On $[a,b]$-Covered Graphs

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Abstract. A graph G is $[a,b]$-covered if each edge of G belongs to an $[a,b]$-factor. Here a necessary and sufficient condition for a graph to be $[a,b]$-covered is given and it is shown that an $[m,n]$-graph is $[a,b]$-covered if $bm-na \geq 2(n-b)$ and $0 \leq a < b \leq n$.

1. Introduction

By a graph we mean a finite, undirected graph with no loops. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. A vertex set $S \subseteq V(G)$ is independent if there are no edges in G whose two end-vertices in S. For a vertex x of G, the degree of x in G is denoted by $d_G(x)$. Let b and a be integers such that $b \geq a \geq 0$. We say that G is an $[a,b]$-graph if $a \leq d_G(x) \leq b$ for all $x \in V(G)$. An $[r,r]$-graph is also called an r-regular graph. Similarly, an $[r,r]$-factor is called an r-factor.

Let G be a graph, and g and f be two integer-valued functions defined on $V(G)$ such that $g(x) \leq f(x)$ for every $x \in V(G)$. Then a (g,f)-factor of G is a spanning subgraph F of G satisfying $g(x) \leq d_F(x) \leq f(x)$ for all $x \in V(G)$. For a subset S of $V(G)$, we write $G-S$ for the subgraph of G obtained by deleting the vertices in S together with their incident edges. If S and T are disjoint subsets of $V(G)$, then $e(S,T)$ denotes the number of edges of G joining S and T. We shall need the (g,f)-factor theorem due to Lovász [6] for which Tutte [7] has given a short proof.

Theorem 1.1. *[6,7] Let G be a graph and g and f be integer-valued functions defined on $V(G)$ such that $g(x) \leq f(x)$ for all $x \in V(G)$. Then G has a (g,f)-factor if and only if*

$$\delta(S,T) = \sum_{x\in T}\{d_G(x) - g(x)\} + \sum_{x\in S} f(x) - e(S,T) - h(S,T) \geq 0$$

for all disjoint subsets S and T of $V(G)$, where $h(S,T)$ denotes the number of components, C, of $G-(S\cup T)$ such that $g(x) = f(x)$ for all $x \in V(C)$ and $e(T,V(C)) + \sum_{x\in V(C)} f(x) \equiv 1 \pmod 2$.

Kano and Saito [4] discussed some sufficient conditions for a graph to have an $[a,b]$-factor. Heinrich and others [3] characterized graphs which have an $[a,b]$-factor. Little [5] introduced the concept of a factor-covered graph, which is a graph

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such that for every edge e there exists a 1-factor containing e. Here we generalize this idea as follows. A graph G is $[a,b]$-factor-covered, or simply $[a,b]$-covered, if for each edge e of G there is an $[a,b]$-factor containing e. Thus, a $[1,1]$-covered graph is a factor-covered graph. In 1974, Little proved the following result.

Theorem 1.2. *[5] Let G be a graph of even order. Then G is factor-covered if and only if*

(1) *$o(G-S) \le |S|$ for all $S \subset V(G)$ and*

(2) *$o(G-S) = |S|$ implies that S is independent*

where $o(G-S)$ is the number of odd components of $G-S$.

The above mentioned results can be found in a survey article [1].

In this paper we shall characterize $[a,b]$-covered graphs and show that an $[m,n]$-graph is $[a,b]$-covered if $bm - na \ge 2(n-b)$ and $0 \le a < b \le n$.

2. The characterization of $[a,b]$-covered graphs

If $a = 0$ and $a < b$, every graph is $[a,b]$-covered. In this section we deduce a characterization of $[a,b]$-covered graphs where $b > a \ge 1$. When $a = b$, the problem is more complicated and will be dealt with in another paper.

Let G be a graph and $S, T \subseteq V(G)$. Then the number of vertices of degree j in $G-S$ is denoted by $p_j(G-S)$ and the number of vertices in T having degree j in $G-S$ by $p_j(G-S|T)$.

Theorem 2.1. *Let $b > a \ge 1$ be integers. Then a graph G is $[a,b]$-covered if and only if for all $S \subseteq V(G)$*

$$\sum_{j=0}^{a-1}(a-j)p_j(G-S) \le b|S| - \varepsilon(S)$$

where $\varepsilon(S) = 2$ if S is not independent, $\varepsilon(S) = 1$ if S is independent and there is at least one edge xy such that $x \in S$, $y \in V(G)\backslash S$ and $d_{G-S}(y) \ge a$, and $\varepsilon(S) = 0$ otherwise.

