ON GRAPHS WITH ADJACENT VERTICES OF LARGE DEGREE

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Abstract. Let $\mathcal{G}(n,m)$ denote the class of simple graphs on n vertices and m edges and let $G \in \mathcal{G}(n,m)$. For suitably restricted values of m, G will necessarily contain certain prescribed subgraphs such as cycles of given lengths and complete graphs. For example, if $m > \frac{1}{4}n^2$ then G contains cycles of all lengths up to $\lfloor \frac{1}{2}(n+3) \rfloor$. Recently we have established a number of results concerning the existence of certain subgraphs (cliques and cycles) in the subgraph of G induced by the vertices of G having some prescribed minimum degree. In this paper, we present some further results of this type. In particular, we prove that every $G \in \mathcal{G}(n,m)$ contains a pair of adjacent vertices each having degree (in G) at least $f(n,m)$ and determine the best possible value of $f(n,m)$. For $m > \frac{1}{4}n^2$ we find that G contains a triangle with a pair of vertices satisfying this same degree restriction. Some open problems are discussed.

1. Introduction.

All graphs considered in this paper are finite, loopless and have no multiple edges. For the most part, our notation and terminology follows that of Bondy and Murty [2]. Thus, a graph G has vertex set $V(G)$, edge set $E(G)$, $\nu(G)$ vertices, $\varepsilon(G)$ edges, maximum degree $\Delta(G)$ and minimum degree $\delta(G)$. K_n denotes the complete graph on n vertices and C_ℓ a cycle of length ℓ. $G + H$ denotes the disjoint union of the graphs G and H. The **join** $G \vee H$ of disjoint graphs G and H is the graph obtained from $G + H$ by joining each vertex of G to each vertex of H.

Let $\mathcal{G}(n,m)$ denote the class of graphs on n vertices and m edges, and let $G \in \mathcal{G}(n,m)$. For suitably restricted values of m, G will necessarily contain certain prescribed subgraphs such as cycles of given lengths, complete graphs, etc. Indeed, given any graph H on n or fewer vertices, then for sufficiently large m all graphs in $\mathcal{G}(n,m)$ will contain a subgraph isomorphic to H. The problem of determining the maximum m such that $\mathcal{G}(n,m)$ contains at least one graph G which has no subgraph isomorphic to H is a fundamental problem in extremal

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graph theory. This maximum m is known for certain H. For example, Turan's theorem gives the maximum m when $H = K_{k+1}$. We refer the reader to the book of Bollobás [1] for an excellent presentation of results of this type.

Recently, we ([3 - 6]) have established a number of results concerning the existence of certain subgraphs (mostly cliques and cycles) in the subgraph of $G \in \mathcal{G}(n, m)$ induced by the vertices of G having some prescribed minimum degree d. We obtained the best possible results when the subgraph H in question was a K_{k+1}, $k \geq 2$, a C_ℓ or a path of specified length.

In this paper, we present some further results of this type. In particular, we prove that every $G \in \mathcal{G}(n, m)$ contains a pair of adjacent vertices each having degree (in G) at least $f(\alpha)n$, where $\alpha = \frac{m}{n^2}$ and

$$f(\alpha) = \begin{cases} \frac{1}{2}\left(1 - \sqrt{1 - 4\alpha}\right), & \text{if } \alpha \leq \frac{2}{9} \\ 2\sqrt{2\alpha} - 1, & \text{otherwise.} \end{cases}$$

Moreover, we establish that this result is best possible. For $\alpha > \frac{1}{4}$, we prove that a pair of adjacent vertices each having degree at least $f(\alpha)n$ is contained in a triangle of G. We conclude the paper with a discussion on some open problems.

2. Main results.

Our first result establishes the existence of a K_2 in the subgraph of $G \in \mathcal{G}(n, m)$ induced by the vertices of degree at least d for sufficiently small d.

Theorem 1. *Let $G \in \mathcal{G}(n, m)$ and let $\alpha = \frac{m}{n^2}$. Then G contains a pair of adjacent vertices each having degree at least $f(\alpha)n$, where*

$$f(\alpha) = \begin{cases} \frac{1}{2}\left(1 - \sqrt{1 - 4\alpha}\right), & \textit{if } \alpha \leq \frac{2}{9} \\ 2\sqrt{2\alpha} - 1, & \textit{otherwise.} \end{cases}$$

Moreover, this result is best possible.

Proof: For a graph $G \in \mathcal{G}(n, \alpha n^2)$, let $d(G)$ be the smallest value such that the subgraph G_1 of G induced by the vertices of degree at least $d(G)$ has no edges. Let

$$\min_{G \in \mathcal{G}(n, \alpha n^2)} \{d(G)\} = d(G^*) = d,$$

and $|V(G_1^*)| = n_1$. We will show that $d > f(\alpha) \cdot n$.

