THREE PAIRWISE ORTHOGONAL DIAGONAL LATIN SQUARES[1]

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Abstract. A diagonal Latin square is a Latin square whose main diagonal and back diagonal are both transversals. It is proved in this paper that there are three pairwise orthogonal diagonal Latin squares of order n for all $n \geq 7$ with 28 possible exceptions, in which 118 is the greatest one.

1. Introduction.

A *diagonal Latin square* of order n is a Latin square each of whose *main* diagonal (the cells $\{(i,i): 1 \leq i \leq n\}$) and *back* diagonal (the cells $\{(i, n+1-i): 1 \leq i \leq n\}$) is a transversal. t *pairwise orthogonal diagonal Latin squares* of order n, denoted briefly by tPODLS(n), are t pairwise orthogonal Latin squares each of which is a diagonal Latin square of order n.

For $t = 1$, it has been shown (see Lindner [9], Hilton [6] and Gergely [3]) that diagonal Latin squares exist for all positive integer n greater than 3. For $t = 2$, it has been shown (see Heinrich and Hilton [5], Wallis and Zhu [12, 13, 16] that a pair of orthogonal diagonal Latin squares of order n exists for all $n \geq 7$ with one possible exception of $n = 10$. For $t \geq 3$, less work has been done. Gergely has proved in [4] that

$$D(n) \geq \begin{cases} \alpha(n) - 3 & \text{if } n \text{ is odd,} \\ \alpha(n) - 2 & \text{if } n \text{ is even,} \end{cases} \tag{1.1}$$

where $D(n)$ denotes the maximum number of PODLS(n), $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}$ is the prime factorization of n and $\alpha(n) = \min_{1 \leq i \leq r}\{p_i^{\alpha_i}\}$. He has also pointed out that

$$D(n) \leq \begin{cases} n-3 & \text{if } n \text{ is odd } (n \geq 3), \\ n-2 & \text{if } n \text{ is even.} \end{cases} \tag{1.2}$$

From (1.1) and (1.2), he has given the result that for any prime power order n,

$$D(n) = \begin{cases} n-3 & \text{if } n \text{ is odd } (n \geq 3), \\ n-2 & \text{if } n \text{ is even.} \end{cases} \tag{1.3}$$

Some bounds have been obtained by using PBD's in [14], that is $D(n) \geq 3$ for $n \geq 447$, $D(n) \geq 4$ for $n \geq 511$, $D(n) \geq 5$ for $n \geq 2724$, and $D(n) \geq 6$ for $n \geq 6278$. Asymptotically, it has been proved in [7] that $D(n) \to \infty$ as $n \to \infty$.

In this paper it is proved, by using an entirely different method from PBD's, that there are three pairwise orthogonal diagonal Latin squares of order n for all $n \geq 7$ with 28 possible exceptions, in which 118 is the greatest one.

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[1] This paper, which appeared in Chinese in *Applied Mathematics, A Journal of Chinese Universities*, No. 1, Vol. 1 (Sept.) 1986, has been translated as a courtesy to our anglophone readers.

2. Preliminaries.

Let $N(n)$ ($I(n)$) be the maximum number of pairwise orthogonal (idempotent) Latin squares of order n. It is obvious that

$$D(n) \leq I(n), \quad \text{and} \tag{2.1}$$

$$N(n) - 1 \leq I(n) \leq N(n). \tag{2.2}$$

Let $IA_t(v,k)$ denote t pairwise orthogonal Latin squares of order v (briefly t POLS(v)) with t sub-POLS(k) missing. Usually we leave the size k hole in the lower right corner. In fact, it is not necessary for such orthogonal subsquares of order k really to exist. For example, Brouwer [1] has obtained an $IA_4(10,2)$ although there do not exist 4POLS(2).

Further denote by $IA_t^*(v,k)$ an $IA_t(v,k)$ in which the first $v-k$ elements in the main diagonal of every square are distinct and different from the missing elements. It is easy to see that the existence of an $IA_{t+1}(v,k)$ implies the existence of an $IA_t^*(v,k)$, and that $IA_t^*(v,1)$ exists if $I(v) \geq t$. From Brouwer's $IA_4(10,2)$ and Seiden and Wu's result on 3POLS(v) of sum composition [11], we have

Example 2.1: An $IA_3^*(10,2)$ exists.