Proof: The condition is necessary. Let G have an $[a,b]$-factor F, and let $S \subseteq V(G)$. If $d_{G-S}(x) = j$, $0 \le j \le a-1$, then in F, the vertex x is incident with at least $a-j$ edges, xx_i, where $x_i \in S$. But $d_F(x) \le b$ for every $x \in V(G)$. Thus,

$$\sum_{j=0}^{a-1}(a-j)p_j(G-S) \le b|S|,$$

since there exist at most $b|S|$ edges of F joining S to $G-S$. Now if S is not independent, the induced subgraph $G[S]$ contains at least one edge e. Consider an $[a,b]$-factor F containing e. We have

$$\sum_{j=0}^{a-1}(a-j)p_j(G-S) \le b|S| - 2.$$

If S is independent and there is an edge xy such that $x \in S$, $y \in V(G)\backslash S$ and $d_{G-S}(y) \ge a$, consider an $[a,b]$-factor F containing xy. We have

$$\sum_{j=0}^{a-1}(a-j)p_j(G-S) \le b|S| - 1.$$

The condition is sufficient. Let $e = uv$ be any edge of G. Define f and g on $V(G)$ by $f(u) = f(v) = b-1$, $f(x) = b$ when $x \neq u,v$, $g(u) = g(v) = a-1$ and $g(x) = a$ when $x \neq u,v$. Clearly, there exists an $[a,b]$-factor of G containing e if there exists a (g,f)-factor of $G' = G - e$, by Theorem 1.1 if for any $S,T \subseteq V(G')$ with $S \cap T = \phi$,

$$\delta(S,T) = \sum_{x\in T}\{d_{G'}(x) - g(x)\} + \sum_{x\in S} f(x) - e(S,T) - h(S,T) \ge 0.$$

Since $g(x) \neq f(x)$ for any $x \in V(G')$, $h(S,T) = o$. Thus, we have

$$\delta(S,T) = \sum_{x\in T} d_{G'-S}(x) - \sum_{x\in T} g(x) + \sum_{x\in S} f(x).$$

We distinguish three cases.

Case 1. $u, v \in S$. In this case we have

$$\delta(S,T) = \sum_{x\in T} d_{G'-S}(x) - a|T| + b|S| - 2$$

and

$$p_j(G-S) = p_j(G'-S).$$

Clearly,

$$\sum_{x\in T} d_{G'-S}(x) \ge a|T| - \sum_{j=0}^{a-1}(a-j)p_j(G'-S|T)$$
$$\ge a|T| - \sum_{j=0}^{a-1}(a-j)p_j(G'-S) = a|T| - \sum_{j=0}^{a-1}(a-j)p_j(G-S).$$

By hypothesis,

$$\sum_{j=0}^{a-1}(a-j)p_j(G-S) \leq b|S|-2,$$

since S is not independent. It follows that

$$\sum_{x\in T} d_{G'-S}(x) \geq a|T| - b|S| + 2,$$

namely, $\delta(S,T) \geq 0$.

Case 2. $u \in S$ and $v \in V(G)\backslash S$. In this case we have

$$p_j(G'-S) = p_j(G-S).$$

Case 2.1. $v \in T$. We have

$$\begin{aligned}\delta(S,T) &= \sum_{x\in T} d_{G'-S}(x) - (a|T|-1) + b|S| - 1 \\ &= \sum_{x\in T} d_{G'-S}(x) - a|T| + b|S|.\end{aligned}$$

By hypothesis,

$$\sum_{j=0}^{a-1}(a-j)p_j(G-S) \leq b|S|.$$

Thus,

$$\begin{aligned}\sum_{x\in T} d_{G'-S}(x) &\geq a|T| - \sum_{j=0}^{a-1}(a-j)p_j(G'-S|T) \\ &\geq a|T| - \sum_{j=0}^{a-1}(a-j)p_j(G'-S) \\ &= a|T| - \sum_{j=0}^{a-1}(a-j)p_j(G-S) \geq a|T| - b|S|.\end{aligned}$$

It follows that $\delta(S,T) \geq 0$.