By simple counting we have

$$\alpha n^2 \leq g(n_1) = \begin{cases} (n - n_1)(d - 1), & \text{if } n_1 \geq d - 1 \\ \frac{1}{2}(n - n_1)(n_1 + d - 1), & \text{otherwise.} \end{cases} \tag{1}$$

For $n_1 \geq d - 1$, $g(n_1)$ is clearly monotonically decreasing in n_1 and hence

$$\max_{n_1 \geq d-1} \{g(n_1)\} = g(d - 1). \tag{2}$$

For $n_1 \leq d-1$,

$$g(n_1+1) - g(n_1) = \frac{1}{2}(n - 2n_1 - d)$$

which is non-negative only when $n_1 \leq \frac{1}{2}(n-d)$. Hence,

$$\max_{n_1 \leq d-1} \{g(n_1)\} = \begin{cases} g(d-1), & \text{if } d-1 \leq \frac{1}{3}(n-1) \\ g\left(\frac{1}{2}(n-d+1)\right) & \text{otherwise.} \end{cases} \qquad (3)$$

From (1), (2) and (3) we conclude that

$$\alpha n^2 \leq \begin{cases} (n-d+1)(d-1), & \text{if } d-1 \leq \frac{1}{3}(n-1) \\ \frac{1}{8}(n+d-1)^2, & \text{otherwise.} \end{cases}$$

Hence,

$$\begin{aligned} d-1 &\geq \begin{cases} a(\alpha) = \frac{n}{2}\left(1-\sqrt{1-4\alpha}\right), & \text{if } d-1 \leq \frac{1}{3}(n-1) \\ b(\alpha) = n\left(2\sqrt{2\alpha}-1\right), & \text{otherwise.} \end{cases} \\ &\geq \min\{a(\alpha),\ b(\alpha)\}. \end{aligned}$$

We note that $a(\alpha) = b(\alpha)$ when $\alpha = \frac{2}{9}$ and $a(\alpha) - b(\alpha)$ is an increasing function in α. Hence

$$d-1 \geq \begin{cases} a(\alpha), & \text{if } \alpha \leq \frac{2}{9} \\ b(\alpha), & \text{otherwise.} \end{cases}$$

Thus

$$d-1 \geq f(\alpha)\cdot n$$

as required.

That the result is best possible follows from the following constructions. For $\alpha \leq \frac{2}{9}$, the graph $K_{u,n-u}$ with $u = \lceil f(\alpha)n \rceil$ has at least αn^2 edges and each edge has one end with degree $\lceil f(\alpha)n \rceil$. A graph $G \in \mathcal{G}(n, \alpha n^2)$ having the required degree property can be obtained from $K_{u,n-u}$ by deleting, if necessary, a few edges. For $\alpha \geq \frac{2}{9}$, let

$$u = \lfloor n\sqrt{2\alpha} \rfloor.$$

Let $R_{u,t}$ denote a graph on u vertices and $\lfloor \frac{1}{2}ut \rfloor$ edges having maximum degree t. Consider the graph

$$H = \overline{K}_{n-u} \vee R_{u,t}$$

with $t = \lceil nf(\alpha) \rceil - (n-u)$, where $\overline{K}_{n-u}$ denotes the complement of K_{n-u}. The graph H has at least αn^2 edges and each edge has at least one end of degree at most $\lceil f(\alpha)n \rceil$. A graph $G \in \mathcal{G}(n, \alpha n^2)$ having the required degree property can be obtained from H by deleting, if necessary, a few edges. This completes the proof of the theorem. ∎

We note that for $\alpha \le \frac{2}{9}$

$$f(\alpha) = \frac{1}{2}\left(1 - \sqrt{1 - \frac{4m}{n^2}}\right)$$
$$> \frac{1}{2}\left(1 - \sqrt{\left(1 - \frac{2m}{n^2}\right)^2}\right) = \frac{m}{n^2},$$

and hence

$$f(\alpha)\cdot n > \frac{m}{n}.$$

Thus, every $G \in \mathcal{G}(n, m)$, $m \le \frac{2}{9}n^2$, contains a pair of adjacent vertices each having degree greater than half the average degree. When m is $0(n)$ we can restate Theorem 1 as:

Corollary. *Let $G \in \mathcal{G}(n, m)$ with $m = 0(n)$. Then G contains a pair of adjacent vertices each having degree (in G) at least $\lfloor \frac{m}{n} \rfloor + 1$ and this bound is attained for sufficiently large n.*

A special case of a result proved in [3] asserts that every $G \in \mathcal{G}\left(n, \lfloor \frac{1}{4}n^2 \rfloor + 1\right)$ contains a triangle each vertex of which has degree greater than $\frac{n}{3}$ and that this result is best possible. It is natural to ask whether anything can be said about the degrees of two vertices in a triangle of G. We now show that for $\alpha > \frac{1}{4}$ there is always a pair of adjacent vertices in $G \in \mathcal{G}(n, \alpha n^2)$ having degree at least $(2\sqrt{2\alpha} - 1)n$ and contained in a triangle of G. Moreover, this result is best possible.

