underline{7}\ 2\ 6\ \underline{11}\ 12\ \underline{5}\ 3\ 1\ \underline{10}\ 8$	A_9
$(A_8)^T$	$3\ 12\ \underline{1}\ \underline{6}\ \underline{11}\ 4\ 8\ \underline{7}\ \underline{10}\ 9\ 5\ \underline{2}$	$\underline{4}\ \underline{10}\ \underline{5}\ 1\ 3\ 8\ \underline{9}\ \underline{6}\ 7\ 2\ 12\ \underline{11}$	A_9

Next, we shall check by the G-table whether A_1, A_3 and A_8 are equivalent or not.

We list the G-table in the following

Matrices	Mult.	0	1	2	3	4	5	6	7
A_1	12	77	64	22	0	2	0	0	0
A_3	12	45	100	20	0	0	0	0	0
A_8	12	45	100	20	0	0	0	0	0

But $A_3 \not\sim A_8$, because let s_i be the ith row vector of A_8. Then there are six pairs of row vectors such that the intersection number of pairs vectors equal 2, i.e. (s_1, s_2), (s_3, s_{12}), (s_4, s_{10}), (s_5, s_{11}), (s_6, s_8) and (s_7, s_9).

For each pair, we shall permutate A_8 to be normal. But for each case we can't obtain A_3 by doing so. This means $A_3 \not\sim A_8$. This completes our proof.

6 Classifications of weighing matrices of order 12 and weight 6

Let A be a weighing matrix of order 12 and weight 6. The IPC are $p_0 + p_2 + p_4 + p_6 = 11$ and $p_2 + 2p_4 + 2p_6 = 15$. Clearly $p_6 \leq 1$ and $p_0 \leq 2$. Thus we obtain six kinds of solutions:

I $p_0 = 2$, $p_2 = 3$, $p_4 = 6$, $p_6 = 0$;
II $p_0 = 1$, $p_2 = 5$, $p_4 = 5$, $p_6 = 0$;
III $p_0 = 2$, $p_2 = 4$, $p_4 = 4$, $p_6 = 1$;
IV $p_0 = 0$, $p_2 = 8$, $p_4 = 2$, $p_6 = 1$;
V $p_0 = 0$, $p_2 = 7$, $p_4 = 4$, $p_6 = 0$ and
VI $p_0 = 1$, $p_2 = 6$, $p_4 = 3$, $p_6 = 1$.

In [1], they called these six cases types D, E, G, H, F and D in order, and showed that a matrix which was constructed from Case I was equivalent to a matrix which was constructed from Case VI. And also, they constructed four weighing matrices, say A_i^* $(1 \leq i \leq 4)$, each of which belongs to one of the types E, F, G and H. We list these matrices for convenience sake.

$$A_1^* = \begin{bmatrix} 0&1&1&1&1&1&1&0&0&0&0&0 \\ 1&0&1&-&-&1&0&1&0&0&0&0 \\ 1&1&0&1&-&-&0&0&1&0&0&0 \\ 1&-&1&0&1&-&0&0&0&1&0&0 \\ 1&-&-&1&0&1&0&0&0&0&1&0 \\ 1&1&-&-&1&0&0&0&0&0&0&1 \\ -&0&0&0&0&0&0&1&1&1&1&1 \\ 0&-&0&0&0&0&1&0&1&-&-&1 \\ 0&0&-&0&0&0&1&1&0&1&-&- \\ 0&0&0&-&0&0&1&-&1&0&1&- \\ 0&0&0&0&-&0&1&-&-&1&0&1 \\ 0&0&0&0&0&-&1&1&-&-&1&0 \end{bmatrix}$$

(Type (E,E))

$$A_2^* = \begin{bmatrix} 0&1&1&0&1&-&0&1&0&0&1&0 \\ 1&0&1&1&-&0&1&0&0&1&0&0 \\ 1&1&0&-&0&1&0&0&1&0&0&1 \\ 0&-&1&0&1&1&0&0&1&0&0&- \\ -&1&0&1&0&1&0&1&0&0&-&0 \\ 1&0&-&1&1&0&1&0&0&-&0&0 \\ 0&-&0&0&0&-&0&1&1&0&-&1 \\ -&0&0&0&-&0&1&0&1&-&1&0 \\ 0&0&-&-&0&0&1&1&0&1&0&- \\ 0&-&0&0&0&1&0&1&-&0&1&1 \\ -&0&0&0&1&0&1&-&0&1&0&1 \\ 0&0&-&1&0&0&-&0&1&1&1&0 \end{bmatrix}$$

(Type (F,F))

$$A_3^* = \begin{bmatrix} 0 & 1 & - & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & - & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ - & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & - & - & 0 & 1 & - & 1 & 0 & 0 & - & 0 & 0 \\ - & 0 & - & 1 & - & 0 & 0 & 1 & 0 & 0 & - & 0 \\ - & - & 0 & - & 0 & 1 & 0 & 0 & 1 & 0 & 0 & - \\ - & 0 & 0 & - & 0 & 0 & 0 & 1 & - & 0 & 1 & 1 \\ 0 & - & 0 & 0 & - & 0 & 1 & - & 0 & 1 & 0 & 1 \\ 0 & 0 & - & 0 & 0 & - & - & 0 & 1 & 1 & 1 & 0 \\ - & 0 & 0 & 1 & 0 & 0 & 0 & - & - & 0 & 1 & - \\ 0 & - & 0 & 0 & 1 & 0 & - & 0 & - & 1 & - & 0 \\ 0 & 0 & - & 0 & 0 & 1 & - & - & 0 & - & 0 & 1 \end{bmatrix}$$

(Type (G,G))

$$A_4^* = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & - & - & - & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & - & 0 & 1 & - & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & - & 0 & - & 1 & 0 & 1 & - & 0 & 0 & 0 & 0 \\ 1 & 0 & - & 0 & 0 & 0 & - & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & - & 0 & 0 & 0 & - & 0 & - & - & - & 0 \\ 0 & 1 & - & 0 & 0 & 0 & 1 & 0 & 1 & - & 0 & 1 \\ 0 & 1 & - & 0 & 0 & 0 & 1 & 0 & - & 1 & 0 & - \\ 0 & 0 & 0 & 1 & 0 & - & 0 & - & 1 & 0 & - & - \\ 0 & 0 & 0 & 1 & 0 & - & 0 & - & - & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & - & 0 & 1 & 0 & 1 & - & 1 \\ 0 & 0 & 0 & 0 & 1 & - & 0 & 1 & 0 & - & 1 & - \end{bmatrix}$$

(Type (H,E))

(Type (H, E) means that A_4^* belongs to type H and A_4^T belongs to type E)

We shall consider in order from our Case I.