Example 2.2: There are $IA_3^*(13+3,3)$, $IA_3^*(16+3,3)$, $IA_3^*(16+5,5)$, $IA_3^*(37+3,3)$, and $IA_3^*(37+9,9)$.

From Example 2.11 in [15] we actually have

Example 2.3: An $IA_3^*(7+2,2)$ exists.

Lemma 2.4. *An $IA_3^*(n,7)$ exists if $n \geq 29$ except possibly for $n \in \{30,31,32,34,37,38,39,41,44,45,46,47,48,52,53\}$.*

Proof: From Theorem 1.5 in [15] we need only prove the existence of $IA_3^*(n,7)$ for $n = 29,33,40,42,43,60$, and 74. Since

$$\begin{aligned}
29 &= 7.4+1,\\
33 &= 7.4+5,\\
40 &= 5.7+(2+2+1),\\
42 &= 5.7+(2+2+2+1),\\
43 &= 5.7+(2+2+2+2),\\
60 &= 7.7+(2.5+1),\\
74 &= 9.7+(2.5+1),
\end{aligned}$$

the conclusion comes from Lemma 2.6 in [15]. ∎

Lemma 2.5. *An $IA_3(n,8)$ exists if $n \geq 32$ and $n \neq 46$.*

Proof: From Theorem 1.6 in [15] we need only prove the existence of $IA_3(35,8)$. Since $35 = 8.4+3$, the conclusion follows from Lemma 2.3 in [15]. ∎

Lemma 2.6. *In* $I(n) \geq 3$ *if* $n \notin \{2,3,4,6,10,14,18,20,22,24,26,28,30,34,38\}$.

Proof: Since $N(n) \geq 4$ implies $I(n) \geq 3$, from the list in [1] we need only prove $I(n) \geq 3$ for $n = 33, 42, 44$, and 52. Orders 33 and 42 come from Lemma 2.4. Orders 44 and 52 come from Lemma 2.6 in [15] and the following expressions,

$$44 = 9.4 + 8,$$
$$52 = 11.4 + 8. \qquad \blacksquare$$

We also need the following lemma (see the list in [1]).

Lemma 2.7. $N(n) \geq 3$ *if* $n \notin \{2,3,6,10,14\}$.

From Gergely [4] we have

Lemma 2.8. *If* $D(m) \geq t$ *and* $D(n) \geq t$, *then* $D(mn) \geq t$.

3. Some constructions.

The following lemma is essentially the Construction 2 in [8].

Lemma 3.1. *Suppose* q *is odd and there is an* $IA_t^*(m+k,k)$. *Then* $\min\{D(q), D(m+k), I(m)\} \geq t$ *implies* $D(qm+k) \geq t$.

Corollary 3.2. $D(n) \geq 3$ *if* $n \in S_1 = \{50,51,78,85,93,94,106,122,134,145,146,162,166,172,177,190,205,210,218,219,250,254,273,302,346,354\}$.

Proof: The existence of $IA_3^*(7+2,2)$ and $IA_3^*(13+3,3)$ comes from Example 2.3 and Example 2.2. From [2] and [10] we have $N(12) \geq 5$ and $N(15) \geq 4$, and then $I(12) \geq 3$ and $I(15) \geq 3$. Now the conclusion follows from Lemma 3.1 and the following expressions.

50	=	7.7+1,	172	=	13.13+3,
51	=	7.7+2,	177	=	11.16+1,
78	=	11.7+1,	190	=	27.7+1,
85	=	7.12+1,	205	=	29.7+2,
93	=	13.7+2,	210	=	19.11+1,
94	=	7.13+3,	218	=	31.7+1,
106	=	7.15+1,	219	=	31.7+2,
122	=	11.11+1,	250	=	19.13+3,
134	=	19.7+1,	254	=	23.11+1,
145	=	9.16+1,	273	=	17.16+1,
146	=	11.13+3,	302	=	43.7+1,
162	=	23.7+1,	346	=	23.15+1,
166	=	11.15+1,	354	=	27.13+3. ∎

Denote by $IA_t^{**}(v,k)$ an $IA_t^*(v,k)$ in which the elements in the cells $(1, v-k), (2, v-k-1), \ldots, (v-k, 1)$ of every square are distinct and different from the missing elements. It is clear that an $IA_t^{**}(v,0)$ exists if $D(v) \geq t$ and that an $IA_t^{**}(v,1)$ exists if v is odd and $D(v) > t$. We now give a construction which is a kind of variation of Lemma 2.6 in [15] and generalizes some constructions in [8].

