Integer-Magic Spectra of Trees with Diameters at most Four

Sin-Min Lee
Department of Computer Science
San Jose State University
San Jose, CA 95192

Ebrahim Salehi
Department of Mathematical Sciences
University of Nevada, Las Vegas
Las Vegas, NV 89154-4020

Hugo Sun
Department of Mathematics
California State University Fresno
Fresno, CA 93740

Abstract

For any $k \in \mathbb{N}$, a graph $G = (V, E)$ is said to be $\mathbb{Z}_k$-magic if there exists a labeling $l : E(G) \longrightarrow \mathbb{Z}_k - \{0\}$ such that the induced vertex set labeling $l^+ : V(G) \longrightarrow \mathbb{Z}_k$ defined by

$$l^+(v) = \sum_{u \in N(v)} l(uv)$$

is a constant map. For a given graph G, the set of all $k \in \mathbb{Z}_+$ for which G is $\mathbb{Z}_k$-magic is called the integer-magic spectrum of G and is denoted by $IM(G)$. In this paper we will consider trees whose diameters are at most 4 and will determine their integer-magic spectra.

Keywords: Integer-magic Spectrum, Magic, and Non-magic graphs.
AMS Subject Classification: 05C15.

1 Introduction

For any abelian group A, written additively, any mapping $l : E(G) \longrightarrow A - \{0\}$ is called a *labeling*, or edge-labeling. Given an edge-labeling of G one can introduce a vertex labeling $l^+ : V(G) \longrightarrow A$ as follows:

$$l^+(v) = \sum_{u \in N(v)} l(uv),$$

where $N(v)$ denotes the set of all vertices of G that are adjacent with v. A graph G is said to be *A-magic* if there is a labeling $l : E(G) \longrightarrow A - \{0\}$ such that for each vertex v, the sum of the labels of the edges incident with v are all equal to the same constant; that is, $l^+(v) = c$ for some fixed $c \in A$. We will call $< G, l >$ an A-magic graph with sum c. In general, a graph G may admit more than one labeling to become an A-magic graph: for example, if $|A| > 2$ and $l : E(G) \longrightarrow A - \{0\}$ is a magic labeling of G with sum c, then $\lambda : E(G) \longrightarrow A - \{0\}$, the inverse labeling of l, defined by $\lambda(uv) = -l(uv)$ will provide another magic labeling of G with sum $-c$.

The original concept of an A-magic graph is due to J. Sedlacek [21, 22], who defined it to be a graph with a real-valued edge labeling such that

1. distinct edges have distinct nonnegative labels; and

2. the sum of the labels of the edges incident to a particular vertex is the same for all vertices.

Given a graph G, the problem of deciding whether G admits a magic labeling is equivalent to the problem of deciding whether a set of linear homogeneous Diophantine equations has a solution [23]. At present, given an abelian group, no general efficient algorithm is known for finding magic labelings for general graphs.

When $A = \mathbb{Z}$, the $\mathbb{Z}$-magic graphs were considered in Stanley [23, 24], he pointed out that the theory of magic labeling can be put into the more general context of linear homogeneous diophantine equations. When the group is $\mathbb{Z}_k$, we shall refer to the $\mathbb{Z}_k$-magic graph as *k-magic*. Graphs which are k-magic had been studied in [6, 9, 10, 12, 14, 16, 18]. For convenience, we will use the notation 1-magic instead of $\mathbb{Z}$-magic. Doob [2, 3, 4], also considered A-magic graphs where A is an abelian group. He determined which wheels are $\mathbb{Z}$-magic.
A graph $G = (V, E)$ is called *fully magic* [14, 16] if it is A-magic for every abelian group A, and it is called *non-magic* if for every abelian group A it is not A-magic. Also, a graph G is said to be $\mathbb{N}$-magic if there exists a labeling $l : E(G) \to \mathbb{N}$ such that $l^+(v)$ is a constant, for every $v \in V(G)$. It is well-known that a graph G is $\mathbb{N}$-magic if and only if each edge of G is contained

in a 1-factor (a perfect matching) or a $\{1,2\}$-factor [11, 20, 29]. Berge [1] called a graph *regularisable* if a regular multigraph could be obtained from G by adding edges parallel to the edges of G. In fact, a graph is regularisable if and only if it is $\mathbb{N}$-magic. For $\mathbb{N}$-magic graphs, readers are referred to [7, 8, 9, 10, 12, 27, 24, 25, 26]. The notion of $\mathbb{Z}$-magic is weaker than $\mathbb{N}$-magic. Figure 1 shows a graph which is $\mathbb{Z}$-magic but not $\mathbb{N}$-magic.