Case I

Lemma 30 *There is no weighing matrix with a row which has Case I for the IPC.*

Proof: Let A be a weighing matrix for Case I. Let $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_3$ be the first three rows of A. Then without loss of generality, we may assume that

$$\begin{aligned}\mathbf{r}_1 &= (1,1,1,1,1,1,0,0,0,0,0,0)\\ \mathbf{r}_2 &= (0,0,0,0,0,0,1,1,1,1,1,1)\\ \mathbf{r}_3 &= (0,0,0,0,0,0,1,1,1,-,-,-)\end{aligned}$$

Let $\mathcal{A},\mathcal{B},\mathcal{C}$ be sets of row vectors, each of which is orthogonal to $\mathbf{r}_1,\mathbf{r}_2$ and $\mathbf{r}_3$, where

$$\begin{aligned}\mathcal{A} &= \{\mathbf{a}=(\alpha_1,\ldots,\alpha_6,\alpha_7,\alpha_8,\alpha_9,0,0,0) \mid \\ &\quad \alpha_i \in \{0,1,-1\}, \alpha_7 \neq \alpha_8 \neq \alpha_9, |\mathbf{a}|=6\}\\ \mathcal{B} &= \{\mathbf{b}=(\beta_1,\ldots,\beta_6,0,0,0,\beta_{10},\beta_{11},\beta_{12}) \mid \\ &\quad \beta_j \in \{0,1,-1\}, \beta_{10} \neq \beta_{11} \neq \beta_{12}, |\mathbf{b}|=6\}\end{aligned}$$

and

$$\begin{aligned}\mathcal{C} &= \{\mathbf{c}=(\gamma_1,\ldots,\gamma_6,\gamma_7,\gamma_8,\gamma_9,\gamma_{10},\gamma_{11},\gamma_{12}) \mid \\ &\quad \gamma_k \in \{0,1,-1\}, \gamma_7 \neq \gamma_8 \neq \gamma_9,\ \gamma_{10} \neq \gamma_{11} \neq \gamma_{12}, |\mathbf{c}|=6\}\end{aligned}$$

By our assumptions, we have to find six vectors from $\mathcal{A}\cup\mathcal{B}$ and three vectors from $\mathcal{C}$, and further calculation shows in fact three vectors must come from each of $\mathcal{A}$ and $\mathcal{B}$.

Let $\bar{A}$ be the (12×6) matrix which is the last six columns of A. Then, from the above paragraph, we may assume that

$$\bar{A} = \left[\begin{array}{cc|c|c|c} 1 & 1 & & & \\ 1 & 1 & P^T & 0_3 & R^T \\ 1 & 1 & & & \\ \hline 1 & - & & & \\ 1 & - & 0_3 & Q^T & S^T \\ 1 & - & & & \end{array}\right]^T,$$

where each row of P, Q, R and S contains exactly one 0, one 1 and one -1. Now, we can easily show that in P and Q exactly one zero must lie in each row and column. As a consequence, without loss of generality, we may assume that

$$P = \begin{bmatrix} 1 & - & 0 \\ 1 & 0 & - \\ 0 & - & 1 \end{bmatrix} = Q = R.$$

But we can't find any S which satisfies our conditions.

Case II

Let A be a weighing matrix for our Case II. Then without loss of generality, we may assume that

$$A = \left[\begin{array}{c|c} E(1) & E(2) \\ \hline E(3) & E(4) \end{array}\right],$$

where each $E(i)$ is a (6×6) matrix, and the first rows of $E(1)$ and $E(4)$ and the first column of $E(1)$ have all 1 entries, and also the first row of $E(2)$ and column of $E(3)$ have all 0 entries.

Moreover, we can assume that there is at least a row vector with weight 2 in $E(1)$, because if all row vectors except the first row in $E(1)$ have weight 4, we may exchange $E(1)$ for $E(4)$. So, we may assume that the second row vector of $E(1)$ is $(1, -, 0, 0, 0, 0)$.

Lemma 31 *There are four weighing matrices which are constructed from Case II, up to equivalence.*

Proof: There are 16 possibilities for $E(1)$ which could be inequivalent, say E_i $(1 \le i \le 16)$. We list these matrices:

$$\underset{E_1}{\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & - & & & & \\ 1 & & - & & & \\ 1 & & & - & & \\ 1 & & & & - & \\ 1 & & & & & - \end{bmatrix}} \quad \underset{E_2}{\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & - & & & & \\ 1 & & - & & & \\ 1 & & & 1 & - & - \\ 1 & & & - & & \\ 1 & & & - & & \end{bmatrix}}$$

$$\underset{E_3}{\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & - & & & & \\ 1 & & - & & & \\ 1 & & & 1 & - & - \\ 1 & & & - & 1 & - \\ 1 & & & - & - & 1 \end{bmatrix}} \quad \underset{E_4}{\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & - & & & & \\ 1 & & 1 & - & - & \\ 1 & & - & - & & 1 \\ 1 & & - & & 1 & - \\ 1 & & & 1 & - & - \end{bmatrix}}$$

