PERFECT MENDELSOHN DESIGNS

L. Zhu

Department of Mathematics
Suzhou University
Suzhou, CHINA

Abstract. Let v, k and λ be positive integers. A perfect Mendelsohn design with parameters v, k and λ, denoted by (v, k, λ)-PMD, is a decomposition of the complete directed multigraph λK_v^* on v vertices into k-circuits such that for any r, $1 \le r \le k-1$, and for any two distinct vertices x and y there are exactly λ circuits along which the (directed) distance from x to y is r. In this survey paper, we describe various known constructions, new results and some further questions on PMDs.

1. Introduction

A set of k distinct elements $\{a_1, a_2, \ldots, a_k\}$ is said to be cyclically ordered by $a_1 < a_2 < \ldots < a_k < a_1$ and two elements a_i, a_{i+t} are said to be *t-apart* in a cyclic k-tuple $(a_1, a_2, \ldots, a_k)$ where $i + t$ is taken modulo k.

Let v, k and λ be positive integers. A (v, k, λ)-Mendelsohn design (briefly (v, k, λ)-MD) is a pair $(X, \mathcal{B})$ where X is a v-set (of *points*) and $\mathcal{B}$ is a collection of cyclically ordered k-subsets of X (called *blocks*) such that every ordered pair of points of X appears consecutively in exactly λ blocks of $\mathcal{B}$. The (v, k, λ)-MD is called *r-fold perfect* if each ordered pair of points of X appears t-apart in exactly λ blocks for all $t = 1, 2, \ldots, r$. A $(k-1)$-fold perfect (v, k, λ)-MD is called *perfect* and is denoted briefly by (v, k, λ)-PMD.

In graph notation, a (v, k, λ)-MD is equivalent to the decomposition of the complete directed multigraph λK_v^* on v vertices into k-circuits. A (v, k, λ)-PMD is equivalent to the decomposition of λK_v^* into k-circuits such that for any r, $1 \le r \le k-1$, and for any two distinct vertices x and y there are exactly λ circuits along which the (directed) distance from x to y is r.

If we ignore the cyclic order of the elements in blocks, a (v, k, λ)-PMD becomes a $(v, k, \lambda(k-1))$-BIBD. Therefore, we can consider perfect Mendelsohn designs as a generalization of balanced incomplete block designs. It was N.S. Mendelsohn who first introduced the cyclic order of the elements into blocks and discussed the existence question of a (v, k, λ)-PMD (see, **[16, 18]**), which was originally called perfect cyclic design.

Since the complete directed multigraph λK_v^* contains $\lambda v(v-1)$ arcs and each block as a circuit contains k arcs, it is easy to see that the number of blocks in a (v, k, λ)-PMD is

$$\lambda v(v-1)/k.$$

This leads to an obvious necessary condition for the existence of a (v, k, λ)-PMD, that is,

$$\lambda v(v-1) \equiv 0 \pmod{k}. \tag{1.1}$$

This condition is not always sufficient, for example, an exhaustive enumeration shows [16] that no $(6,3,1)$-PMD can exist.

In this paper, we shall survey the recent results (an earlier survey can be found in [3]) for the existence of (v,k,λ)-PMD, some old and new constructions, and some further questions.

2. Further necessary conditions from orthogonal array

An *orthogonal array* $OA(k,v^2)$ is a $v^2 \times k$ array whose entries come from a v-set X, and such that for any pair of columns every ordered pair of elements of X (not necessarily distinct) appear in the same row exactly once.

The connection between $(v,k,1)$-PMD and $OA(k,v^2)$ is described in [6].

Theorem 2.1. *If a $(v,k,1)$-PMD exists, then also an $OA(k,v^2)$ exists.*

Proof: Suppose the PMD is based on the symbols $1,2,\ldots,v$. Form the array with k columns as follows. Take the first v rows to be

$$\begin{matrix} 1 & 1 & 1 & \ldots & 1 \\ 2 & 2 & 2 & \ldots & 2 \\ \vdots & & & & \\ v & v & v & \ldots & v. \end{matrix}$$

The array is completed by taking each k-cyclic block and use it to fill k rows of the array by writing down all its cyclic permutations. It is readily verified that the $v^2 \times k$ array is an $OA(k,v^2)$. ∎

It is well known (see, for example, [12]) that the existence of an $OA(k,v^2)$ is equivalent to the existence of $k-2$ mutually orthogonal Latin squares (MOLS) of order v. We then have the following.