Case 2.2. $v \notin T$. We have

$$\delta(S,T) = \sum_{x\in T} d_{G'-S}(x) - a|T| + b|S| - 1.$$

If $d_{G-S}(v) \geq a$, by hypothesis,

$$\sum_{j=0}^{a-1}(a-j)p_j(G-S) \leq b|S|-1.$$

Thus,

$$\begin{aligned}\sum_{x\in T} d_{G'-S}(x) &\geq a|T| - \sum_{j=0}^{a-1}(a-j)p_j(G'-S|T) \\ &\geq a|T| - \sum_{j=0}^{a-1}(a-j)p_j(G'-S) \\ &= a|T| - \sum_{j=0}^{a-1}(a-j)p_j(G-S) \geq a|T| - b|S| + 1.\end{aligned}$$

So $\delta(S,T) \geq 0$.

If $d_{G-S}(v) < a$, we have

$$\begin{aligned}\sum_{x\in T} d_{G'-S}(x) &\geq a|T| - \sum_{j=0}^{a-1}(a-j)p_j(G'-S|T) \\ &\geq a|T| - \sum_{j=0}^{a-1}(a-j)p_j(G'-S) + 1 \\ &= a|T| - \sum_{j=0}^{a-1}(a-j)p_j(G-S) + 1 \geq a|T| - b|S| + 1.\end{aligned}$$

Therefore $\delta(S,T) \geq 0$.

Case 3. $u, v \in V(G)\backslash S$.

Case 3.1. $u, v \notin T$. In this case

$$\delta(S,T) = \sum_{x\in T} d_{G'-S}(x) - a|T| + b|S|$$

and

$$p_j(G'-S|T) = p_j(G-S|T).$$

By hypothesis,

$$\sum_{j=0}^{a-1}(a-j)p_j(G-S) \leq b|S|.$$

Thus,

$$\begin{aligned}\sum_{x\in T} d_{G'-S}(x) &\geq a|T| - \sum_{j=0}^{a-1}(a-j)p_j(G'-S|T) \\ &= a|T| - \sum_{j=0}^{a-1}(a-j)p_j(G-S|T) \\ &\geq a|T| - \sum_{j=0}^{a-1}(a-j)p_j(G-S) \geq a|T| - b|S|.\end{aligned}$$

So $\delta(S,T) \geq 0$.

Case 3.2. $u \in T$ and $v \notin T$. In this case

$$\delta(S,T) = \sum_{x\in T} d_{G'-S}(x) - (a|T|-1) + b|S|.$$

If $d_{G-S}(u) > a$, we have $p_j(G'-S|T) = p_j(G-S|T)$ for $0 \leq j \leq a-1$. If $d_{G-S}(u) = a$, we have $p_{a-1}(G'-S|T) = p_{a-1}(G-S|T) + 1$. So

$$\sum_{j=0}^{a-1}(a-j)p_j(G'-S|T) = \sum_{j=0}^{a-1}(a-j)p_j(G-S|T) + 1.$$

If $d_{G-S}(u) = j_0 < a$, then $p_{j_0}(G'-S|T) = p_{j_0}(G-S|T) - 1$ and $p_{j_0-1}(G'-S|T) = p_{j_0-1}(G-S|T) + 1$. It is easy to verify that

$$\sum_{j=0}^{a-1}(a-j)p_j(G'-S|T) = \sum_{j=0}^{a-1}(a-j)p_j(G-S|T) + 1.$$

Thus, in all cases we have

$$\sum_{j=0}^{a-1}(a-j)p_j(G'-S|T) \leq \sum_{j=0}^{a-1}(a-j)p_j(G-S|T) + 1.$$

Therefore

$$\begin{aligned}\sum_{x\in T} d_{G'-S}(x) &\geq a|T| - \sum_{j=0}^{a-1}(a-j)p_j(G'-S|T) \\ &\geq a|T| - \sum_{j=0}^{a-1}(a-j)p_j(G-S|T) - 1 \\ &\geq a|T| - \sum_{j=0}^{a-1}(a-j)p_j(G-S) - 1 \geq a|T| - b|S| - 1.\end{aligned}$$

Hence, $\delta(S,T) \geq 0$.

Case 3.3. $u, v \in T$. In this case

$$\delta(S,T) = \sum_{x \in T} d_{G'-S}(x) - (a|T| - 2) + b|S|.$$

By an argument similar to that used in *Case 3.2*, it is not difficult to verify that

$$\sum_{j=0}^{a-1} (a-j)p_j(G' - S|T) \le \sum_{j=0}^{a-1} (a-j)p_j(G - S|T) + 2.$$

As in *Case 3.2*, we can prove that

$$\sum_{x \in T} d_{G'-S}(x) \ge a|T| - b|S| - 2.$$

So $\delta(S,T) \ge 0$.