Lemma 3.3. *Suppose there are $t+1$ POLS(q) such that t of them are tPODLS(q). Suppose $2\,|qmk$, $D(k) \geq t$ and that $IA_t^*(m+\ell_i, \ell_i)$ exist for $0 \leq i \leq q-1$, where $k = \ell_0 + \ell_1 + \ldots + \ell_{q-1}$. Further suppose an $IA_t^{**}(m+\ell_0, \ell_0)$ exists if $2 \not| q$. then $D(qm+k) \geq t$.*

Proof: Since tPODLS(q) have an extra orthogonal mate, they have q disjoint common transversals each of which is determined by an element in the extra square. Label these transversals as $T_0, T_1, \ldots, T_{q-1}$, provided that T_0 contains the central cell if $2 \not| q$.

Begin with the tPODLS(q) and replace each of its cells with an $m \times m$ array labelled by the elements in the cell, the array is the upper left part of $IA_t^*(m+\ell_i, \ell_i)$ if the cell is contained in T_i, $0 \leq i \leq q-1$. But if the cell is in the back diagonal of the tPODLS(q), it will be filled with a modified $IA_t^*(m+\ell_i, \ell_i)$, that is, by permuting the first m columns the main diagonal of the upper left part in the $IA_t^*(m+\ell_i, \ell_i)$ becomes its back diagonal. Furthermore, the central cell will be filled with the upper left part of an $IA_t^{**}(m+\ell_0, \ell_0)$ if $2 \not| q$. Suppose every $IA_t^*(m+\ell_i, \ell_i)$ is based on certain m elements and ℓ_i new elements, and the new elements remain unchanged when labelling. Then we obtain the upper left part of an $IA_t^{**}(qm+k, k)$ whose right part consists of the columns $C_0, C_1, \ldots, C_{q-1}$ where C_i comes from the right part of the $IA_t^*(m+\ell_i, \ell_i)$ in T_i, and the lower part is obtained in a similar fashion, see Figure 1.

Now fill the size k hole in the lower right corner of the $IA_t^{**}(qm+k, k)$ with the given tPODLS(k). Since $2\,|qmk$, $2\,|qm$ or $2\,|k$. So we can divide the upper left part of the $IA_t^{**}(qm+k, k)$ or the tPODLS(k) into four parts and put them in the four corners of some tPODLS($qm+k$) by permuting the corresponding rows and columns. The remaining verification is a routine matter and the proof is complete. ∎

We remark that the Constructions 1, 3, and 4 in [8] are essentially the particular cases of Lemma 3.3 with only one $\ell_i \neq 0$ when $2\,|q$, $2\,|m$ and $2\,|n$, respectively.

Corollary 3.4. $D(n) \geq 3$ *if* $n \in S_2 = \{40, 60, 80, 92, 100, 120, 123, 124, 168, 188, 220, 231, 236, 237, 252, 265, 266, 280, 282, 284, 285, 305, 312, 315, 316\}$.