$$\underset{E_5}{\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & - & & & & \\ 1 & & 1 & - & - & \\ 1 & & - & 1 & & - \\ 1 & & - & & & \\ 1 & & & - & & \end{bmatrix}} \quad \underset{E_6}{\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & - & & & & \\ 1 & & 1 & - & - & \\ 1 & & - & 1 & & - \\ 1 & & - & - & & 1 \\ 1 & & & & & - \end{bmatrix}}$$

$$\underset{E_7}{\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & - & & & & \\ 1 & & 1 & - & - & \\ 1 & & - & 1 & - & \\ 1 & & - & & 1 & - \\ 1 & & & - & & \end{bmatrix}} \quad \underset{E_8}{\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & - & & & & \\ 1 & 1 & - & - & & \\ 1 & - & & & & \\ 1 & & 1 & & - & - \\ 1 & & - & & & \end{bmatrix}}$$

$$E_9 = \left[\begin{array}{cccccc} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & - & & & & \\ 1 & 1 & - & - & & \\ 1 & - & & & & \\ 1 & & 1 & - & - & \\ 1 & & - & 1 & & - \end{array}\right] \qquad E_{10} = \left[\begin{array}{cccccc} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & - & & & & \\ 1 & 1 & - & - & & \\ 1 & - & 1 & - & & \\ 1 & & - & & & \\ 1 & & & 1 & - & - \end{array}\right]$$

$$E_{11} = \left[\begin{array}{cccccc} 1 & 1 & 1 & 1 & 1 & \\ 1 & - & & & & \\ 1 & 1 & - & - & & \\ 1 & - & & & 1 & - \\ 1 & & & & - & \\ 1 & & & & - & \end{array}\right] \qquad E_{12} = \left[\begin{array}{cccccc} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & - & & & & \\ 1 & 1 & - & - & & \\ 1 & - & & & 1 & - \\ 1 & & - & & - & 1 \\ 1 & & 1 & & - & - \end{array}\right]$$

$$E_{13} = \left[\begin{array}{cccccc} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & - & & & & \\ 1 & 1 & - & - & & \\ 1 & - & & & 1 & - \\ 1 & & 1 & - & - & \\ 1 & & - & 1 & - & \end{array}\right] \qquad E_{14} = \left[\begin{array}{cccccc} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & - & & & & \\ 1 & 1 & - & - & & \\ 1 & 1 & & & - & - \\ 1 & - & & & & \\ 1 & - & & & & \end{array}\right]$$

$$E_{15} = \left[\begin{array}{cccccc} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & - & & & & \\ 1 & 1 & - & - & & \\ 1 & 1 & & & - & - \\ 1 & - & 1 & - & & \\ 1 & - & - & 1 & & \end{array}\right] \qquad E_{16} = \left[\begin{array}{cccccc} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & - & & & & \\ 1 & 1 & - & - & & \\ 1 & 1 & & & - & - \\ 1 & - & 1 & & - & \\ 1 & - & - & & 1 & \end{array}\right]$$

But $E_6 \sim E_7$ (we obtain E_7 by operating with $\pi($ 1 2 5 4 6 3) and $\rho($ 1 2 6 4 3 5) on E_6) and $E_8 \sim E_{11}$ (similarly $\pi($ 1 6 5 3 2 4) and $\rho($ 1 3 5 6 2 4) on E_8).

It is easily shown that we can't construct weighing matrices from E_3, E_{10}, E_{12} and E_{15}, because we can't decide the $E(2)$ parts of A for them.

We can uniquely and up to equivalence, construct a weighing matrix, say E_i^*, from E_i, where i=1, 2, 4, 5, 7, 9, 11, 13, 14 and 16. We list these matrices:

$$E_1^* = \left[\begin{array}{cccccccccccc} 1 & 1 & 1 & 1 & 1 & 1 & & & & & & \\ 1 & - & & & & & 1 & 1 & - & - & & \\ & 1 & & - & & & 1 & - & 1 & & - & \\ 1 & & & - & & & - & & 1 & - & 1 & \\ 1 & & & & - & & & - & - & 1 & 1 & \\ 1 & & & & & - & - & 1 & & 1 & - & \\ & & & & & & 1 & 1 & 1 & 1 & 1 & 1 \\ & - & - & 1 & & 1 & - & & & & & 1 \\ & - & 1 & & 1 & - & & - & & & & 1 \\ & 1 & - & - & 1 & & & & - & & & 1 \\ & 1 & & 1 & - & - & & & & - & & 1 \\ & & 1 & - & - & 1 & & & & & - & 1 \end{array}\right]$$

$$
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & & & & & & \\
1 & - & & & & & 1 & 1 & - & - & & \\
1 & & - & & & & 1 & - & 1 & & - & \\
1 & & & 1 & - & - & - & & & & & 1 \\
1 & & & - & & & & - & - & 1 & 1 & \\
1 & & & - & & & - & 1 & 1 & & & - \\
 & & & & & & 1 & 1 & 1 & 1 & 1 & 1 \\
 & 1 & 1 & & - & - & 1 & & & & & - \\
 & 1 & - & & & & & 1 & - & 1 & - & \\
 & 1 & - & & - & 1 & & & & - & 1 & \\
 & 1 & & - & 1 & - & & & & - & & 1 \\
 & & 1 & - & - & 1 & & & & & - & 1
\end{bmatrix}
$$

E_2^*

$$
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & & & & & & \\
1 & - & & & & & 1 & 1 & - & - & & \\
1 & & 1 & - & - & & - & & & & 1 & \\
1 & & - & - & & 1 & & & 1 & & - & \\
1 & & - & & 1 & - & & - & & & 1 & \\
 & & & 1 & - & - & & & & 1 & - & \\
 & & & & & & 1 & 1 & 1 & 1 & 1 & 1 \\
 & 1 & - & & & & - & 1 & - & & & 1 \\
 & 1 & & - & & & 1 & & - & 1 & & - \\
 & 1 & & & - & & 1 & - & & - & & 1 \\
 & 1 & & & & - & & 1 & 1 & - & & - \\
 & & 1 & - & 1 & - & & & & & - & 1
\end{bmatrix}
$$