Corollary 2.2. *If there exists a $(v,k,1)$-PMD, then there exist $k-2$ MOLS of order v.*

The non-existence of two MOLS of order six has been known for a long time (see [22]), a short proof can be found in [20]. We then have

Corollary 2.3. *There does not exist a $(6,5,1)$-PMD, nor a $(6,6,1)$-PMD.*

3. Constructions using finite fields and groups

The following two theorems were found by N.S. Mendelsohn in [18] under the language of quasigroups. We give the direct proof here.

Theorem 3.1. *Let p be an odd prime. Then there exists a $(p,p,1)$-PMD.*

Proof: Let $X = GF(p) = \{0,1,2,\ldots,p-1\}$. Let $\mathcal{B} = \{(0,j,2j,\ldots,(p-1)j) \mid j \in GF(p)\backslash\{0\}\}$. Then, $(X,\mathcal{B})$ is the required $(p,p,1)$-PMD. In fact, for any two elements $x, y \in X$ and $x \neq y$, and for any $1 \leq t \leq p-1$, the equations

$$x = ij,$$
$$y = (i+t)j,$$

have the unique solution, $j = (y-x)/t$ and $i = xt/(y-x)$. ∎

Theorem 3.2. *Let $v = p^r$ be any prime power and $k > 2$ be such that k is a divisor of $v-1$, then there exists a $(v, k, 1)$-PMD.*

Proof: Let $X = GF(v)$. For any $\lambda, u, z \in GF(v)$ satisfying $u \neq z, \lambda \neq 1$ and $\lambda^k = 1$, form a block $(u, \lambda u + (1-\lambda)z, \lambda^2 u + (1-\lambda^2)z, \ldots, \lambda^{k-1}u + (1-\lambda^{k-1})z)$. For any element u' in this block, the triple u', z, λ will determine the same (cyclic) block. It is easy to see that the elements in the block are indeed distinct, therefore, for fixed z and λ, we can obtain $(v-1)/k$ blocks. And for fixed λ, we can obtain $v(v-1)/k$ blocks, which we denote by $\mathcal{B}$. Then, $(X, \mathcal{B})$ is the required $(v, k, 1)$-PMD. In fact, for any two distinct elements x and y, and for any $1 \leq t \leq k-1$, the equations

$$x = u,$$
$$y = \lambda^t u + (1-\lambda^t)z,$$

have the unique solution $u = x$ and $z = (y - \lambda^t x)/(1-\lambda^t)$, which gives the block containing x and y t-apart. ∎

A direct construction using groups below is a variation of the method using difference sets in the construction of BIBDs (see, for example, [12]). Instead of listing all of the blocks of a design, it suffices to give the group G acting on a set of base blocks. We shall adapt the following notation: dev $\mathcal{B} = \{B + g \mid B \in \mathcal{B}$ and $g \in G\}$, where $\mathcal{B}$ is the collection of base blocks of the design.

The following results come from the recent work [9], where the groups are the additive groups of some finite fields.

Theorem 3.3. *If v is a prime power and $v \geq k$, then there exists a (v, k, k)-PMD.*

Proof: Take $X = GF(v)$ and $\mathcal{B} = \{(0, \xi^i, \xi^{i+1}, \ldots, \xi^{i+k-2}) \mid 1 \leq i \leq v-1\}$, where ξ is a primitive element of $GF(v)$. It is easy to check that $(X, \text{dev } \mathcal{B})$ is the required (v, k, k)-PMD. ∎

Theorem 3.4. *If $q = kn+1$ is a prime power, then there exists a $(q+1, k, k)$-PMD.*

