A SMALL EMBEDDING FOR PARTIAL 4-CYCLE SYSTEMS

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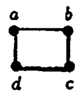
CANADA

1. Introduction.

A *4-cycle system* (FCS) is a pair (K_n, C), where K_n is the complete undirected graph on n vertices and C is an edge disjoint collection of 4-cycles which *partition* K_n. The *number* n is called the *order* of the FCS (K_n, C) and it is well-known that the spectrum (= set of all orders for which a FCS exists) is *precisely* the set of all $n \equiv 1 \pmod 8$. If (K_n, C) is a FCS then $|C| = n(n-1)/8$. In what follows we will denote the 4-cycle



by any cyclic shift of (a, b, c, d) or (b, a, d, c).

Example 1.1: (FCS of order 9). Let K_9 be based on Z_9 and define $C = \{(i, 1+i, 5+i, 2+i) \mid i \in Z_9\}$. Then (K_9, C) is a FCS of order 9.

A *partial* FCS is a pair (K_n, P), where P is an edge disjoint collection of 4-cycles of K_n.

Example 1.2: (Partial FCS of order 6). Let K_6 be based on $\{1,2,3,4,5,6\}$ and $P = \{(1,2,3,4), (1,3,5,6), (2,6,4,5)\}$.

Now given a *partial* FCS (K_n, P_1) there is the obvious problem of *completion*. That is, does there exist a FCS (K_n, P_2) such that $P_1 \subseteq P_2$? In general, the answer to this question is no. For example, the partial FCS (K_6, P) in Example 1.2 cannot be completed to a FCS (since among other reasons $6 \not\equiv 1 \pmod 8$). The (partial) FCS (K_n, P_1) is said to be *embedded* in the FCS (K_m, P_2) provided that $m \geq n$ *and* $P_1 \subseteq P_2$. Since a partial FCS cannot generally be completed to a FCS the problem of whether or not a partial FCS can always be embedded in some FCS is immediate. In 1974 Richard Wilson [3] showed that this is always possible. Actually, Wilson proved a much broader result on the embedding of partial graph designs in general, not just for 4-cycle systems. Nevertheless, the embedding guaranteed in [3] is exponentially large. Recently a much smaller embedding was obtained. In [1] it is shown that a partial FCS of order n can always be embedded in a FCS of order m for every admissable $m \geq 8n+1$. The object of this note

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is to very much improve this bound to *approximately* $2n+\sqrt{n}$. In particular, we prove that a partial FCS of order n can be embedded in a FCS of order m for every admissable $m \geq 2\binom{x}{2} + x = x^2$, where x is the smallest *odd* integer such that $\binom{x}{2} \geq n$.

2. The main constructions.

We begin with a theorem due to Dominique Sotteau which is the main ingredient in our construction.

Theorem 2.1. *(D. Sotteau [2]). The complete bipartite graph $K_{x,y}$ can be decomposed into cycles of length $2k$ if and only if $x \geq k$, $y \geq k$, $2k$ divides xy, and x and y are even.*

Since we are dealing with 4-cycle systems only, Sotteau's Theorem reduces to the *single requirement* that x and y are *even*. That is to say, $K_{x,y}$ can be decomposed into 4-cycles if and only if x and y are even.

A modification of the following construction will give us our main embedding result.

The $2\binom{x}{2} + x$ construction. Let X be a set of size $|X| = x$, where x is *odd.* Let S be a set of size $|S| = \binom{x}{2}$, set $M = (S \times \{1,2\}) \cup X$, and define a collection C of 4-cycles of M as follows:

(1) For each 2-element subset $\{a,b\}$ of S, place the 4-cycle $((a,1),(b,1),(a,2),(b,2))$ in C.
(2) Denote by $T(X)$ the set of $\binom{x}{2}$ 2-element subsets of X and let α be any 1 - 1 mapping of S onto $T(X)$. For each element $a \in S$ place *exactly one* of the two 4-cycles $((a,1),(a,2),c,d)$ or $((a,1),(a,2),d,c)$ in C, where $a\alpha = \{c,d\}$.
(3) For each element $c \in X$ denote by $D(c)$ the set of all $(a,i) \in S \times \{1,2\}$ such that the edge $\{(a,i),c\}$ belongs to a 4-cycle of type (2). Since c belongs to $x-1$ 2-element subsets of $T(X)$, $|D(c)| = x-1$. Furthermore, and this is *important*, the collection $\pi = \{D(c) \mid c \in X\}$ is a partition of $S \times \{1,2\}$. By Sotteau's Theorem $K_{D(c),X\setminus\{c\}}$ can be partitioned into 4-cycles. Denote this collection of $(x-1)^2/4$ 4-cycles by $F(c)$ and place these 4-cycles in C.