E_4^*

$$
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & & & & & & \\
1 & - & & & & & 1 & 1 & - & - & & \\
1 & & 1 & - & - & & - & & & & 1 & \\
1 & & - & 1 & & - & & - & & & 1 & \\
1 & & - & & & & - & 1 & & 1 & - & \\
1 & & & - & & & 1 & - & 1 & & - & \\
 & & & & & & 1 & 1 & 1 & 1 & 1 & 1 \\
 & 1 & & - & 1 & - & & & - & & & 1 \\
 & 1 & - & & - & 1 & & & & - & & 1 \\
 & 1 & & & - & & 1 & & - & 1 & & - \\
 & 1 & & & & - & & 1 & 1 & - & & - \\
 & & - & - & 1 & 1 & & & & & 1 & -
\end{bmatrix}
$$

E_5^*

$$
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & & & & & & \\
1 & - & & & & & 1 & 1 & - & - & & \\
1 & & 1 & - & - & & - & & & & 1 & \\
1 & & - & 1 & - & & & & & 1 & & - \\
1 & & - & & 1 & - & - & & & & & 1 \\
1 & & & - & & & 1 & - & 1 & & - & \\
& & & & & & 1 & 1 & 1 & 1 & 1 & 1 \\
& 1 & - & & - & 1 & & & & - & & 1 \\
& 1 & & & & - & 1 & - & - & & 1 & \\
& 1 & & & & - & & 1 & 1 & - & & - \\
& 1 & & - & & & & 1 & - & 1 & - & \\
& & - & - & 1 & 1 & & & & & 1 & -
\end{bmatrix}
$$

E_7^*

$$
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & & & & & & \\
1 & - & & & & & 1 & 1 & - & - & & \\
1 & 1 & - & - & & & & & & & 1 & - \\
1 & - & & & & & - & - & & & 1 & 1 \\
1 & & 1 & - & - & & & & 1 & & - & \\
1 & & - & 1 & & - & & & & 1 & - & \\
& & & & & & 1 & 1 & 1 & 1 & 1 & 1 \\
& & - & - & 1 & 1 & & & & & - & 1 \\
& & - & 1 & - & 1 & & & 1 & - & & \\
& & & & - & 1 & 1 & - & - & 1 & & \\
& - & & & & 1 & - & 1 & & 1 & & - \\
& - & & & 1 & & 1 & - & 1 & & & -
\end{bmatrix}
$$

E_9^*

$$
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & & & & & & \\
1 & - & & & & & 1 & 1 & - & - & & \\
1 & 1 & - & - & & & & & & & 1 & - \\
1 & - & & & 1 & - & - & & 1 & & & \\
1 & & & & - & & 1 & - & 1 & & - & \\
1 & & & & - & & - & & - & 1 & & 1 \\
& & & & & & 1 & 1 & 1 & 1 & 1 & 1 \\
& & 1 & 1 & - & - & & & & & 1 & - \\
& - & 1 & - & & 1 & & & & 1 & & - \\
& 1 & 1 & - & & - & & 1 & & & - & \\
& & 1 & - & & & & - & & - & 1 & 1 \\
& & & & - & 1 & - & 1 & 1 & - & &
\end{bmatrix}
$$

E_{11}^*

$$
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & & & & & & \\
1 & - & & & & & 1 & 1 & - & - & & \\
1 & 1 & - & - & & & & & & & 1 & - \\
1 & - & & & 1 & - & - & & 1 & & & \\
1 & & 1 & - & - & & & - & & & & 1 \\
1 & & - & 1 & - & & & & & 1 & - & \\
 & & & & & & 1 & 1 & 1 & 1 & 1 & 1 \\
 & - & & & & 1 & - & & - & 1 & 1 & \\
 & - & & & & 1 & 1 & - & 1 & & & - \\
 & & - & - & 1 & 1 & & & & & - & 1 \\
 & & & & - & 1 & - & 1 & 1 & - & & \\
 & & 1 & - & & & & 1 & & 1 & - & -
\end{bmatrix}
$$

E^*_{13}

$$
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & & & & & & \\
1 & - & & & & & 1 & 1 & - & - & & \\
1 & 1 & - & - & & & 1 & - & & & & \\
1 & 1 & & & - & - & - & 1 & & & & \\
1 & - & & & & & & & 1 & 1 & - & - \\
1 & - & & & & & - & - & & & 1 & 1 \\
 & & & & & & 1 & 1 & 1 & 1 & 1 & 1 \\
 & & - & - & 1 & 1 & - & 1 & & & & \\
 & & - & 1 & - & 1 & & & 1 & - & & \\
 & & 1 & - & - & 1 & & & & & 1 & - \\
 & & & & - & 1 & & & - & 1 & - & 1 \\
 & & 1 & - & & & & & 1 & - & - & 1
\end{bmatrix}
$$

E^*_{14}

$$
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & & & & & & \\
1 & - & & & & & 1 & 1 & - & - & & \\
1 & 1 & - & - & & & 1 & - & & & & \\
1 & 1 & & & - & - & - & 1 & & & & \\
1 & - & 1 & & - & & & - & 1 & & & \\
1 & - & - & & 1 & & - & & & 1 & & \\
 & & & & & & 1 & 1 & 1 & 1 & 1 & 1 \\
 & & - & 1 & - & 1 & & & & & 1 & - \\
 & & & - & & 1 & - & & & - & 1 & 1 \\
 & & & - & & 1 & & 1 & 1 & & - & - \\
 & & & & - & 1 & & & - & 1 & - & 1 \\
 & & - & 1 & & & & & 1 & - & - & 1
\end{bmatrix}
$$

E^*_{16}

Next, we list an equivalence table for these matrices.