Proof: Suppose ξ is a primitive element of $GF(q)$. Let $X = GF(q) \cup \{\infty\}$. Let $\mathcal{B}$ consist of the following blocks:

$$\begin{aligned}
&(z, z\xi^n, z\xi^{2n}, \ldots, z\xi^{(k-1)n}), \\
&\qquad\qquad z \in GF(q)\backslash\{0, 1, \xi^n, \ldots, \xi^{(k-1)n}\}, \\
&(1, \xi^n, \xi^{2n}, \ldots, \xi^{(k-1)n}), \quad \text{twice}, \\
&(\infty, z, z^{\xi n}, \ldots, z\xi^{(k-2)n}), \\
&\qquad\qquad z \in \{1, \xi^n, \ldots, \xi^{(k-1)n}\}.
\end{aligned}$$

It is checked that $(X, \text{dev } \mathcal{B})$ is the required $(q+1, k, k)$-PMD. ∎

4. Constructions using pairwise balanced designs

Let K be a set of positive integers. A *pairwise balanced design* (PBD) of index λ $B(K, \lambda; v)$ is a pair $(X, \mathcal{B})$ where X is a v-set (of points) and $\mathcal{B}$ is a collection of subsets of X (called blocks) with sizes from K such that every pair of distinct points of X is contained in exactly λ blocks of $\mathcal{B}$. We shall denote by $B(K, \lambda)$ the set of all integers v for which there exists a PBD $B(K, \lambda; v)$. A $B(K, 1)$ will be denoted simply by $B(K)$. A PBD $B(\{k\}, \lambda; v)$ is essentially a *balanced incomplete block design* (BIBD) with parameters v, k and λ.

The following recursive construction is a general form of Theorem 2.9 in [**18**].

Theorem 4.1. *Let v, k and λ_1 and λ_2 be positive integers. Suppose there exists a PBD $B(\{k_1, k_2, \ldots, k_r\}, \lambda_1; v)$ and for each k_i there exists a (k_i, k, λ_2)-PMD, then there exists a $(v, k, \lambda_1\lambda_2)$-PMD.*

Proof: Let $(X, \mathcal{A})$ be the PBD. For any block $A \in \mathcal{A}$ construct on A a $(|A|, k, \lambda_2)$-PMD $(A, \mathcal{B}_A)$. Then $(X, \cup_{A \in \mathcal{A}} \mathcal{B}_A)$ is the required $(v, k, \lambda_1 \lambda_2)$-PMD. ∎

Since for any prime p and any integer $r \geq 1$ there exists a PBD $B(\{p\}, 1; p^r)$ which is indeed an affine space, we then apply Theorem 4.1 with the $(p, p, 1)$-PMD from Theorem 3.1 to obtain a $(p^r, p, 1)$-PMD. Therefore, Theorem 3.1 can be strengthened as follows.

Theorem 4.2. *Let p be an odd prime and $r \geq 1$ be an integer. Then there exists a $(p^r, p, 1)$-PMD.*

Using the theory of PBD closed set of R. M. Wilson, some asymptotic results were established in [**6, 18**].

Theorem 4.3. *A $(v, k, 1)$-PMD exists for all sufficiently large v with $k \geq 3$ and $v \equiv 1 \pmod{k}$.*

Theorem 4.4. *A $(v, k, 1)$-PMD exists with $v(v-1) \equiv 0 \pmod{k}$ for the case when k is an odd prime and v is sufficiently large.*

We remark that the term "sufficiently large" in Theorem 4.3 and Theorem 4.4 is unspecified and the problem of finding a concrete bound for v in both cases remains to be solved.

5. Construction by filling in holes

We denote by $K_{n_1,n_2,\dots,n_h}$ the complete multipartite directed graph with vertex set $X = \cup_{1\le i\le h} X_i$, where X_i $(1 \le i \le h)$ are disjoint sets with $|X_i| = n_i$, $v = \sum_{1\le i\le h} n_i$, and where two vertices x and y from different sets X_i and X_j are joined by exactly two arcs (x, y) and (y, x).