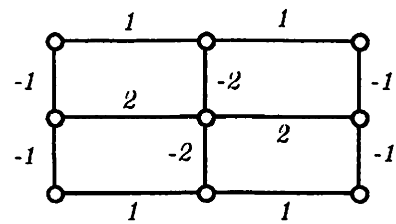

Figure 1: The graph $P_3 \times P_3$ is $\mathbb{Z}$-magic but is not $\mathbb{N}$-magic.

Observation 1.1. *For any $n \geq 3$, the path of order n is non-magic.*

Observation 1.2. *Any graph with a pendant path of length two is non-magic.*

In this paper, we will denote the set of positive integers by $\mathbb{N}$, and for any $k > 0$,

$$k\mathbb{N} = \{ kn \ : \ n \in \mathbb{N} \}, \quad \text{also} \quad k + \mathbb{N} = \{ k + n \ : \ n \in \mathbb{N} \}.$$

For a given graph G the set of all positive integers h for which G is $\mathbb{Z}_h$-magic (or simply h-magic) is called the *integer-magic spectrum* of G and is denoted by $IM(G)$. Since any regular graph is fully magic, then it is h-magic for all positive integers $h \geq 2$; therefore, $IM(G) = \mathbb{N}$. In what follows we will consider trees whose diameters are at most 4 and will determine their integer-magic spectra.

2 Trees with diameters two; Stars

For any $n \geq 1$, the complete bipartite graph $K(1,n)$ is called a *star* and is denoted by $ST(n)$. Note that $K(1,1)$ is the same as P_2, the path of order two, and it is fully magic. Also, $K(1,2)$ is the same as P_3, the path of order three, which is non-magic. To study the integer-magic spectrum of $ST(n)$ we will assume that $n \geq 3$.

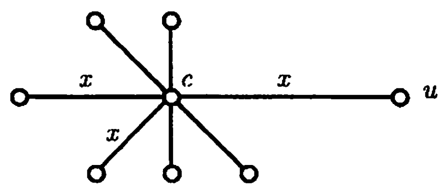

Figure 2: A typical labeling of $ST(n)$.

Theorem 2.1. *Let $n \geq 3$, and $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ be the prime factorization of $n-1$. Then $IM(\,ST(n)\,) = \bigcup_{i=1}^{k} p_i I\!N$.*

Proof. In a typical magic labeling of $ST(n)$, as illustrated in the Figure 2, all the edges are labeled with the same element x of the group $\mathbb{Z}_h$. The required condition is $l^+(c) = l^+(u)$ or

$$(n-1)x \equiv 0 \pmod{h}. \tag{1}$$

If $\gcd(n-1, h) = 1$, then Equation (1) will become $x \equiv 0 \pmod{h}$, which is not an acceptable solution. Therefore, we need $\gcd(n-1, h) = \delta > 1$, which implies $h \in \bigcup_{i=1}^{k} p_i I\!N$, and Equation (1) will have a non-zero solution for x. On the other hand, let p be a prime factor of $n-1$ and let $h \in pI\!N$. Then the choice of $x = \frac{h}{p}$ will work: Because, $l^+(c) = nx = (n-1)\frac{h}{p} + \frac{h}{p} \equiv \frac{h}{p} \pmod{h}$. □

Examples 2.2.

(a) $IM(\,ST(33)\,) = 2I\!N$: *here* $n-1 = 32 = 2^5$.

(b) $IM(\,ST(25)\,) = 2I\!N \cup 3I\!N$; *here* $n-1 = 24 = 2^3 \times 3$.

(c) $IM(\,ST(361)\,) = 2I\!N \cup 3I\!N \cup 5I\!N$: *here* $n-1 = 360 = 2^3 \times 3^2 \times 5$.