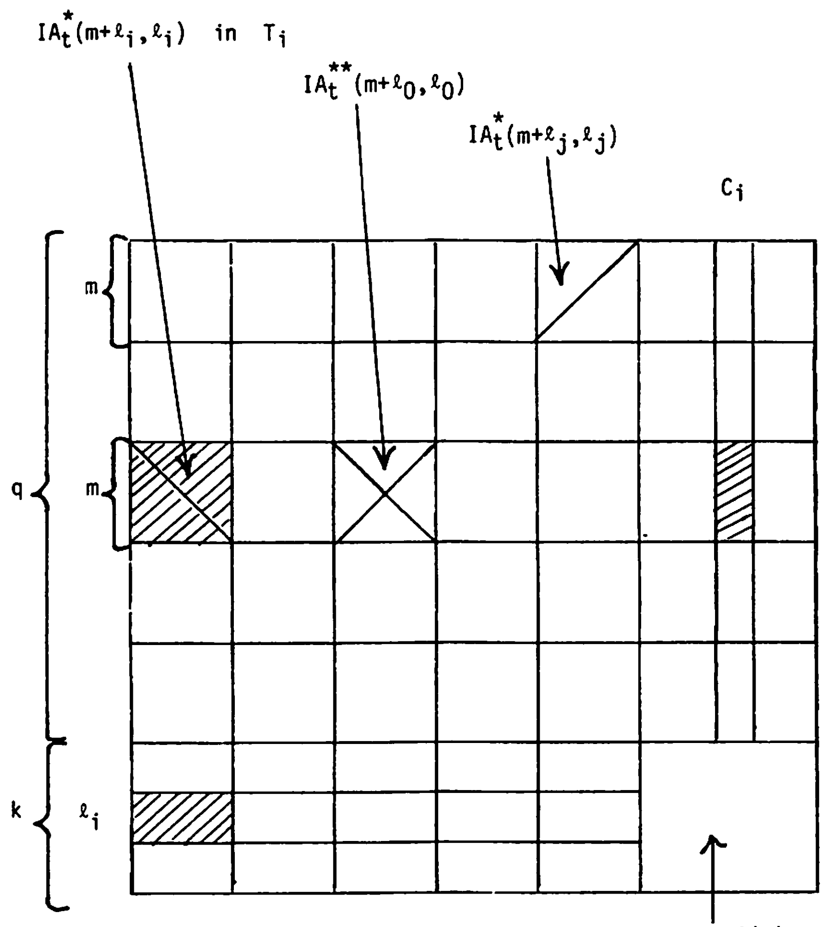

Figure 1

Proof: From [13] we have $D(12) \geq 4$. In Lemma 3.3, it is easy to check from (1.3) with the following expressions that $D(q) \geq 4$. Other conditions are also satisfied from Example 2.1 along with the known fact that $D(50) \geq 4$ in [8] and that $I(21) \geq 3$ and $I(39) \geq 3$ in Lemma 2.6. The following are the expressions.

$$
\begin{aligned}
40 &= 8.4+8, \\
60 &= 12.4+12, \\
80 &= 9.8+8, \\
92 &= 12.7+8, \\
100 &= 12.7+16, \\
120 &= 16.7+8, \\
123 &= 13.8+19, \qquad 19 = 2.9+1, \\
124 &= 16.7+12,
\end{aligned}
$$

168	=	8.20+8,				
188	=	16.11+12,				
220	=	13.16+12,				
231	=	32.7+7,				
236	=	32.7+12,				
237	=	19.12+9,				
252	=	16.15+12,				
265	=	16.16+9,				
266	=	27.8+50,		50	=	2.24+1+1,
280	=	17.16+8,				
282	=	29.8+50,		50	=	2.24+1+1,
284	=	23.12+8,				
285	=	17.16+13,				
305	=	8.37+9,				
312	=	8.38+8,				
315	=	19.16+11,				
316	=	25.12+16.	∎			

Corollary 3.5. $D(n) \geq 3$ *if* $n \in S_3 = \{183, 314\}$.

Proof: We intend to use Lemma 3.3 for $183 = 8.22 + 7$. The only trouble is the lack of $IA_3^*(22+0,0)$. Notice that the six PODLS(8), $L_k = \xi_k\xi_i + \xi_j, \xi_k \in GF(8)\backslash\{0,1\}$, have the orthogonal mate $L_1 = \xi_i + \xi_j$ which is symmetric. Then we can permute rows and columns simultaneously to make the orthogonal mate having constant in either main diagonal or back diagonal, so that the six PODLS(8) have two common transversals in both main and back diagonals and other six disjoint common transversals elsewhere. It is easy to see that only $IA_3(m + \ell_i, \ell_i)$, instead of $IA_3^*(m + \ell_i, \ell_i)$, is needed for these six common transversals. From Lemma 2.7 we know $N(22) \geq 3$, thus from Lemma 3.3 we have $D(183) \geq 3$.

From [2] we know that 4PODLS(12) have the same property as six PODLS(8) mentioned above. Write $314 = 12.22 + 50, 50 = 7.7 + 1$. Since an $IA_3^*(22+7,7)$ exists from Lemma 2.4 and $N(22) \geq 3$, then $D(314) \geq 3$. ∎

Lemma 3.6. $D(n) \geq 3$ *for* $n \in S_4 = \{238, 262\}$.