If $K_{n_1,n_2,\dots,n_h}$ can be decomposed into k-circuits such that for any r, $1 \le r \le k-1$, and for any two vertices x and y from different sets X_i and X_j, there is exactly one circuit along which the (directed) distance from x to y is r, we call $(X, \mathcal{B})$ a *holey perfect Mendelsohn design*, where $\mathcal{B}$ is the collection of all circuits. We denote the design by $(v, k, 1)$-HPMD. The set X_i $(1 \le i \le h)$ is called a *hole* and the vector $(n_1, n_2, \dots, n_h)$ is called the *type* of the HPMD.

A $(v, k, 1)$-HPMD of type $(1, 1, \dots, 1, n)$ is called an *incomplete perfect Mendelsohn design*, denoted by $(v, n, k, 1)$-IPMD. It is easy to see that a $(v, k, 1)$-PMD is indeed a $(v, k, 1)$-HPMD of type $(1, 1, \dots, 1)$. We can construct PMD from IPMD by filling in hole.

Theorem 5.1. *If there exist both $(v, n, k, 1)$-IPMD and $(n, k, 1)$-PMD, then there exists a $(v, k, 1)$-PMD.*

Proof: Let $(X, Y, \mathcal{B})$ be the $(v, n, k, 1)$-IPMD where Y is the hole of size n. Let $(Y, \mathcal{B}_0)$ be the $(n, k, 1)$-PMD. Then, $(X, \mathcal{B} \cup \mathcal{B}_0)$ is the required $(v, k, 1)$-PMD. ∎

Example 5.2: Let $G = Z_{14}$ be the residue classes of integers modulo 14. Let $X = Z_{14} \cup \{\infty_1, \infty_2, \infty_3, \infty_4\}$ and $\mathcal{B} = \{(0, 1, 4), (0, 4, 1), (0, 5, 7), (\infty_1, 0, 6), (\infty_2, 0, 8), (\infty_3, 0, 9), (\infty_4, 0, 12)\}$. It is readily checked that $(X, \{\infty_1, \infty_2, \infty_3, \infty_4\}, \text{dev } \mathcal{B})$ is a $(18, 4, 3, 1)$-IPMD. We also have a $(4, 3, 1)$-PMD from Theorem 3.2. Applying Theorem 5.1 we obtain a $(18, 3, 1)$-PMD.

Theorem 5.3. *If there exists a $(v, k, 1)$-HPMD of type $(n_1, n_2, \dots, n_h)$ and an $(n_i + m, m, k, 1)$-IPMD for $2 \le i \le h$, then there exists a $(v + m, n_1 + m, k, 1)$-IPMD. Moreover, if there exists an $(n_1 + m, m, k, 1)$-IPMD, then there exists a $(v + m, m, k, 1)$-IPMD.*

Proof: Let $(X, \mathcal{B})$ be the given HPMD. X is partitioned into $X_1, X_2, \dots, X_h$, $|Y| = m$, $Y \cap X = \phi$. Let $(X_i \cup Y, Y, \mathcal{B}_i)$ be the given IPMD for $2 \le i \le h$. Then $(X \cup Y, X_1 \cup Y, (\cup_{2\le i\le h} \mathcal{B}_i) \cup \mathcal{B})$ is the required $(v + m, n_1 + m, k, 1)$-IPMD. If we further have the IPMD $(X_1 \cup Y, Y, \mathcal{B}_1)$, then $(X \cup Y, Y, (\cup_{1\le i\le h} \mathcal{B}_i) \cup \mathcal{B})$ is a $(v + m, m, k, 1)$-IPMD. ∎

6. Existence of $(v, 3, \lambda)$-PMD

The existence question for $(v, 3, \lambda)$-PMD has been solved in [1] and [16] by using quasigroup composition techniques and some direct constructions. Since PBD construction plays an important role in uniformly solving the existence questions for various kinds of combinatorial designs, we would like to give an alternative proof here for the existence of $(v, 3, \lambda)$-PMD by using PBD constructions. Denote

$$H_3 = \{v \geq 3 \mid v \equiv 0 \text{ or } 1 \pmod 3\},$$
$$H_3^* = \{v \geq 3 \mid v \text{ is an integer}\}.$$

It is proved in [13] that

$$H_3 = B(\{3,4,6\}). \tag{6.1}$$

By a similar proof (6.1) can be strengthened to

$$B(\{3,4,\}) \supset H_3 \backslash \{6,18\}. \tag{6.2}$$

From [12] we have

$$H_3^* = B(\{3,4,5,6,8\}). \tag{6.3}$$

We are now in a position to prove the following.