3 Trees with diameters three; Double-Stars

Trees with diameter 3 are called double-stars. These graphs have two central vertices u and v plus leaves. We will use $DS(m,n)$ to denote the double-star whose two central vertices have degrees m and n, respectively. By the Observation 1.1, if $m=2$ or $n=2$, then $DS(m,n)$ is non-magic.

therefore the integer-magic spectrum will be $\emptyset$. As a result, in what follows, we will assume that $m \geq n \geq 3$. Moreover, being a tree, $DS(m,n)$ is 2-magic if and only if m and n are odd numbers.

Note that in any magic labeling of a graph the end-edges (edges incident with the end-vertices) are labeled with the same element of the group A. Therefore, for any magic labeling of a double star, as illustrated in Figure 3, we use at most two non-zero group elements x and y.

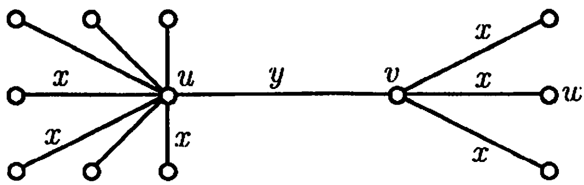

Figure 3: A typical magic labeling of $DS(m,n)$.

In particular, we need to have $l^+(u) = l^+(v)$ and $l^+(v) = l^+(w)$. These equations, when we are using $\mathbb{Z}_h$ will translate to:

$$(m-n)x \equiv 0 \pmod{h}; \tag{2}$$

$$(n-2)x + y \equiv 0 \pmod{h}. \tag{3}$$

Theorem 3.1. *The graph $DS(m,n)$ is $\mathbb{Z}$-magic (or 1-magic) if and only if $m = n$.*

Proof. If $DS(m,n)$ is $\mathbb{Z}$-magic, then Equation (2) would imply that $(m-n)x = 0$. Since x is non-zero, then we will have $m = n$. Conversely, if $m = n$, then the choices of $x = 1$ and $y = 2 - m$ provide a magic labeling with $l^+ \equiv 1$. □

Theorem 3.2. $IM(\,DS(m,m)\,) = \mathbb{N} - \{\, h \in \mathbb{N} \; : \; h > 1 \;\&\; h|(m-2) \,\}$.

Proof. By the Theorem 3.1, $DS(m,m)$ is 1-magic and if $h > m-2$, then the choices of $x = 1$, $y = h - m + 2$ will work with $l^+ \equiv 1$. Now assume that $1 < h \leq m-2$. Since $m = n$, Equation (2) holds. It is enough to show that (3) is true. Note that $DS(m,m)$ is h-magic if and only if h is not a divisor of $(m-2)$. Because, if $h|(m-2)$, then (3) becomes $y \equiv 0 \pmod{h}$, which is not an acceptable answer. On the other hand, if h does not divide $(m-2)$, then the choices of $x = 1$, $y = 2 - m \pmod{h}$ will work with $l^+ \equiv 1$. □

Examples 3.3.

(a) $IM(DS(3,3)) = \mathbb{N}$; here $m-2=1$ does not have any divisor bigger than 1. In fact, $DS(3,3)$ is the only double-star whose integer-magic spectrum is $\mathbb{N}$.

(b) $IM(DS(11,11)) = \mathbb{N} - \{3,9\}$; here $m-2=9$, and we need to exclude its divisors that are bigger than one: namely, 3, 9.

(c) $IM(DS(26,26)) = \mathbb{N} - \{2,3,4,6,8,12,24\}$; here $m-2=24$, and we need to exclude its divisors that are bigger than 1.

Theorem 3.4. *Let $p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$ and $p_1^{\beta_1}p_2^{\beta_2}\cdots p_k^{\beta_k}$ be the prime factorizations of $m-n$ and $n-2$, respectively. Then $IM(DS(m,n)) = \cup_{i=1}^{k} A_i$, where*

$$A_i = \begin{cases} p_i^{1+\beta_i}\mathbb{N} & \text{if } \alpha_i > \beta_i \geq 0; \\ \emptyset & \text{if } \beta_i \geq \alpha_i \geq 0. \end{cases}$$

Proof. We will prove the statement through four steps.

