# COMPUTATIONAL CONSIDERATIONS FOR EXCLUSIVE $(M,N)$-TRANSITIVE AUGMENTATIONS

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**Abstract.** Digraph $D$ is defined to be exclusive $(M,N)$-transitive if, for each pair of vertices $x$ and $y$, for each $xy$-path $P_1$ of length $M$ there is an $xy$-path $P_2$ of length $N$ such that $P_1 \cap P_2 = \{x,y\}$. It is proved that computation of a minimal edge augmentation to make $K$ exclusive $(M,N)$-transitive is $NP$-hard for $M > N \geq 2$, even if $D$ is acyclic. The corresponding decision problems are $NP$-complete. For $N = 1$ and $D = (V,E)$ with $|V| = n$, an $O(n^{M+3})$ algorithm to compute the exclusive $(M,1)$-transitive closure of an arbitrary digraph is provided.

## 1. Introduction and terminology.

Digraph $D$ is said to be *exclusive* $(M,N)$*-transitive* if, for each pair of vertices $x$ and $y$, for each $xy$-path, (that is, simple path), $P_1$ of length $M$ there is an $xy$-path $P_2$ of length $N$ such that $P_1 \cap P_2 = \{x,y\}$. $D$ is $(M,N)$*-transitive* if for each path of length $M$ from vertex $x$ to vertex $y$ there is a subset of the vertices on the path which are the vertices of a path of length $N$ from $x$ to $y$. $D$ is *free* $(M,N)$*-transitive* if for each path of length $M$ from vertex $x$ to vertex $y$ there is a path (unspecified as to what vertices may be used) from $x$ to $y$ of length $N$.

Exclusive $(2,1)$-transitivity, $(2,1)$-transitivity and free $(2,1)$-transitivity are all equivalent to "normal" transitivity. For all positive integers $M$, exclusive $(M,1)$-transitivity, $(M,1)$-transitivity and free $(M,1)$-transitivity are equivalent. $(M,N)$-transitivity has recently been examined in [1], [2] and [7]. Free $(M,N)$-transitivity is investigated in [2].

An *exclusive* $(M,N)$*-transitive closure*, $D_T$, of a digraph $D$, is a new digraph formed by augmenting $D$ with a minimal set of new edges so that $D_T$ is exclusive $(M,N)$-transitive. It is easy to show that the exclusive $(M,1)$-transitive closure of any digraph must be unique.

Further notation used is generally consistent with [3] and [5]. For a digraph $D = (V,E)$, the *vertex set* and *ordered edge set* are denoted by $V$ and $E$ respectively. A *path* of $D$ is formed by a sequence of distinct vertices $P = (x_0, x_1, \ldots, x_j)$, such that $\forall i \in \{1,2,\ldots,j\}$ $(x_{i-1}, x_i) \in E$. We refer to $P$ as an $x_0x_j$*-path* of *length* $j$. A *cycle* is an $x_0x_j$-path together with the edge $(x_j, x_0)$. A digraph which contains no cycles is said to be *acyclic*.

JCMCC 5(1989), pp. 55–60

## 2. Exclusive $(M, N)$-transitivity is $NP$-hard.

Let $M > N \geq 2$ be positive integers. We consider the following decision problem:

**Exclusive $(M, N)$-transitive closure.**
Instance: A digraph $(V, A)$ and a positive integer $k \leq |V|^2 - |V|$.
Question: Does there exist a set of edges $A'$ with $|A'| \leq k$ such that $(V, A \cup A')$ is exclusive $(M, N)$-transitive?

**Theorem 1.** *The exclusive $(3, 2)$-transitive closure problem is $NP$-complete.*

Proof: Clearly the problem is in $NP$ since exclusive $(M, N)$-transitivity can be checked in polynomial time. To establish the theorem we show a transformation from SATISFIABILITY (SAT) (see [4] and [6]) to EXCLUSIVE $(3, 2)$-TRANSITIVE CLOSURE.