Theorem 6.1. *A necessary and sufficient condition for the existence of a $(v, 3, \lambda)$-PMD is $\lambda v(v-1) \equiv 0 \pmod 3$, except for the non-existing design $(6, 3, 1)$-PMD.*

Proof: (i) The case $\lambda = 1$. A $(3, 3, 1)$-PMD exists from Theorem 3.1. A $(4, 3, 1)$-PMD exists from Theorem 3.2. Applying Theorem 4.1 with (6.2) we obtain a $(v, 3, 1)$-PMD for any $v \in H_3 \backslash \{6, 18\}$. A $(18, 3, 1)$-PMD is given in Example 5.2. The non-existence of a $(6, 3, 1)$-PMD is mentioned in Section 1.

(ii) The case $\lambda = 2$. Applying Theorem 4.1 with (6.1) we need only to prove the existence of $(v, 3, 2)$-PMD for $v \in \{3, 4, 6\}$. $v = 3$ and 4 can be done by taking repeated blocks in $(3, 3, 1)$-PMD and $(4, 3, 1)$-PMD. A $(6, 3, 2)$-PMD is given as $(X, \text{dev } \mathcal{B})$ where

$$G = Z_5, \; X = Z_5 \cup \{\infty\},$$
$$\mathcal{B} = \{(0,1,2), (0,2,1), (\infty,0,2), (\infty,0,3)\}.$$

(iii) The case $\lambda = 3$. Applying Theorem 4.1 with (6.3) we need only to prove the existence of $(v, 3, 3)$-PMD for $v \in \{3, 4, 5, 6, 8\}$. The case

$v = 3, 4, 5, 8$ can be done by applying Theorem 3.3. A $(6, 3, 3)$-PMD is given as $(X, \text{dev } \mathcal{B})$ where

$$G = Z_5,\ X = \{0, 1, 2, 3, 4, 5\},$$
$$\mathcal{B} = \{(0, 1, 2), (0, 2, 4), (0, 3, 1), (\infty, 0, 2), (\infty, 0, 4), (\infty, 0, 4)\}.$$

(iv) The case $\lambda > 3$. If λ is even, we can use (ii) by taking repeated blocks. Otherwise, we take the blocks in a $(v, 3, 2)$-PMD $(\lambda - 3)/2$ times each, and together with the blocks in a $(v, 3, 3)$-PMD. This completes the proof. ∎

7. Constructions by weighting

In constructions of group divisible designs (GDDs), or PBDs, techniques of giving weight are frequently used (see [23]). It needs a master GDD to start with and also some small GDDs as input designs in order to end up with a new GDD. To generalize this method to end up with an HPMD, we can either start with an HPMD and use some GDDs as input designs, or start with a GDD and use some HPMD as input designs. For construction of the first type, [7] gives the following.

Theorem 7.1. *Suppose there exists a* $TD[k; m]$. *Then*

(1) *there exists a* $(mn, k, 1)$*-HPMD of type* $(mn_1, mn_2, \ldots, mn_h)$ *if there exists an* $(n, k, 1)$*-HPMD of type* $(n_1, n_2, \ldots, n_h)$*; and*
(2) *there exists a* $(mn, k, 1)$*-HPMD of type* $(m, m, \ldots, m)$ *if there exists an* $(n, k, 1)$*-PMD.*

For the second type of construction we have

Theorem 7.2. *Suppose there is a* $GDD[K, 1, M; v]$ *with groups* $G_1, G_2, \ldots, G_h$, *where* $|G_i| = n_i, 1 \le i \le h$. *If for any block size* $u \in K$ *there is a* $(mu, k, 1)$*-HPMD of type* $(m, m, \ldots, m)$, *then there exists a* $(mv, k, 1)$*-HPMD of type* $(mn_1, mn_2, \ldots, mn_h)$.