Theorem 2. *Let $G \in \mathcal{G}(n, \alpha n^2)$, $\alpha > \frac{1}{4}$. Then G contains a triangle two vertices of which have degree at least $(2\sqrt{2\alpha} - 1)n$ and this result is best possible.*

Proof: Let u be a vertex of G having maximum degree Δ. We denote by $N(u)$ and $\overline{N}(u)$ the set of neighbours and non-neighbours, respectively, of u in G. If every vertex of $N(u)$ has degree at most $n - \Delta$, then

$$\varepsilon(G) \le \Delta(n - \Delta)$$
$$\le \frac{1}{4}n^2.$$

Hence, at least one vertex of $N(u)$ has degree greater than $n - \Delta$ and, thus, G has a triangle with two of its vertices having degree greater than $n - \Delta$. Thus, we may suppose that

$$\Delta > 2n\left(1 - \sqrt{2\alpha}\right) + 1.$$

Now since

$$|\overline{N}(u)| = n - \Delta - 1 \leq (2\sqrt{2\alpha} - 1)n - 2,$$

G contains a triangle with the required degree property if a vertex of $N(u)$ ha degree at least $(2\sqrt{2\alpha} - 1)n$. So suppose all the vertices of $N(u)$ have degre less than $(2\sqrt{2\alpha} - 1)n$. Then

$$\sum_{v \in V(G)} d(v) < f(\Delta) = (n - \Delta)\Delta + \Delta\left(2\sqrt{2\alpha} - 1\right)n.$$

For fixed n, $f(\Delta)$ attains its maximum value at $\Delta = n\sqrt{2\alpha}$. But $f\left(n\sqrt{2\alpha}\right) = 2\alpha n^2$ and hence

$$\varepsilon(G) < \frac{1}{2} f(\Delta) \leq \alpha n^2.$$

This contradiction establishes the existence of a triangle in G, two vertices (which have degree at least $(2\sqrt{2\alpha} - 1)n$.

That the result is best possible follows from the following construction. Let

$$d = \left\lceil (2\sqrt{2\alpha} - 1)n \right\rceil \text{ and } t = \left\lfloor n\sqrt{2\alpha} \right\rfloor.$$

Let $R_{t,d+t-n}$ be a graph on t vertices, $\lfloor \frac{1}{2} t(d + t - n) \rfloor$ edges having maximu degree $d + t - n$. The graph

$$H = R_{t,d+t-n} \vee \overline{K}_{n-t}$$

has at least αn^2 edges and every triangle contains at least two vertices in $R_{t,d+t-}$ A graph $G \in \mathcal{G}(n, \alpha n^2)$ having the degree property can be obtained from H l deleting, if necessary, a few edges. This completes the proof of the theorem.

3. Discussion.

We conclude this paper with an exposition of some open problems. Our fi problem was noted in [3].

Problem 1. Let $f(n, r)$ denote the largest integer such that every G contained $\mathcal{G}(n, \lfloor \frac{1}{4} n^2 \rfloor + 1)$ contains an r-cycle the sum of the degrees of its vertices bei at least $f(n, r)$. Determine $f(n, r)$.

Theorem 4 of [3] asserts that $f(n, r) > \frac{nr}{3}$ for $3 \leq r \leq \lfloor \frac{n}{6} \rfloor + 2$. Erdös a Laskar [7] proved that

$$(1 + c)n < f(n, 3) < \left(\frac{3}{2} - c\right)n,$$

where c is a positive constant. This result has recently been improved by Fan [8] who proves that for every $G \in \mathcal{G}(n, m)$

$$f(n,3) \geq \begin{cases} \frac{5m}{n}, & \text{if } \frac{1}{4}n^2 < m < n^2(10-\sqrt{32})/17 \\ 2n+\frac{4}{n}\sqrt{m(4m-n^2)}-m, & \text{if } n^2(10-\sqrt{32})/17 \leq m \leq \frac{1}{3}n^2 \\ (3\Delta - 2n + 4m)/\Delta, & \text{otherwise.} \end{cases}$$

Determining $f(n,3)$ exactly seems to be difficult.

Our next problem is suggested by the results of this paper.

Problem 2. Let $G \in \mathcal{G}\left(n, \lfloor\frac{1}{4}n^2\rfloor + 1\right)$. A triple (a, b, c) of non-negative reals is said to be **feasible** if every G contains a triangle $x_1x_2x_3$ with

$$d_G(x_1) > an,\ d_G(x_2) > bn \text{ and } d_G(x_3) > cn.$$

Characterize the set of feasible triples.

Trivially $\left(\frac{1}{2}, 0, 0\right)$ is feasible. We know ([3], Theorem 4) that $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ is feasible and is best possible. Theorem 2 asserts that $\left(\sqrt{2}-1, \sqrt{2}-1, 0\right)$ is feasible; can we say anything about $d_G(x_3)$? Problem 2 can be generalized to larger cycles, in particular odd cycles C_{2r+1}. The results of [3] - [6] yield some feasible tuples. More generally, we can ask the same questions when $m = \alpha n^2$, $\alpha > 0$.

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