Let $U = \{u_1, u_2, \ldots, u_p\}$ be the set of variables and $C = \{c_1, \ldots, c_q\}$ be the set of clauses making up an instance of SAT. Define an instance of EXCLUSIVE $(3, 2)$-TRANSITIVE CLOSURE as follows: The vertices, $V$, consist of the union of the following sets:

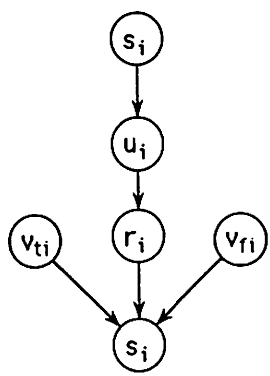


Figure 1

$$\{s_0\} \cup \{u_i, r_i, s_i, v_{ti}, v_{fi} \mid i = 1, \ldots, p\} \cup \{c_j \mid j = 1, \ldots, q\}.$$

For each variable $i$, $A$ includes the edges:

$$\{(s_0, u_i), (u_i, r_i), (r_i, s_i), (v_{bi}, s_i) \mid i = 1, \ldots, p, \ b = t, f\}.$$

Thus, each variable provides a subdigraph as in Figure 1.

The complete digraph is formed so that the vertex $s_0$ is connected to each vertex $u_i$ and if variable $u_i$ is contained in the clause $c_j$ then either $v_{ti}$ or $v_{fi}$, depending on the sign of $u_i$ in clause $c_j$, is connected to vertex $c_j$. $r_i$ is also connected to vertex $c_j$. This means that $A$ also includes the edges:

$$\{(v_{ti}, c_j) \mid u_i \text{ is a literal in the clause } c_j\}$$
$$\{(v_{fi}, c_j) \mid \neg u_i \text{ is a literal in the clause } c_j\}$$
$$\{(r_i, c_j) \mid u_i \text{ or } \neg u_i \text{ is a literal in the clause } c_j\}.$$

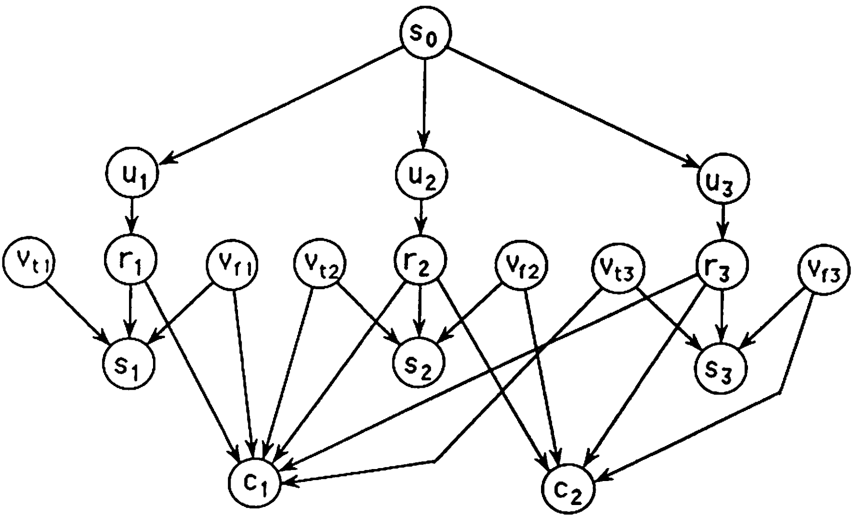


Figure 2

Figure 2 provides an example with three variables and two clauses:

$$(\neg u_1 \vee u_2 \vee u_3) \wedge (\neg u_2 \vee \neg u_3).$$

$k$ is defined to be $p$, the number of variables. If there is an assignment of the variables $B$: $\{u_1, u_2, \ldots, u_p\} \rightarrow \{t, f\}$ which results in each of the clauses having the value true, then the digraph obtained by adding the $p$ edges $\{(s_0, v_{gi}) \mid i = 1, \ldots, p \text{ and } g = B(u_i)\}$ is exclusive $(3,2)$-transitive. Thus, the exclusive $(3,2)$-transitive closure will not require more than $k$ edges for $k = p$.

Conversely, suppose that $(V, A \cup A')$ is the exclusive $(3,2)$-transitive closure of $(V, A)$ with $|A'| \leq p$.

Since there exist at least $p$ distinct paths of length 3 such that any two paths have only $s_0$ in common, at least $p$ edges must be added to have the needed paths of length 2. Therefore $|A'| = p$.

The exclusive $(3,2)$-transitive closure of a digraph as in Figure 1 must contain one of two extra edges for each $i$: $(s_0, v_{ti})$ or $(s_0, v_{fi})$, so $A'$ must contain one such edge for each $i$. Define an assignment of the variables $u_1, u_2, \ldots, u_p$, that is, $B: \{u_1, u_2, \ldots, u_p\} \rightarrow \{t, f\}$ by

$$B(u_i) = \begin{cases} t; & \text{if } (s_0, v_{ti}) \in A' \\ f; & \text{if } (s_0, v_{fi}) \in A'. \end{cases}$$

Since $(V, A \cup A')$ is exclusive $(3,2)$-transitive, each path of length 3 from $s_0$ to $c_j$, for $j = 1, \ldots, q$ requires a path of length 2 from $s_0$ to $c_j$. Thus, each clause in the instance of SAT will have the value true with the assignment $B$.