Proof: Let $(X, \mathcal{G}, \mathcal{A})$ be the given GDD. $I_m = \{1, 2, \ldots, m\}$. Let $A \in \mathcal{A}$ be any block with size $|A| = u$. Construct on $A \times I_m$ a $(mu, k, 1)$-HPMD with holes $\{x\} \times I_m (x \in A)$ and blocks $\mathcal{B}_A$. Then $(X \times I_m, \cup_{A \in \mathcal{A}} \mathcal{B}_A)$ is the required HPMD with holes $G_i \times I_m$ $(1 \le i \le h)$. ∎

8. Constructions using k-difference sequence

Let k be an odd integer. Let $S = (s_0, s_1, \ldots, s_{k-1})$, $s_i \in Z_k$. If for any $r \in Z_k \backslash \{0\}$,

$$\{s_{i+r} - s_i \mid i \in Z_k\} = Z_k,$$

where the sum $i + r$ is calculated in Z_k, we call S a *k-difference sequence.*

We also need the concept of MOLS with holes (see, for example, [21]). Let $P = \{X_1, X_2, \ldots, X_h\}$ be a partition of an n-set X. A *holey Latin square*, having partition P, is an $n \times n$ array L, indexed by X, satisfying the following properties:

(1) a cell of L either contains an element of X or is empty;

(2) the subarray indexed by $X_i \times X_i$ are empty, for $1 \leq i \leq h$ (these subarrays are called *holes*);

(3) the elements occurring in row (or column) $x \in X_i$ of L are precisely those in $X \backslash Xi$. The *type* of L is the vector $(n_1, n_2, \ldots, n_h)$ where $n_i = |X_i|$.

Suppose $L_1, L_2, \ldots, L_t$ are holey Latin squares having partition P. If for any $1 \leq i < j \leq t$ the superposition of L_i and L_j yields every ordered pair in $X^2 \backslash \cup_{1 \leq i \leq h} X_i^2$, we call them t *holey mutually orthogonal Latin squares* of order n with type $(n_1, n_2, \ldots, n_h)$, denoted by t HMOLS(n).

The following construction is proved in [7].

Theorem 8.1. *Let k be an odd prime. Suppose that there exist $k-2$ HMOLS(n) of type $(n_1, n_2, \ldots, n_h)$. Then there exists an $(kn, k, 1)$-HPMD of type $(kn_1, kn_2, \ldots, kn_h)$.*

From Theorem 5.3 and Theorem 8.1 the following corollary is also obtained in [7].

Corollary 8.2. *Let k be an odd prime. Suppose that there exist $k-2$ HMOLS(n) of type $(n_1, n_2, \ldots, n_h)$. Then we have the following constructions.*

(i) *The* ***kn-construction*** *: A $(kn, k, 1)$-PMD exists if a $(kn_i, k, 1)$-PMD exists for all i, $1 \leq i \leq h$; and*

(ii) *The* ***$kn+1$-construction*** *: A $(kn+1, k, 1)$-PMD exists if a $(kn_i+1, k, 1)$-PMD exists for all i, $1 \leq i \leq h$.*

9. Construction from Steiner pentagon systems

A *Steiner pentagon system* (SPS) is a pair (K_n, P) where K_n is the complete undirected graph (based on the set V), P is a collection of pentagons in K_n such that each edge of K_n belongs to exactly one pentagon of P, and each pair of distinct elements of V are joined by a path of length 2 in exactly one pentagon of P. The number n is called the order of the SPS (K_n, P) and, of course, $|P| = n(n-1)/10$.

An observation in [7] shows that the existence of an SPS of order n implies the existence of an $(n, 5, 1)$-PMD. By assigning to each pentagon $\langle a, b, c, d, e \rangle$ of the SPS of order n, the two directed cycles (a, b, c, d, e) and (a, e, d, c, b), these directed cycles form not only a partition of arcs for the complete directed graph, but also a perfect Mendelsohn design. It is proved in [15] that a SPS (K_n, P) exists if and only if $n \equiv 1$ or $5 \pmod{10}$, $n \neq 15$. This implies the following.