3. $[a, b]$-covered graphs

In [2, Chapter 8, Theorem 13] Berge proved that if G is a graph of even order which is regular of degree $r > 1$ and if G is $(r-1)$-edge-connected, then G is factor-covered. Here we present a similar result about $[a, b]$-covered graphs. In particular, we have the following general result.

Theorem 3.1. *Let G be an $[m, n]$-graph. If a and b are integers such that $bm - na \ge 2(n-b)$ and $0 \le a < b \le n$, then G is $[a, b]$-covered.*

Proof: When $a = 0$, the theorem is trivial. Otherwise, by Theorem 2.1 it is sufficient to prove that for all $S \subseteq V(G)$,

$$\sum_{j=0}^{a-1} (a-j)p_j(G - S) \le b|S| - \varepsilon(S) \tag{1}$$

where $\varepsilon(S)$ is defined as in Theorem 2.1. For all $S \subseteq V(G)$, if $p_j(G-S) = 0$ for all $j = 0, 1, \ldots, a-1$, clearly, inequality (1) is true. In the following we assume that $p_j(G - S) \neq 0$ for some j and some $S \subseteq V(G)$. We consider the following three cases.

Case 1. S is not independent. Since $m \le d_G(x) \le n$ for every $x \in V(G)$, it follows that

$$\sum_{j=0}^{a-1} (m-j)p_j(G - S) \le n|S| - 2.$$

Thus,

$$b|S| \geq \frac{b}{n}\left[\sum_{j=0}^{a-1}(m-j)p_j(G-S)+2\right].$$

Since $bm-na \geq 2(n-b)$, $(bm-na)p_j(G-S) \geq 2(n-b)$ when $p_j(G-S) \neq 0$. Also $n-b \geq 0$. Thus, we have

$$\frac{b}{n}\sum_{j=0}^{a-1}(m-j)p_j(G-S)+\frac{2b}{n}-\left[\sum_{j=0}^{a-1}(a-j)p_j(G-S)+2\right]$$
$$=\frac{1}{n}\sum_{j=0}^{a-1}\left[bm-na+(n-b)j\right]p_j(G-S)-\frac{2}{n}(n-b)\geq 0.$$

namely,

$$b|S| \geq \sum_{j=0}^{a-1}(a-j)p_j(G-S)+2.$$

Case 2. S is independent and there is an edge xy such that $x \in S,\ y \in V(G)\backslash S$ and $d_{G-S}(y) \geq a$. In this case

$$\sum_{j=0}^{a-1}(m-j)p_j(G-S) \leq n|S|-1.$$ Thus,

$$b|S| \geq \frac{b}{n}\left[\sum_{j=0}^{a-1}(m-j)p_j(G-S)+1\right].$$

As in Case 1, we can prove that

$$\frac{b}{n}\left[\sum_{j=0}^{a-1}(m-j)p_j(G-S)+1\right] \geq \sum_{j=0}^{a-1}(a-j)p_j(G-S)+1.$$

So

$$b|S| \geq \sum_{j=0}^{a-1}(a-j)p_j(G-S)+1.$$

Case 3. Case 1 and *Case 2* are not true. In this case

$$\sum_{j=0}^{a-1}(m-j)p_j(G-S) \leq n|S|.$$

It follows that

$$b|S| \geq \frac{b}{n}\sum_{j=0}^{a-1}(m-j)p_j(G-S).$$

Since $bm - na \geq 2(n-b) \geq 0$, we have

$$\frac{b}{n}\sum_{j=0}^{a-1}(m-j)p_j(G-S) - \sum_{j=0}^{a-1}(a-j)p_j(G-S)$$
$$= \frac{1}{n}\sum_{j=0}^{a-1}\left[bm - na + (n-b)j\right]p_j(G-S) > 0.$$

Thus inequality (1) is satisfied and the proof is complete.

The next result follows immediately from Theorem 3.1.

Corollary 3.2. *Let b, a and r be integers such that $r(b-a) \geq 2(r-b)$ and $0 \leq a < b \leq r$. Then if graph G is r-regular, G is $[a,b]$-covered.*

From Corollary 3.2 we can easily obtain the following corollaries.

Corollary 3.3. *Let a and b be non-negative integers and let G be an r-regular graph. Then G is $[a,b]$-covered if $b - a \geq 2$ and $b \leq r$.*

Corollary 3.4. *Every r-regular graph is $[k-1, k]$-covered for every integer k such that $\frac{r}{2} \leq k \leq r$.*

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