Since the above transformation can be done in polynomial time the exclusive $(3,2)$-transitive closure problem is $NP$-complete. ∎

The above proof can be modified by inserting additional vertices on paths to obtain the following:

**Corollary 1.** *The EXCLUSIVE $(M, N)$-TRANSITIVE CLOSURE problem is $NP$-complete for all $M > N \geq 2$.*

**Corollary 2.** *For $M > N \geq 2$, computation of the exclusive $(M, N)$-transitive closure of a general digraph is $NP$-hard.*

Note that Theorem 1 and Corollaries 1 and 2 are true even for acyclic digraphs, since the digraph constructed in the proof of Theorem 1 is acyclic.

## 3. Computing exclusive $(M, 1)$-transitive closures for arbitrary digraphs.

When computing exclusive $(M, 1)$-transitive closures on arbitrary (possibly cyclic) digraphs it becomes necessary to grow trees describing paths. For a given node, say $x_0$, of digraph $D = (V, E)$ with $|V| = n$, we may grow a tree, $T$, of depth $M$ with $x_0$ as the root so that vertex $k$ is a child of vertex $j$ iff $(j, k) \in E$ and, furthermore, $k$ is not an ancestor of $j$. This tree, which provides all paths of length $M$ from $x_0$, will be grown in a depth first, recursive manner. Each root-to-frontier path of length $M$ in $T$, then represents a path $P = (x_0, x_1, \ldots, x_M)$ in the digraph which must be checked to see if there is a corresponding edge $(x_0, x_M)$ in $D$. If not, edge $(x_0, x_M)$ must be added to fulfill the exclusive $(M, 1)$-requirements for $P$. This process of growing a tree with root $x_0$ and adding edges if necessary will be called GROW-TREE-AND-AUGMENT($x_0$).

When growing the above tree on a digraph with $n$ vertices there may be $n - 1$ branches from the root, and $n - 2$ branches from each vertex at depth 1, etc. In total, there may be $P = (n-1)\cdot(n-2)\cdot \ldots \cdot(n-M)$ vertices to place on the tree. $P$ must be less than $n^M$ so for a given root vertex, $x_0$, the time complexity of GROW-TREE-AND-AUGMENT($x_0$) is bounded by $n^M$. Of course, it is possible that the tree from each vertex will need to be reconstructed several times to reflect changes made by adding subsequent edges.

Algorithm 1: Computes $G_T$, the exclusive $(M, 1)$-transitive closure for arbitrary digraph $G$ with $n$ vertices, for $M \geq 2$.

begin

$G_T \leftarrow G$

repeat

Apply GROW-TREE-AND-AUGMENT($i$) to each vertex $i$ of $G_T$

until no edges are added in loop

end of algorithm.

**Theorem 2.** *Algorithm 1 computes the exclusive $(M, 1)$-transitive closure for any digraph.*

Proof: Upon completion of Algorithm 1 $G_T$ must be exclusive $(M, 1)$-transitive or the algorithm would not have halted.

Suppose $G_T$ is not the exclusive $(M, 1)$-transitive closure of $G$. Thus, $\exists$ edge set $E' = \{e_i = (a_i, b_i) \mid \text{for } i = 1, \ldots, k\}$ in $E(G_T) - E(G)$ such that the digraph $G_T^*$ obtained by removing $E'$ from $G_T$ is exclusive $(M, 1)$-transitive. Let $e_j$ be the first such edge added to $G_T$ by the algorithm. Let $G_T'$ be the version of $G_T$ just prior to adding $e_j$. Note that $G_T'$ is a subdigraph of $G_T^*$. Thus, $G_T^*$ contains a path of length $M$ from $a_j$ to $b_j$ but does not contain $e_j = (a_j, b_j)$, which contradicts the exclusive $(M, 1)$-transitivity of $G_T^*$. ∎

**4. Summary.**

This paper has defined the exclusive $(M, N)$-transitive property for directed graphs. It has established that, for $M > N \geq 2$, computation of an exclusive $(M, N)$-transitive closure of a digraph is $NP$-hard and the related decision problem is $NP$-complete. An algorithm has been presented which computes the exclusive $(M, 1)$-transitive closure of an arbitrary digraph $D = (V, E)$ in $O(n^{M+3})$ time, where $|V| = n$.

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