It is straightforward to see that (K_m, C) is a 4-cycle system of order $2\binom{x}{2} + x = x^2$, where $|M| = m$ and K_m is based on M.

The following construction is a lot simpler than the $2\binom{x}{2} + x$ construction and will be used to extend the main embedding result.

The $n + 8$ construction. Let (K_n, C) be a FCS of order n based on the set $\{\infty\} \cup S$ and let X be a set of size $|X| = 8$. Let $M = \{\infty\} \cup S \cup X$ and define a collection of 4-cycles C^* as follows:

(1) $C \subseteq C^*$.
(2) Let $(K_9, C(9))$ be any FCS of order 9 (Example 1.1) based on $\{\infty\} \cup X$

and place the 4-cycles in $C(9)$ in C^*.

(3) Use Sotteau's Theorem to decompose $K_{S,X}$ into 4-cycles and place these 4-cycles in C^*.

Then (K_{n+8}, C^*), based on M, is a FCS of order $n+8$ *and* contains the FCS (K_n, C) as a subsystem.

3. Embedding partial 4-cycle systems.

Before plunging into the main embedding theorem we will need one more idea. Two partial FCSs (K_n, P_1) and (K_n, P_2) are *mutually balanced* provided they cover exactly the same edges. That is to say, the edge $\{a, b\}$ belongs to a 4-cycle of P_1 if and only if it belongs to a 4-cycle of P_2.

Example 3.1: Let $E = \{a, b, c, d\} \times \{1, 2\}$ and define P_1 and P_2 as follows:

$$\begin{aligned} P_1 =&\{((a,1),(b,1),(a,2),(b,2)),((b,1),(c,1),(b,2),(c,2)),\\ &((c,1),(d,1),(c,2),(d,2)),((d,1),(a,1),(d,2),(a,2))\} \quad \textit{and} \\ P_2 =&\{((a,1),(b,1),(c,1),(d,1)),((a,2),(b,2),(c,2),(d,2)),\\ &((a,1),(b,2),(c,1),(d,2)),((a,2),(b,1),(c,2),(d,1))\}. \end{aligned}$$

If K_8 is based on E, then (K_8, P_1) and (K_8, P_2) are a pair of mutually balanced *partial* FCSs of order 8.

Theorem 3.2. *A partial FCS of order n can be embedded in a FCS of order x^2 where x is the smallest* odd *positive integer such that $\binom{x}{2} \geq n$.*

Proof: Let (K_n, P) be a *partial* FCS of order n based on $N = \{1, 2, 3, \ldots, n\}$. Let x be the smallest *odd* positive integer such that $\binom{x}{2} \geq n$. Let X be a set of size $|X| = x$ and S a set of size $|S| = \binom{x}{2}$ containing N. Let $M = (S \times \{1,2\}) \cup X$ and let (K_m, C) be the FCS constructed on M using the $2\binom{x}{2} + x$ construction. If $P = \{(a_1, b_1, c_1, d_1), (a_2, b_2, c_2, d_2), \ldots, (a_t, b_t, c_t, d_t)\}$ we will denote the copies of (K_8, P_1) and (K_8, P_2) based on $\{a_i, b_i, c_i, d_i\} \times \{1, 2\}$ by (K_8, P_{1i}) and (K_8, P_{2i}). If $i \neq j$, then P_{1i} and P_{1j} contain no common edges. As a consequence $(K_m, C^* = (C \setminus \cup_{i=1}^{t} P_{1i}) \cup (\cup_{i=1}^{t} P_{2i}))$ is a FCS system of order $m = 2\binom{x}{2} + x$. Clearly C^* contains two disjoint copies of (K_n, P). This completes the proof. ∎

Corollary 3.3. *A* partial *FCS of order n can be embedded in a FCS of order m for* every admissable $m \geq x^2$, *where x is the smallest* odd *integer such that* $\binom{x}{2} \geq n$.

Proof: Let (K_n, P) be a partial FCS of order n. By Theorem 3.2 (K_n, P) can be embedded in a FCS (K_{x^2}, P^*) of order x^2, where x is the smallest odd positive integer such that $\binom{x}{2} \geq n$. Iteration of Corollary 3.3 embeds (K_{x^2}, P^*), and therefore (K_n, P), in a FCS (K_m, C) for every $m = x^2 + 8k$; that is, for every admissable $m \geq x^2$. ∎

4. Concluding remarks.

Although the result in this note dramatically improves the known bound for embedding partial FCSs, it is still not the best possible embedding. The best possible bound is *approximately* $n + \sqrt{n}$. Couched in the vernacular of this note, *approximately* $\binom{x}{2} + x$, where x is the smallest positive integer such that $\binom{x}{2} \geq n$.

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