Step 1. Note that if $m-n=1$ or $m-n$ be a divisor of $n-2$ or $m-2$, then $IM(DS(m,n)) = \emptyset$. Because, $m-n=1$ will convert Equation (2) to $x=0$, which is not an acceptable solution. Also, if $m-2=q(m-n)$, then Equation (3) becomes $y=-(m-2)x=-q(m-n)x=0$, again an unacceptable solution. This is consistent with the statement of the theorem, for in either cases $\beta_i \geq \alpha_i$.

Therefore, we may assume that $m-n>1$ and that $m-n$ is not a divisor of $n-2$.

Step 2. Let p be a prime number, $\alpha > \beta$, $p^\alpha | (m-n)$, and $p^\beta | (n-2)$, but $p^{\beta+1}$ does not divide $(n-2)$. Then the graph $DS(m,n)$ is $p^{\beta+1}$-magic. Here, since $m-n \equiv 0 \pmod{p^\alpha}$, Equation (2) holds. Choose $x=1$. Note that $y \equiv 2-n \equiv 2-m \pmod{p^{\beta+1}}$ is non-zero and these labels work with $l^+ \equiv 1$; $l^+(v) = m-1+2-n \equiv m-n+1 \equiv 1 \pmod{p^{\beta+1}}$.

Step 3. If the graph $DS(m,n)$ is h-magic, then for every $r \in \mathbb{N}$ the graph is hr-magic. To see this, we simply observe that if x, y are the non-zero labels modulo h, then rx, ry will be non-zero modulo hr and will be valid magic labelings. Combination of these steps shows that the integer-magic spectrum of $DS(m,n)$ contains $p^{\beta+1}\mathbb{N}$.

Step 4. The previous steps show that $\cup_{i=1}^{k} A_i \subset IM(DS(m,n))$. To show that

$$IM(DS(m,n)) \subset \cup_{i=1}^{k} A_i,$$

suppose $DS(m,n)$ is h-magic. Then from Equation (3), h does not divide $(n-2)x$; otherwise, y will be zero. But from Equation (2),

$h|(m-n)x$. Therefore, in the prime factorization of h there exists a prime factor p^γ with the property that $p^\gamma|(m-n)x$, while p^γ does not divide $(n-2)x$. Choose $\beta \geq 0$ such that $p^{\beta+1}|(m-n)$, but $p^{\beta+1}$ does not divide $n-2$. By step 2, the graph $DS(m,n)$ is $p^{\beta+1}$-magic, and by step 3, $h \in p^{\beta+1}\mathbb{N}$. This completes the proof of the theorem.

□

Corollary 3.5. *$IM(\, DS(m,n)\,) = \emptyset$ if and only if $(m-n)|(n-2)$.*

Proof. If $IM(\, DS(m,n)\,) = \emptyset$, then by 3.4. $\beta_i \geq \alpha_i$ $(1 \leq i \leq k)$ and hence $(m-n)|(n-2)$. Conversely, if $(m-n)$ divides $n-2$, then $y \equiv -(n-2)x \equiv 0$, not an acceptable solution. □

Corollary 3.6. *If $|m-n| = 1$, then $DS(m,n)$ is non-magic.*

Examples 3.7.

(a) $IM(\, DS(9,3)\,) = 2\mathbb{N} \cup 3\mathbb{N}$: here $m-n = 6 = 2 \times 3$, while $n-2 = 1$.

(b) $IM(\, DS(6,4)\,) = \emptyset$: here $m-n = 2$ is a divisor of $n-2 = 2$.

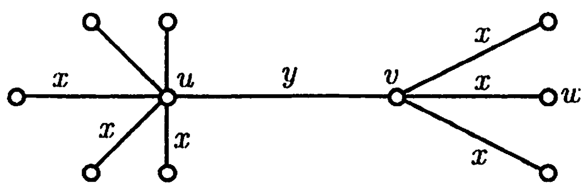

Figure 4: $IM(\, DS(6,4)\,) = \emptyset$.

(c) $IM(\, DS(28,4)\,) = 4\mathbb{N} \cup 3\mathbb{N}$; here $m-n = 24 = 2^3 \times 3$, while $n-2 = 2$.

(d) $IM(\, DS(16,10)\,) = 3\mathbb{N}$; here $m-n = 6 = 2 \times 3$, while $n-2 = 2^3$.

(e) $IM(\, DS(20,14)\,) = \emptyset$; here $m-n = 6$ is a divisor of $n-2 = 12$.