Old	π-operations	ρ-operations	New
E_4^*	$\underline{3}$ 7 2 10 9 $\underline{8}$ 11 5 $\underline{4}$ 6 12 $\underline{1}$	7 $\underline{11}$ $\underline{1}$ 5 4 $\underline{3}$ 8 9 $\underline{10}$ $\underline{12}$ 2 $\underline{6}$	E_1^*
E_5^*	1 2 3 4 5 6 9 8 $\underline{11}$ $\underline{10}$ $\underline{12}$ 7	Identity	Sym
E_7^*	11 $\underline{1}$ $\underline{4}$ 6 12 3 $\underline{5}$ $\underline{2}$ 8 10 9 7	$\underline{4}$ 2 10 $\underline{11}$ $\underline{9}$ 8 $\underline{1}$ $\underline{5}$ 3 6 $\underline{12}$ 7	E_2^*
E_9^*	11 8 1 10 $\underline{6}$ 9 $\underline{5}$ 12 3 $\underline{4}$ $\underline{7}$ 2	6 $\underline{12}$ $\underline{2}$ 10 $\underline{7}$ 8 5 $\underline{3}$ 4 11 $\underline{1}$ $\underline{9}$	E_2^*
E_{11}^*	6 $\underline{3}$ $\underline{8}$ 7 $\underline{9}$ 11 10 5 1 4 2 12	12 1 $\underline{5}$ 10 $\underline{9}$ $\underline{7}$ $\underline{11}$ 3 $\underline{4}$ 2 $\underline{6}$ 8	E_2^*
E_{13}^*	6 $\underline{2}$ $\underline{11}$ 12 8 7 9 5 4 1 3 $\underline{10}$	10 1 $\underline{5}$ $\underline{11}$ $\underline{3}$ 4 $\underline{8}$ 9 7 $\underline{2}$ 6 $\underline{12}$	E_2^*
E_{16}^*	5 $\underline{11}$ 12 $\underline{2}$ 10 7 9 $\underline{6}$ 8 $\underline{1}$ 4 $\underline{3}$	9 $\underline{5}$ 3 $\underline{8}$ 1 $\underline{2}$ $\underline{10}$ 11 12 6 $\underline{4}$ $\underline{7}$	E_2^*
E_1^*	1 2 4 6 3 5 11 10 8 9 12 7	1 2 4 6 3 5 12 10 9 7 8 11	$(E_1^*)^T$

Case III

Let A be a weighing matrix for our Case III. Then, similarly to Lemma 31, we can assume, without loss of generality, that

$$A = \left[\begin{array}{c|c} G(1) & G(2) \\ \hline G(3) & G(4) \end{array}\right],$$

where

$$G(1) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & - & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & - & - & - \\ 1 & & & & & \\ 1 & & & * & & \\ 1 & & & & & \end{bmatrix}, \quad G(2) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & - & 0 & 1 & - & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & & \\ & & & * & & \\ & & & & & \end{bmatrix},$$

$$G(3) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & & & & & \\ 0 & & & & & \\ 0 & & & * & & \\ 0 & & & & & \end{bmatrix} \text{ and } G(4) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & - & - & - \\ & & & & & \\ & & & * & & \\ & & & & & \\ & & & & & \end{bmatrix}.$$

Lemma 32 *There are three weighing matrices, up to equivalence, which are constructed from Case III.*

Proof: There are four possibilities for $G(1)$, say G_i $(1 \leq i \leq 4)$, up to equivalence, We list them:

$$\underset{G_1}{\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & - & & & & \\ 1 & 1 & 1 & - & - & - \\ 1 & & - & & & \\ 1 & & - & & & \\ 1 & - & & & & \end{bmatrix}} \quad \underset{G_2}{\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & - & & & & \\ 1 & 1 & 1 & - & - & - \\ 1 & & - & 1 & - & \\ 1 & & - & - & 1 & \\ 1 & - & & & & \end{bmatrix}}$$

$$\underset{G_3}{\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & - & & & & \\ 1 & 1 & 1 & - & - & - \\ 1 & & - & 1 & - & \\ 1 & & - & - & & 1 \\ 1 & - & & & 1 & - \end{bmatrix}} \quad \underset{G_4}{\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & - & & & & \\ 1 & 1 & 1 & - & - & - \\ 1 & & - & & & \\ 1 & & - & & 1 & - \\ 1 & - & & & - & 1 \end{bmatrix}}$$

But we can't find the $G(3)$-part of A for G_1. On the other hand, we can, uniquely and up to equivalence, construct weighing matrices from G_i, say G_i^*, where $i = 2, 3, 4$.

We list them:

$$\underset{G_2^*}{\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & & & & & & \\ 1 & - & & & & & 1 & - & & 1 & - & \\ 1 & 1 & 1 & - & - & - & & & & & & \\ 1 & & - & 1 & - & & & 1 & - & & & \\ 1 & & - & - & 1 & & & & & & 1 & - \\ 1 & - & & & & & - & & 1 & - & & 1 \\ & & & & & & 1 & 1 & 1 & 1 & 1 & 1 \\ & & & & & & 1 & 1 & 1 & - & - & - \\ & - & 1 & & - & 1 & & & & & 1 & - \\ & 1 & - & & - & 1 & & - & 1 & & & \\ & & & - & & 1 & 1 & & - & - & & 1 \\ & & & - & & 1 & - & 1 & & 1 & - & \end{bmatrix}}$$