Theorem 9.1. *A* $(v, 5, 1)$*-PMD exists for any integer* $v \geq 5$, *and* $v \equiv 1$ *or 5* $\pmod{10}$, *except possibly* $v = 15$.

The connections between SPS and $(v, 5, 1)$-PMD suggest that the existence question for Steiner n-gon system would be of interest not only by itself but also in solving the existence question of $(v, k, 1)$-PMD where k is an odd integer.

10. Some recent results

N. S. Mendelsohn started in [17] the investigation of the existence of $(v, 4, 1)$-PMD noticing that a $(v, 4, 1)$-PMD is equivalent to a quasigroup of order v satisfying certain identities. A partial solution for $v \equiv 1 \pmod 4$ was obtained in Bennett [2]. Zhang [24] discussed the remaining case $v \equiv 0 \pmod 4$. A non-existence of a $(8, 4, 1)$-PMD was pointed out by K. Heinrich through an exhaustive computer search, which is further confirmed by an independent investigation in [10]. The general case for $\lambda > 1$ is also discussed in [10]. Apart from the non-existence of a $(8, 4, 1)$-PMD, the non-existence of a $(4, 4, \lambda)$-PMD for λ odd is also obtained. Combining all these we have an almost complete solution in the following form, leaving only one case of $(12, 4, 1)$-PMD unsettled.

Theorem 10.1. *The necessary condition for the existence of a* $(v, 4, \lambda)$*-PMD, namely,* $\lambda v(v-1) \equiv 0 \pmod 4$, *is also sufficient, except for* $v = 4$ *and* λ *odd,* $v = 8$ *and* $\lambda = 1$, *and possibly excepting* $v = 12$ *and* $\lambda = 1$.

The most recent work on $(v, 5, \lambda)$-PMD [7, 9] involves the constructions by filling in holes, by weighting and by k-difference sequence. As we mentioned in Corollary 2.3, there does not exist a $(6, 5, 1)$-PMD. An almost complete solution for $(v, 5, 1)$-PMD leaves 21 possible exceptions of v. And for $(v, 5, \lambda)$-PMD we have 7 unknown cases left all for $\lambda = 5$. The following results are obtained in [7, 9].

Theorem 10.2. *The necessary condition for the existence of a* $(v, 5, \lambda)$*-PMD, namely,* $\lambda v(v-1) \equiv 0 \pmod 5$ *is also sufficient, except for* $v = 6$ *and* $\lambda = 1$, *and the possible exceptions of* (v, λ) *where* $\lambda = 1$ *and* $v \in \{10, 15, 20, 26, 30, 36, 46, 50, 56, 66, 86, 90, 110, 126, 130, 140, 146, 186, 206, 246, 286\}$, *and* $\lambda = 5$ *and* $v \in \{14, 18, 22, 24, 28, 34, 39\}$.

For $v = 7$, it is also pointed out in [7, Theorem 6.2] that for any integer $v \geq 539$ where $v \equiv 0$ or $1 \pmod 7$, there exists an $(v, 7, 1)$-PMD. Some work on $(v, 7, 1)$-PMD by using PBDs is also done in [5].

11. Some further questions

(1) **PMD.** As we mentioned above the existence of (v, k, λ)-PMD has been solved completely for $k = 3$. An almost complete solution is also obtained for $k = 4$ and 5. For block size four the existence of a $(12, 4, 1)$-PMD is the only

one in doubt. For block size five, there are 21 possible exceptions when $\lambda = 1$ and 7 possible exceptions when $\lambda = 5$. It would be nice to solve these cases before working on larger block sizes.

The case when $k = 6$ is more interesting. In this case, the necessary condition for the existence of a $(v, 6, 1)$-PMD, that is, $v(v-1) \equiv 0 \pmod{6}$, gives four residue classes $v \equiv 0, 1, 3, 4 \pmod{6}$. For the last two residue classes, we don't have even one single example so far. Therefore, the existence of $(v, 6, \lambda)$-PMD seems more challenging.