4 Trees of diameter four

Definition 4.1. A tree of diameter four, denoted by $TF(n; a_1, a_2, \cdots, a_n)$, consists of n stars $ST(a_1), ST(a_2), \cdots, ST(a_n)$ one of their edges is incident with a common vertex. The common vertex will be called the center of the tree and will be denoted by c.
In other words, $TF(n; a_1, a_2, \cdots, a_n)$ is a tree with center-vertex c, in which n edges $\{cu_1, cu_2, \cdots, cu_n\}$ are emanated from c, and $\deg(u_i) = a_i$ for each

$i = 1, 2, \cdots, n$, as illustrated in the Figure 5. In order to have a tree of diameter four, one needs $n \geq 2$ and $a_i \geq 2$ for at least two values of i.

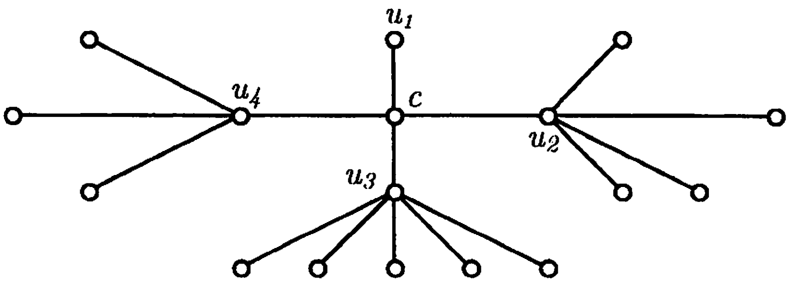

Figure 5: $TF(4;\ 1, 5, 6, 4)$; An example of a tree of diameter 4.

Observation 4.2. *If one of $a_1, a_2, \cdots, a_n$ is 2, then $IM(TF(n;\ a_1, a_2, \cdots, a_n)) = \emptyset$.*

Observation 4.3. *Let $b_1, b_2, \cdots, b_n$ be any permutation of $a_1, a_2, \cdots, a_n$. Then $TF(n;\ a_1, a_2, \cdots, a_n)$ is isomorphic with $TF(n;\ b_1, b_2, \cdots, b_n)$.*

As a result of these two observations, in any tree $TF(n;\ a_1, a_2, \cdots, a_n)$ with diameter four, we may assume that $1 < n$, $a_1 \leq a_2 \leq \cdots \leq a_n$, and $a_i \neq 2$.

Theorem 4.4. *Suppose $3 \leq m < n$ and let $m + n - 3 = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ be the prime factorization of $m + n - 3$. Then*

$$IM(TF(2;\ m, n)) = \bigcup_{i=1}^{k} p_i \mathbb{N} - \{\ d \mid d \text{ is a divisor of } m-1,\ m-2,\ n-1, \text{ or } n-2$$

Proof. For any magic labeling of this graph, as illustrated in the Figure 6, one needs two distinct non-zero elements $x, y \in \mathbb{Z}_h$.
The condition $l^+(u) = x$ implies

$$(m-1)x + y \equiv x \pmod{h}. \tag{4}$$

Since $x \neq y$, h cannot be a divisor of $m - 1$. Also Equation (4) can be written as $y \equiv -(m-2)x \pmod{h}$, which implies that h cannot be a divisor of $m - 2$ either. Similarly, from $l^+(v) = x$ we will have

$$(n-1)x + x - y \equiv x \pmod{h}. \tag{5}$$

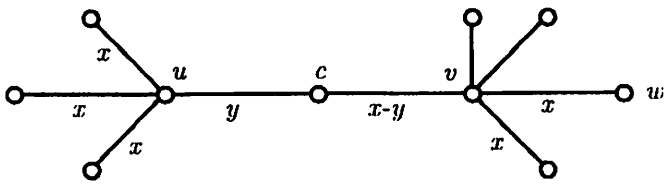

Figure 6: A typical magic labeling of $SF(2; 4, 5)$. Here $x \neq y$.