$$\underset{G_3^*}{\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & & & & & & \\ 1 & - & & & & & 1 & - & & 1 & - & \\ 1 & 1 & 1 & - & - & - & & & & & & \\ 1 & & - & 1 & - & & & & & & 1 & - \\ 1 & & - & - & & 1 & & & & - & & 1 \\ 1 & - & & & 1 & - & - & 1 & & & & \\ & & & & & & 1 & 1 & 1 & 1 & 1 & 1 \\ & & & & & & 1 & 1 & 1 & - & - & - \\ & 1 & - & & 1 & - & 1 & & - & & & \\ & 1 & - & & & & - & & 1 & 1 & - & \\ & & & 1 & - & & & 1 & - & & - & 1 \\ & & & 1 & & - & & - & 1 & - & & 1 \end{bmatrix}}$$

$$
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & & & & & & \\
1 & - & & & & & 1 & - & & 1 & - & \\
1 & 1 & 1 & - & - & - & & & & & & \\
1 & & - & & & & & - & 1 & - & 1 & \\
1 & & - & & 1 & - & & 1 & - & & & \\
1 & - & & & - & 1 & - & 1 & & & & \\
& & & & & & 1 & 1 & 1 & 1 & 1 & 1 \\
& & & & & & 1 & 1 & 1 & - & - & - \\
& & & 1 & & - & - & & 1 & & - & 1 \\
& 1 & - & - & & 1 & & & & & - & 1 \\
& & & 1 & - & & 1 & & - & - & & 1 \\
& - & 1 & - & 1 & & & & & - & & 1
\end{bmatrix}
$$

G_4^*

Case IV

Let A be a weighing matrix for our Case IV. Then we can assume, without loss of generality, that

$$A = \left[\begin{array}{c|c} H(1) & H(2) \\ \hline H(3) & H(4) \end{array}\right],$$

where

$$H(1) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & - & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & - & - & - \\ 1 & & & & & \\ 1 & & & * & & \\ 1 & & & & & \end{bmatrix},$$

$$H(2) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & & \\ & & * & & & \\ & & & & & \end{bmatrix} \text{ and } H(3) = \begin{bmatrix} 0 & & & & & \\ 0 & & & & & \\ 0 & & & & & \\ 0 & & & * & & \\ 0 & & & & & \\ 0 & & & & & \end{bmatrix}.$$

Lemma 33 *There is a unique weighing matrix, up to equivalence, which is constructed from Case IV.*

Proof: There are three possibilities for $H(1)$, say H_1, H_2 and H_3, up to equivalence. We list them:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & - & & & & \\ 1 & 1 & 1 & - & - & - \\ 1 & - & & & & \\ 1 & & - & & & \\ 1 & & - & & & \end{bmatrix}$$

H_1

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & - & & & & \\ 1 & 1 & 1 & - & - & - \\ 1 & - & & & & \\ 1 & & - & 1 & - & \\ 1 & & - & - & 1 & \end{bmatrix}$$

$$H_2$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & - & & & & \\ 1 & 1 & 1 & - & - & - \\ 1 & & - & & & \\ 1 & - & & 1 & - & \\ 1 & & - & - & 1 & \end{bmatrix}$$

$$H_3$$

For H_3, we can't find the $H(3)$-part. For H_2, the $H(3)$-part is unique, say $\bar{H}_2$, up to equivalence, where

$$\bar{H}_2 = \begin{bmatrix} 0 & 1 & - & 0 & 0 & 0 \\ 0 & 1 & - & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & - \\ 0 & 0 & 0 & 1 & 0 & - \\ 0 & 0 & 0 & 0 & 1 & - \\ 0 & 0 & 0 & 0 & 1 & - \end{bmatrix}$$

. But this case is reduced to the case of H_1 by operating with $\pi(1\ \underline{2}\ 3\ \underline{4}\ 7\ 8\ 5\ 6\ 9\ 10\ 11\ 12)$ and $\rho(2\ 1\ 3\ 4\ 5\ 6)$ on the matrix $[H_2^T, \bar{H}_2^T]^T$.

We can uniquely construct the weighing matrix, say H_1^*, from H_1 up to equivalence, where

$$H_1^* = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & & & & & & \\ 1 & - & & & & & 1 & 1 & 1 & 1 & & \\ 1 & 1 & 1 & - & - & - & & & & & & \\ 1 & - & & & & & 1 & - & - & - & & \\ 1 & & - & & & & - & 1 & - & & 1 & \\ 1 & & - & & & & - & - & 1 & & - & \\ & 1 & - & 1 & - & & 1 & & & & & 1 \\ & 1 & - & - & 1 & & 1 & & & & & - \\ & & & 1 & & - & & 1 & & - & - & - \\ & & & 1 & & - & & - & & 1 & 1 & - \\ & & & & 1 & - & & & 1 & - & 1 & 1 \\ & & & & 1 & - & & & - & 1 & - & 1 \end{bmatrix}.$$

Case V

Let A be a weighing matrix for our Case V. Then, without loss of generality, we can assume that

$$A = \left[\begin{array}{c|c} F(1) & F(2) \\ \hline F(3) & F(4) \end{array}\right],$$

where

$$F(1) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & - & 0 & 0 & 0 & 0 \\ 1 & & & & & \\ 1 & & & & & \\ 1 & & & * & & \\ 1 & & & & & \end{bmatrix},$$

$$F(2) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ & & & & & \\ & & & & & \\ & & * & & & \\ & & & & & \end{bmatrix} \text{ and } F(3) = \begin{bmatrix} 0 & & & & & \\ 0 & & & & & \\ 0 & & & & & \\ 0 & & & * & & \\ 0 & & & & & \\ 0 & & & & & \end{bmatrix}.$$

Lemma 34 *All weighing matrices which are constructed from Case V are equivalent to matrices which we have constructed previously.*

