A NOTE ON GROUP LATIN SQUARES

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Let H be a group of order n written multiplicatively, and let $\underline{h} = (h_1, h_2, \ldots, h_n)$ and $\underline{k} = (k_1, k_2, \ldots, k_n)$ be arrangements of the elements of H. Then we can form a Latin square $L(\underline{h}, \underline{k})$ of side n whose (i, j) entry is $h_i k_j$. We say that $L(\underline{h}, \underline{k})$ and any Latin square obtained from $L(\underline{h}, \underline{k})$ by permutation of the symbols is a *group Latin square based on the group* H.

A Latin square L of side n may be considered as a set of n^2 cells with three block systems upon it (namely, those corresponding to rows, columns and symbols), such that any two blocks from different systems contain exactly one cell in common. The *automorphism group of* L is then defined as the group of all permutations of the n^2 cells of L which preserve the three block systems. The automorphism group of a Latin square L based on a group H of order n has a subgroup isomorphic to $H \times H$ acting regularly on the n^2 cells of L, and such that one of the direct factors fixes each row and acts regularly on the columns (and on the symbols) while the other direct factor fixes each column and acts regularly on the rows (and on the symbols), (see Bailey (1982) for example). We show here that if a Latin square L admits a group H of automorphisms such that H fixes each row and is regular on the columns, then L is a group Latin square based on H:

Theorem. *Let L be a Latin square of side n which admits a group H of automorphisms such that H fixes each row of L and H acts regularly on the columns of L. Then L is a group Latin square based on H.*

This is a contribution to the problem of identifying Latin squares by their automorphism groups. Bailey (1982) showed that the only Latin squares with automorphism group having as little as 4 orbits on unordered pairs of cells were those based on an elementary abelian 2-group (not $\mathbf{Z}_2$) or on $\mathbf{Z}_3$. Further, Bailey (1987) asked whether any Latin square whose automorphism group is transitive on cells must be a group Latin square. We note that, for such a square, the subgroup H of automorphisms fixing each row acts semiregularly on the columns and on the symbols (that is, only the identity of H fixes a column or a symbol). Thus, the result of this paper gives an affirmative answer to the question when this group H is regular on columns. The question in general remains unanswered.

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Proof of the Theorem: Let L, H satisfy the hypotheses of the theorem and let R, C, S be the set of rows, columns, symbols of L, respectively. We shall label R, C and S with elements of H in such a way that the symbol in row h_1 and column h_2 is symbol $h_1 h_2$, for each $h_1, h_2 \in H$. We represent the cell in row i, column j and symbol k by the triple (i, j, k).

Choose a cell. Label its row, column, and symbol with the identity 1 of H; thus this cell is labelled as $(1, 1, 1)$. Now for each $h \in H$ label the image of column 1 under h as column h and the image of symbol 1 under h as symbol h. Thus, since H is regular on columns and symbols, we have labelled C and S with elements of H, and we have labelled the cells in row 1 by triples $(1, h, h)$ for $h \in H$.

Now we label the rows as follows: in column 1, for each $h \in H$, find the cell containing the symbol h and label the row of this cell as row h. Then we have labelled the cells of column 1 by the triples $(h, 1, h)$ for $h \in H$.

Let ℓ_{hk} be the symbol in row h, and column k; consider the cell (h, k, ℓ_{hk}). Since according to our labelling H fixes all rows and acts on columns and symbols by right multiplication, the image of the cell (h, k, ℓ_{hk}) under $k^{-1} \in H$ is the cell $(h, 1, \ell_{hk}k^{-1})$. However, the cell in row h and column 1 is $(h, 1, h)$ and so $\ell_{hk}k^{-1} = h$, that is $\ell_{hk} = hk$. Thus, L is a group Latin square based on H.

References

1. R.A. Bailey, J. Austral. Math. Soc. (A) 33 (1982), 18-22.
2. R.A. Bailey. (1987), private communication.