(2) **RPMD.** A (v, k, λ)-PMD is called *resolvable*, denoted by (v, k, λ)-RPMD, if the block set can be divided into some subsets such that (i) each subset of blocks forms a partition of the v-set when $v \equiv 0 \pmod{k}$, or (ii) each subset of blocks forms a partition of the v-set with one point put aside when $v \equiv 1 \pmod{k}$. The following result is contained in **[6, 8, 11]**.

Theorem 11.1. *A* $(v, 3, 1)$*-RPMD exists if and only if* $v \equiv 0$ *or* $1 \pmod{3}$, $v \not\equiv 6$.

For $k = 4$, there is a partial result (see [2]) as follows.

Theorem 11.2. *A* $(v, 4, 1)$*-RPMD exists for every positive integer* $v \equiv 1 \pmod{4}$ *with the possible exception of* $v = 33, 57, 93$ *and 133.*

Although the existence of a $(v, 4, 1)$-PMD is almost completely solved, the existence of a $(v, 4, 1)$-RPMD is still open. We don't know any example of $(v, 4, 1)$-RPMD when $v \equiv 0 \pmod{4}$. Compared with (v, k, λ)-PMD, the existence question for (v, k, λ)-RPMD seems much more open.

(3) **IPMD.** The idea of using IPMD to construct PMD was developed in a seminar at Suzhou University, during which Zhang [24] found the first example of using a $(16, 5, 4, 1)$-IPMD to construct a $(16, 4, 1)$-PMD, which is in the class of $v \equiv 0 \pmod{4}$. This started the progress made in the recent couple of years on the existence of (v, k, λ)-PMDs. But, IPMD itself is also an interesting question. One special case of IPMD is a PMD with subdesign deleted. The subdesign problem often appears as an embedding problem. For example, Hoffman and Lindner obtained in [14] the following.

Theorem 11.3. *Any* $(u, 3, 1)$*-PMD can always be embedded in a* $(v, 3, 1)$*-PMD as a subdesign for every* $v \geq 2u + 1$ *and* $v \equiv 0$ *or* $1 \pmod{3}$.

However, the necessary condition for the existence of $(v, u, 3, 1)$-IPMD is $(v - u)(v - 2u - 1) \equiv 0 \pmod{3}$ and $v \geq 2u + 1$ (from [4]), which contains more cases than the subdesign problem does. For example, a $(8, 2, 3, 1)$-IPMD $(X, \text{dev}\, \mathcal{B})$ can be easily found by taking

$$G = Z_6,\ X = Z_6 \cup \{\infty_1, \infty_2\},$$
$$\mathcal{B} = \{(0, 1, 3), (\infty_1, 0, 4), (\infty_2, 0, 5)\}.$$

In fact, a complete solution for the existence of a $(v, u, 3, 1)$-IPMD has been found recently in [4].

The existence of a $(v, u, 4, 1)$-IPMD is still widely open. Notice that there is a recent result on the existence of incomplete (BIB) designs of block size four having one hole by Rees and Stinson [19]. If we had a $(4, 4, 1)$-PMD it would then be possible to use their result to discuss the existence of a $(v, u, 4, 1)$-IPMD. Unfortunately, a $(4, 4, 1)$-PMD does not exist. This makes the existence question of $(v, u, 4, 1)$-IPMD more difficult.

(4) **HPMD.** The concept of HPMD has played an important role in solving the existence question for $(v, 5, 1)$-PMD in [7]. It seems interesting to look at the existence for HPMD itself, say a $(v, k, 1)$-HPMD of type $(m, m, \ldots, m)$. If $v = mu$, we write briefly m^u for the type. One necessary condition for the existence of a $(v, k, 1)$-HPMD of type m^u is

$$u(u-1)m^2 \equiv 0 \pmod{k}. \qquad (11.1)$$

For block size $k = 3$, it is likely that (11.1) is also a sufficient condition. However, we are still unable to construct a $(v, 3, 1)$-HPMD of type either 5^6, or 7^6. The existence question of (v, k, λ)-HPMD is very much open indeed.

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