Proof: There are twelve possibilities for $F(1)$ which could be inequivalent, say F_i $(1 \le i \le 12)$. We list these matrices:

$$\underset{F_1}{\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & - & & & & \\ 1 & & - & & & \\ 1 & & & 1 & - & - \\ 1 & & & - & & \\ 1 & & & - & & \end{bmatrix}} \quad \underset{F_2}{\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & - & & & & \\ 1 & & - & & & \\ 1 & & & 1 & - & - \\ 1 & & & - & 1 & - \\ 1 & & & - & - & 1 \end{bmatrix}}$$

$$\underset{F_3}{\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & - & & & & \\ 1 & & 1 & - & - & \\ 1 & & - & 1 & & - \\ 1 & & - & & - & 1 \\ 1 & & & - & 1 & - \end{bmatrix}} \quad \underset{F_4}{\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & - & & & & \\ 1 & & 1 & - & - & \\ 1 & & - & 1 & & - \\ 1 & & - & & & \\ 1 & & & - & & \end{bmatrix}}$$

$$\underset{F_5}{\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & - & & & & \\ 1 & & 1 & - & - & \\ 1 & & - & 1 & - & \\ 1 & & - & & & \\ 1 & & & - & 1 & - \end{bmatrix}} \quad \underset{F_6}{\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & - & & & & \\ 1 & 1 & - & - & & \\ 1 & - & 1 & - & & \\ 1 & & - & & & \\ 1 & & & 1 & - & - \end{bmatrix}}$$

$$
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
1 & - & & & & \\
1 & 1 & - & - & & \\
1 & - & & & 1 & - \\
1 & & & & - & \\
1 & & & & - &
\end{bmatrix}
\qquad
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
1 & - & & & & \\
1 & 1 & - & - & & \\
1 & - & & & 1 & - \\
1 & & - & & - & 1 \\
1 & & 1 & & - & -
\end{bmatrix}
$$

$$
\quad F_7 \qquad\qquad\qquad\qquad F_8
$$

$$
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
1 & - & & & & \\
1 & 1 & - & - & & \\
1 & - & & & 1 & - \\
1 & & 1 & - & - & \\
1 & & - & 1 & - &
\end{bmatrix}
\qquad
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
1 & - & & & & \\
1 & 1 & - & - & & \\
1 & 1 & & & - & - \\
1 & - & & & & \\
1 & - & & & &
\end{bmatrix}
$$

$$
\quad F_9 \qquad\qquad\qquad\qquad F_{10}
$$

$$
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
1 & - & & & & \\
1 & 1 & - & - & & \\
1 & 1 & & & - & - \\
1 & - & 1 & - & & \\
1 & - & - & 1 & &
\end{bmatrix}
\qquad
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
1 & - & & & & \\
1 & 1 & - & - & & \\
1 & 1 & & & - & - \\
1 & - & 1 & - & & \\
1 & - & - & & 1 &
\end{bmatrix}
$$

$$
\quad F_{11} \qquad\qquad\qquad\qquad F_{12}
$$

It is easily shown that we can't construct weighing matrices from F_4, F_6, F_8 and F_9, because we can't find the $F(2)$-parts for them. And also, we can't construct weighing matrices from F_5 and F_9, because we can't find the $F(4)$-parts for them.

If there exist weighing matrices which are constructed from F_{10}, F_{11} and F_{12}, then they are equivalent to the transpose of matrices which are constructed from Cases III and IV. For F_2 and F_3, we can uniquely find the $F(2)$-parts, say $\bar{F}_2$ and $\bar{F}_3$, up to equivalence. Namely,

$$
\bar{F}_2 = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & & \\
1 & - & - & & 1 & \\
- & & & & & 1 \\
- & & & & & - \\
 & & & - & - &
\end{bmatrix}
\text{ and } \bar{F}_3 = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 \\
- & & & & 1 & 0 \\
 & - & & & 1 & 0 \\
 & & - & & - & 0 \\
 & & & - & - & 0
\end{bmatrix}.
$$

But matrices which are constructed from the matrix $[F_3, \bar{F}_3]$ are equivalent to the transpose of matrices which are constructed from Case II.

Also, matrices which are constructed from the matrix $[F_2, \bar{F}_2]$ are equivalent to matrices which are constructed from Cases III or IV.

For F_1, up to equivalence, there are three possibilities for the $F(2)$-part, say X_1, X_2 and X_3, where

$$X_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & - & - & 0 & 1 & 0 \\ 0 & 0 & 0 & - & - & 0 \\ - & 1 & - & 0 & 0 & 1 \\ - & - & 1 & 0 & 0 & - \end{bmatrix}, \quad X_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & - & - & 0 & 1 & 0 \\ 0 & 0 & 0 & - & - & 0 \\ - & 0 & 0 & 1 & - & 0 \\ - & 1 & 0 & - & 1 & 0 \end{bmatrix}$$

$$\text{and } X_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & - & - & 0 & 1 & 0 \\ - & 0 & 0 & 0 & 0 & 1 \\ - & 1 & - & 0 & 0 & - \\ 0 & - & 1 & - & - & 0 \end{bmatrix}.$$

But matrices which are constructed from the matrices $[F_1, X_1]$ or $[F_1, X_2]$ must be equivalent to matrices which are constructed from cases III or IV, because the fifth and sixth row vectors of these matrices have the intersection number 6. Also, it is easily shown that we can't find the $F(4)$-part for the matrix $\begin{bmatrix} F_1 & X_3 \\ X_3^T & * \end{bmatrix}$.

For F_7, up to equivalence, there are two possibilities for the $F(2)$-part, say Y_1 and Y_2, where

$$Y_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & - & 0 & 0 & 0 & 0 \\ 0 & 0 & - & - & 0 & 0 \\ - & 0 & 1 & - & 1 & 0 \\ - & 0 & - & 1 & - & 0 \end{bmatrix} \text{ and } Y_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ - & - & 0 & 0 & 0 & 0 \\ 1 & - & - & 0 & - & 0 \\ - & 1 & 0 & - & 0 & - \end{bmatrix}.$$

But similarly to the above paragraph, we can't find a new weighing matrix from the matrices $[F_7, Y_1]$ and $[F_7, Y_2]$.