Proof: Write $238 = 7.33 + 7$ and $262 = 7.37 + 3$. Since $D(40) \geq 3$ from Corollary 3.4 and there are $IA_3^*(33+7,7)$ and $IA_3^*(37+3,3)$ from Lemma 2.4 and Example 2.2, then the conclusion follows from Lemma 3.1. ∎

We also have some PODLS analoques to the POLS constructions of Lemma 2.3 and Lemma 2.4 in [15] like Lemma 3.3 to the Lemma 2.6 in [15].

Lemma 3.7. *Suppose* $2 \nmid q$ *and there are* $t + k$ *POLS*(q) *such that* t *of them are* *tPODLS*(q). *Suppose there are* $IA_t^*(m+\ell_i,\ell_i)$, $1 \le i \le k$, $\ell_1+\ldots+\ell_k = W$. *Then* $\min\{I(m), D(m+w)\} \ge t$ *implies* $D(qm+w) \ge t$.

Proof: The central element in each of the k extra order q orthogonal mates determines a common transversal in the tPODLS(q). For each of such k transversals intersecting in the central cell, fill its cells with an $IA_t^*(m+\ell_i,\ell_i)$, $1 \le i \le k$, but leave the central cell empty. Fill other cells with $I(m)$. Notice that the cells in the back diagonal of tPODLS(q) are filled with some modified $IA_t^*(m+\ell_i,\ell_i)$ or $I(m)$ whose back diagonal of the order m subarray is occupied by different elements. Label the elements and get the right and lower parts as we did in Lemma 3.3. Then we get an $IA_t^{**}(qm+w, m+w)$ shown in Figure 2. To get the tPODLS$(qm+w)$ we make a size $m+w$ hole in the center by permuting rows and columns and fill the hole with the given tPODLS$(m+w)$. The proof is complete. ∎

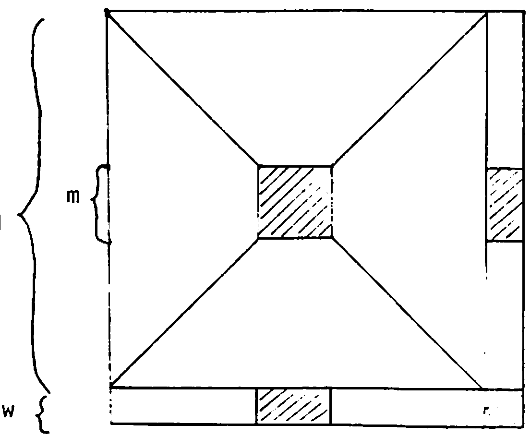

Figure 2

Corollary 3.8. $D(n) \ge 3$ *for* $n \in S_5 = \{39, 48, 54, 55, 82, 126, 138, 155, 180, 214, 215, 235\}$.

Proof: The conclusion comes from Lemma 3.7 and the following expressions,

39	=	9.4+3,	138	=	19.7+5,
48	=	11.4+4,	155	=	19.8+3,
54	=	7.7+(2+2+1),	180	=	25.7+5,
55	=	13.4+3,	214	=	19.11+5,
82	=	11.7+5,	215	=	53.4+3,
126	=	11.11+5,	235	=	29.8+3.

where we have some trouble with $m = 4$. In this case, let $GF(q) = \{0, \pm a, \pm b, \ldots\}$ and label the rows and columns with the order $a, b, \ldots, 0, \ldots, -b-a$, we get $q-1$ POLS(q) as follows: $L_\lambda = \lambda\xi + \eta$, $\lambda, \xi, \eta \in$ GF(q) and $\lambda \neq 0$. Then L_1

and L_{-1} as two extra orthogonal squares will determine two common transversals down the back and main diagonals. So instead of the condition $I(4) \geq 3$ we need only $N(4) \geq 3$ for the other common transversals, thus the Lemma 3.7 still works for the case $m = 4$. This completes the proof. ∎

Lemma 3.9. *Suppose* $2 \not| q$ *and there are* $t + w + 1$ *POLS*(q) *such that* t *of them are* t*PODLS*(q). *Suppose there exist* $IA_t^*(m + h_i, h_i)$ *and* $IA_t^*(m + 1 + h_i, h_i)$, $1 \leq i \leq q - 1$, $h_1 + \ldots + h_{q-1} = h$. *Then* $\min\{I(m), D(m + w), D(h)\} \geq t$ *implies* $D(qm + w + h) \geq t$, *provided* $2 \mid m + w$ *or* $2 \mid h$.

Proof: First do the construction as in Lemma 3.7 with $\ell_i = 1$, $1 \leq i \leq w$. After getting the Figure 2 do the Construction 3.3 again with $h_0 = 0$. For the cells which are filled with $IA_t^*(m + 1, 1)$ in the first step, we should use $IA_t^*(m + 1 + h_i, h_i)$, then we get an array shown in Figure 3. Since $2 \mid m + w$ or $2 \mid h$ we can get some tPODLS$(qm + w + h)$ from the array in Figure 3. ∎

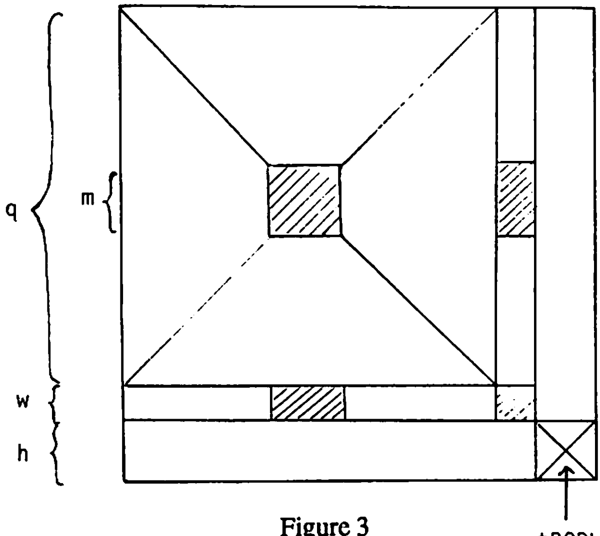

Figure 3

Corollary 3.10. $D(n) \geq 3$ *for* $n \in S_6 = \{58, 62, 86, 87, 90, 98, 130, 142, 150, 154, 158, 170, 174, 178, 182, 186, 194, 198, 202, 206, 222, 226, 230, 234, 310, 318, 334, 358, 366, 374, 382, 422, 430\}$.

Proof: Since $IA_3^*(7 + 2, 2)$ and $IA_3^*(7 + 1 + 2, 2)$ exist from Examples 2.3 and 2.1, the conclusion comes from Lemma 3.9 and the following expressions.

58	=	7.7+1+8,	198	=	27.7+1+8,
62	=	7.7+1+12,	202	=	23.7+1+40,
86	=	11.7+1+8,	206	=	27.7+1+16,
87	=	11.7+2+8,	222	=	27.7+1+32,
90	=	11.7+1+12,	226	=	31.7+1+8,

98	=	11.7+5+16,	230	=	27.7+1+40,
130	=	11.11+1+8,	234	=	31.7+1+16,
142	=	19.7+1+8,	270	=	23.11+1+16,
150	=	19.7+1+16,	310	=	43.7+1+8,
154	=	19.7+5+16,	318	=	43.7+1+16,
158	=	19.7+9+16,	334	=	43.7+1+32,
170	=	23.7+1+8,	358	=	43.7+1+56,
174	=	11.15+1+8,	366	=	43.7+1+64,
178	=	23.7+1+16,	374	=	43.7+1+72,
182	=	23.7+5+16,	382	=	43.7+1+80,
186	=	23.7+9+16,	422	=	59.7+1+8,
194	=	23.7+1+32,	430	=	59.7+1+16. ∎

Corollary 3.11. *$D(n) \geq 3$ for $n \in S_7 = \{246, 278, 390\}$.*

Proof: Write $246 = 12.16 + 54$, $54 = 5.10 + 3.1 + 1$. From Corollary 3.8, $D(54) \geq 3$. From Example 2.2 there is an $IA_3^*(16+5,5)$ and an $IA_3^*(16+3,3)$. Then we have from Lemma 3.3 that $D(246) \geq 3$. Since $278 = 32.7 + 54$ and $54 = 2.27$, $D(279) \geq 3$ from Lemma 3.3. Write $390 = 41.8 + 62$, $62 = 2.31$. From Corollary 3.10 $D(62) \geq 3$. From Example 2.1 there is an $IA_3^*(8+2,2)$. Then we have from Lemma 3.3 that $D(390) \geq 3$. ∎

4. Existence of 3PODLS(n).

To give the existence of 4PODLS(n) we first consider the different residue classes of n modulo 8.