Theorem 5 *There are seven weighing matrices of order 12 and weight 6, up to equivalence.*

Proof: It is sufficient to check whether $E_1^*, E_2^*, E_5^*, E_{14}^*, G_2^*, G_3^*, G_4^*$ and H_1^* are equivalent or not. We list an equivalence table:

Old	π-operations	ρ-operations	New
$(E_2^*)^T$	$5\ 1\ 6\ 3\ 2\ 4\ 8\ 9\ \underline{12}\ 7\ \underline{11}\ 10$	$1\ \underline{4}\ \underline{8}\ \underline{10}\ \underline{12}\ 11\ 6\ \underline{5}\ 7\ 2\ \underline{3}\ 9$	G_3^*
$(E_{14}^*)^T$	$1\ 3\ 2\ 4\ 5\ 6\ 7\ \underline{8}\ \underline{9}\ \underline{10}\ 11\ 12$	$1\ 3\ 4\ 2\ 6\ 5\ \underline{8}\ \underline{9}\ 10\ 12\ \underline{11}\ 7$	H_1^*
$(G_2^*)^T$	$7\ 1\ 10\ \underline{8}\ \underline{11}\ \underline{2}\ 3\ 5\ 12\ \underline{9}\ 6\ \underline{4}$	$2\ \underline{6}\ 7\ \underline{12}\ 8\ 11\ 1\ \underline{4}\ \underline{10}\ 3\ \underline{5}\ 9$	G_2^*
$(G_4^*)^T$	$1\ 4\ 2\ 3\ 6\ 5\ \underline{12}\ \underline{8}\ \underline{10}\ 11\ 9\ 7$	$1\ 3\ 6\ 2\ 5\ 4\ \underline{12}\ \underline{10}\ \underline{9}\ \underline{11}\ \underline{7}\ 8$	E_{16}^*

Next, we shall check by the G-table whether $E_1^*, E_2^*, E_5^*, E_{14}^*$ and G_2^* are equivalent or not. We list the G-table in the following:

Matrices	Mult.	0	1	2	3	4	5	6
E_1^*	12	115	40	10	0	0	0	0
E_2^*	4	125	30	8	2	0	0	0
	8	124	33	5	3	0	0	0
$(E_2^*)^T$	4	117	38	8	2	0	0	0
	8	112	47	5	1	0	0	0
E_5^*	12	115	40	10	0	0	0	0
E_{14}^*	12	148	0	16	0	1	0	0
$(E_{14}^*)^T$	12	112	48	4	0	1	0	0
G_2^*	4	125	32	6	0	2	0	0
	8	122	36	6	0	1	0	0

In order to check whether E_1^* and E_5^* are equivalent or not, we count the distributions (when fixing each row) of the intersection numbers of three row vectors for E_1^* and E_5^*. We list them.

	Mult.	0	1	2	3	4	5	6
E_1^*	12	15	30	0	10	0	0	0
E_5^*	12	19	18	12	6	0	0	0

So, $E_1^*, E_2^*, (E_2^*)^T, E_5^*, E_{14}^*, (E_{14}^*)^T$ and G_2^* are inequivalent to each other. This completes our proof.

Remark: We check relations between $\{E_1^*, E_2^*, (E_2^*)^T, E_5^*, E_{14}^*, (E_{14}^*)^T, G_2^*\}$ and $\{A_1^*, A_2^*, A_3^*, A_4^*\}$. We shall show them by an equivalence table.

Old	π-operations	ρ-operations	New
A_1^*	2 7 $\underline{11}$ 12 9 $\underline{10}$ 8 $\underline{3}$ $\underline{6}$ 4 5 $\underline{1}$	8 1 $\underline{5}$ 6 3 $\underline{4}$ 9 12 $\underline{10}$ $\underline{11}$ 7 $\underline{2}$	E_1^*
A_2^*	2 $\underline{5}$ 6 $\underline{11}$ $\underline{8}$ 3 7 10 $\underline{12}$ $\underline{9}$ 4 1	1 4 7 $\underline{5}$ 10 3 $\underline{2}$ 8 12 $\underline{6}$ $\underline{11}$ 9	G_2^*
A_3^*	1 6 $\underline{4}$ 12 $\underline{9}$ 2 7 $\underline{10}$ 11 $\underline{8}$ $\underline{5}$ 3	6 2 10 $\underline{3}$ 7 5 $\underline{4}$ 12 8 $\underline{1}$ $\underline{9}$ 11	G_2^*
A_4^*	1 $\underline{5}$ 2 $\underline{6}$ $\underline{7}$ $\underline{8}$ 3 4 10 9 12 11	3 1 2 4 5 6 7 $\underline{9}$ $\underline{10}$ $\underline{11}$ $\underline{12}$ 8	H_1^*
	$H_1^* \sim (E_{14}^*)^T$		

7 Summary

We summarize the equivalence results for weighing matrices of weight k and order 12.

k	number of inequivalent matrices	Reference
1	unique	obvious
2	unique	Chan, Rodger, Seberry
3	unique	" " "
4	five	" " "
5	two	" " "
6	seven	Theorem 5
7	three	Theorem 4
8	six	Theorem 3
9	four	Theorem 2
10	four	Theorem 1
11	unique	Chan, Rodger, Seberry
12	unique	Husain

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REFERENCES

[1] H.C.Chan, C.A.Rodger and J.Seberry, On inequivalent weighing matrices, *Ars Combinatoria*, (1986), 21-A, 299-333.
[2] M.Hall Jr, Hadamard matrices of order 16, *J.P.L Research Summary* No 36-10, 1 (1961), 21-26.
[3] Q.M.Husain, On the totality of the solutions for the symmetric incomplete block designs: $\lambda = 2, k = 5$ or 6, *Sankyā* 7 (1945), 204-208.