Lemma 4.1. *$D(n) \geq 3$ for all $n = 8m + k$, $m \geq 0$ and $k = 1, 7, 8$ except possibly $n \in P_1 = \{15, 24, 33\}$.*

Proof: In Lemma 3.3, let $q = 8$, $\ell_i = 0$ or 1. From Lemma 2.7 we know that $I(m) \geq 3$ if $m \notin \{2,3,4,6,10,14,18,20,22,24,26,28,30,34,38\}$. If $I(m) \geq 3$ and $I(m+1) \geq 3$, then from Lemma 3.3 we have $D(8m+k) \geq 3$.

But for $m \in \{9,13,17,19,23,25,27,29,37\}$ we can write $8m + k$ as $m8 + k$ and use the same Lemma to show that $D(8m + k) \geq 3$. Now we need only consider 20 remaining cases of m which are listed in Table 1, where "?" indicates an unknown order, "$\star$" indicates $D(n) \geq 3$ since n is usually a prime power, "$\star\star$" indicates $D(n) \geq 3$ since $n = ab$ with $D(a) \geq 3$ and $D(b) \geq 3$, "Si" indicates that $D(n) \geq 3$ since $n \in S_i$.

Table 1

m		$8m+1$		$8m+7$		$8m+8$
0	$\star$	1	$\star$	7	$\star$	8
1	$\star$	9	?	15	$\star$	16
2	$\star$	17	$\star$	23	?	24
3	$\star$	25	$\star$	31	$\star$	32
4	?	33	S5	39	S2	40
5	$\star$	41	$\star$	47	S5	48
6	$\star$	49	S5	55	$\star\star$	56 = 7.8
10	$\star$	81	S6	87	$\star\star$	88 = 11.8
14	$\star$	113	$\star\star$	119 = 7.17	S2	120
18	S1	145	$\star$	151	$\star\star$	152 = 8.19
20	$\star\star$	161 = 7.23	$\star$	167	S2	168
21	$\star$	169	$\star\star$	175 = 7.25	$\star\star$	176 = 16.11
22	S1	177	S3	183	$\star\star$	184 = 23.8
24	$\star$	193	$\star$	199	$\star\star$	200 = 25.8
26	$\star\star$	209 = 11.19	S5	215	$\star\star$	216 = 27.8
28	$\star\star$	225 = 9.25	S2	231	$\star\star$	232 = 29.8
30	$\star$	241	$\star\star$	247 = 13.19	$\star\star$	248 = 31.8
33	S2	265	$\star$	271	$\star\star$	272 = 16.17
34	S1	273	$\star\star$	279 = 9.31	S2	280
38	S2	305	$\star$	311	S2	312 ∎

Lemma 4.2. *$D(n) \geq 3$ for all $n = 8m + k$, $m \geq 0$ and $k = 11, 12, 13$, except possibly $n \in P_2 = \{20, 21, 28, 35, 36, 44, 45, 52\}$.*

Proof: The proof is similar to that in Lemma 4.1, but we have to consider the case $m = 9$ separately. For these exceptional m we have the following table.

Table 2

m		$8m+11$			$8m+12$			$8m+13$	
0	$\star$	11		$\star$	12		$\star$	13	
1	$\star$	19		?	20		?	21	
2	$\star$	27		?	28		$\star$	29	
3	?	35		?	36		$\star$	37	
4	$\star$	43		?	44		?	45	
5	S1	51		?	52		$\star$	53	
6	$\star$	59		S2	60		$\star$	61	
9	$\star$	83		$\star\star$	84	= 7.12	S1	85	
10	$\star\star$	91	= 7.13	S2	92		S1	93	
14	S2	123		S2	124		$\star$	125	
18	S5	155		$\star\star$	156	= 13.12	$\star$	157	
20	$\star\star$	171	= 9.19	S1	172		$\star$	173	
21	$\star$	179		S5	180		$\star$	181	
22	$\star\star$	187	= 11.17	S2	188		$\star\star$	189	= 7.27
24	$\star\star$	203	= 7.29	$\star\star$	204	= 17.12	S1	205	
26	S1	219		S2	220		$\star\star$	221	= 13.17
28	S5	235		S2	236		S2	237	
30	$\star$	251		S2	252		$\star\star$	253	= 11.23
33	$\star\star$	275	= 11.25	$\star\star$	276	= 12.23	$\star$	277	
34	$\star$	283		S2	284		S2	285	
38	S2	315		S2	316		$\star$	317	∎

Lemma 4.3. *$D(n) \geq 3$ for all $n = 8m + 50$, $m \geq 0$, except possibly $n \in P_3 = \{66, 74, 114\}$.*

Proof: In Lemma 3.3, let $q = 8$. The proof of Corollary 3.5 shows that there are 4POLS(8) containing 3PODLS(8) which have two common transversals down the main diagonal and back diagonal. For these diagonals we use $IA_3^*(m+1,1)$, while for the other six common transversals use only $IA_3(m+8,8)$. From Lemma 2.5 an $IA_3(m+8,8)$ exists if $m \geq 24$ and $m \neq 38$. From Lemma 2.6 $I(m+1) \geq 3$, then an $IA_3^*(m+1,1)$ exists, if $m \notin \{1,2,3,5,9,13,17,19,21,23,25,27,29,33,37\}$. So $D(8m+50) \geq 3$ for all $m \geq 24$ and $m \neq 25,27,29,33,37,38$. For the remaining m we have the following table.

Table 3

m	$8m + 50$				
0	S1	50	15	S6	170
1	S6	58	16	S6	178
2	?	66	17	S6	186
3	?	74	18	S6	194
4	S5	82	19	S6	202
5	S6	90	20	S1	210
6	S6	98	21	S1	218
7	S1	106	22	S6	226
8	?	114	23	S6	234
9	S1	122	25	S1	250
10	S6	130	27	S2	266
11	S5	138	29	S2	282
12	S1	146	33	S3	314
13	S6	154	37	S1	346
14	S1	162	38	S1	354

∎

Lemma 4.4. *$D(n) \geq 3$ for all $n = 8m + 62$, $m \geq 0$, except possibly $n \in P_4 =$* $\{70, 102, 110, 118\}$.

Proof: In Lemma 3.3, let $q = 8$. As the above lemma we need 3PODLS(8) with 8 disjoint common transversals two of which are the main and back diagonals. We use $IA_3^*(m + 7, 7)$ for the diagonals and $IA_3(m + 8, 8)$ for other transversals. Then from Lemma 2.4 and 2.5 we know that $D(8m+62) \geq 3$ for all $m \geq 24$ and $m \neq 24, 25, 27, 30, 31, 32, 34, 37, 38, 39, 40, 41, 45, 46$. For the remaining m we have the following table.

Table 4

m	$8m + 62$				
0	S6	62	19	S5	214
1	?	70	20	S6	222
2	S1	78	21	S6	230
3	S6	86	22	S4	238
4	S1	94	23	S7	246
5	?	102	24	S1	254
6	?	110	25	S4	262
7	?	118	27	S7	278
8	S5	126	30	S1	302
9	S1	134	31	S6	310
10	S6	142	32	S6	318
11	S6	150	34	S6	334
12	S6	158	37	S6	358
13	S1	166	38	S6	366

14	S6	174	39	S6	374
15	S6	182	40	S6	382
16	S1	190	41	S7	390
17	S6	198	45	S6	422
18	S6	206	46	S6	430

From Lemma 4.1 - 4.4, we have considered all the positive integers except $n = 2,3,4,5,6,10,14,18,22,26,30,34,38,42,46,54$. From (1.3) there do not exist 3PODLS(n) for $n = 2,3,4$ and 5. The nonexistence of 3PODLS(6) is obvious since $N(6) = 1$. For the others the existence is unknown except that $54 \in S_5$. Summarizing all these results we have: ∎

Theorem 4.5. *There exist three pairwise orthogonal diagonal Latin squares of every order n where $n > 118$. Orders $2,3,4,5$ and 6 are impossible; the only orders for which the existence is undecided are:*

10, 14, 15, 18, 20, 21, 22, 24, 26, 28,
30, 33, 34, 35, 36, 38, 42, 44, 45, 46,
52, 66, 70, 74, 102, 110, 114, 118.

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