If we add the two Equations (4) and (5) we get

$$(m + n - 3)x \equiv 0 \pmod{h}. \tag{6}$$

This last equation has a non-zero solution for x if and only if $\gcd(m + n - 3, h) > 1$; that is, $h \in \bigcup_{i=1}^{k} p_i \mathbb{N}$. □

Corollary 4.5. *Let $m \geq 3$, and $2m - 3 = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ be the prime factorization of $2m - 3$. Then $IM(TF(2; m, m)) = \bigcup_{i=1}^{k} p_i \mathbb{N}$.*

Proof. Here $m = n$ and

$$\bigcup_{i=1}^{k} p_i \mathbb{N} \bigcap \{ \, d \mid d \text{ is a divisor of } m - 1 \text{ or } m - 2 \, \} = \emptyset.$$

□

Theorem 4.6. *Consider the tree of diameter four $G = TF(n; a_1, a_2, \cdots, a_n)$ $(n \geq 3)$, and let $\pm p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ be the prime factorization of $\sigma = 1 - 2n + \sum_{i=1}^{n} a_i$. Then the integer-magic spectrum of $G = TF(n; a_1, a_2, \cdots, a_n)$ is*

$$IM(G) = \begin{cases} \emptyset & \text{if } \sigma | (a_i - 2) \text{ for some } i > k \\ \mathbb{N} - D & \text{if } \sigma = 0 \\ \bigcup_{i=1}^{k} p_i \mathbb{N} - D & \text{otherwise,} \end{cases}$$

where $D = \{ \, d \mid d$ is a divisor of $a_i - 2$ for some i $(k + 1 \leq i \leq n) \, \}$.

Proof. First, note that if $a_i = 2$ for some i, then G is non-magic, and in this case $\sigma | (a_i - 2)$, which is consistent with the statement of the Theorem. Therefore, we may assume that $a_i \neq 2$. Let $a_1 = a_2 = \cdots = a_k = 1$ and $a_i \geq 3$ $(k + 1 \leq i \leq n)$.

In labeling of $G = TF(n; a_1, a_2, \cdots, a_n)$, we will use $x \in \mathbb{Z}_h$ for all the leaves and if $a_i \geq 3$, we will use $y_i \in \mathbb{Z}_h$ to label the edge cu_i (See the Figure 5). The condition $l^+(u_i) = x$ implies $(a_i - 1)x + y_i \equiv x \pmod{h}$ or

$$y_i \equiv -(a_i - 2)x \pmod{h} \quad (k + 1 \leq i \leq n). \tag{7}$$

Since y_i is a non-zero element of the group $\mathbb{Z}_h$, h cannot be a divisor of $a_i - 2$. Also, the condition $l^+(c) = x$ implies that $kx + y_{k+1} + y_{k+2} + \cdots + y_n \equiv x \pmod{h}$, or

$$\sum_{i=k+1}^{n} y_i \equiv (1 - k)x \pmod{h}. \tag{8}$$

Now if we add the Equations (7) for $i = k + 1, k + 2, \cdots, n$, we will get $\sum_{i=k+1}^{n} y_i \equiv - \sum_{i=k+1}^{n} (a_i - 2)x \pmod{h}$, which together with (8) will give us $(1 - 2n + \sum_{i=1}^{n} a_i)x \equiv 0 \pmod{h}$, or

$$\sigma x \equiv 0 \pmod{h}. \tag{9}$$

Now we will consider the following cases:

Case 1. If for some value of i, $\sigma | (a_i - 2)$, then $a_i - 2 = q\sigma$ and Equation (7) gives us $y_i \equiv -q\sigma x \equiv 0 \pmod{h}$, which is not an acceptable answer. Therefore, $IM(G) = \emptyset$.

Case 2. If $\sigma = 0$, Equation (9) is satisfied for all values of h, and one only needs to exclude divisors of $a_i - 2$ to assure that Equations (7) will provide non-zero solutions for y_i. Therefore, $IM(G) = \mathbb{N} - D$.

Case 3. Suppose $\sigma \neq 0$ and σ is not divisor of any of $a_i - 2$. Then Equation (9) has non-zero solution for x if and only if $\gcd(\sigma, h) > 1$; that is, $h \in \bigcup_{i=1}^{k} p_i\mathbb{N}$. However, we need to exclude divisors of $a_i - 2$ to make sure that Equations (7) will provide non-zero solutions for y_i. Therefore, $IM(G) = \bigcup_{i=1}^{k} p_i\mathbb{N} - D$.

This complete the proof of the Theorem. □

Corollary 4.7. *With the notation of Theorem 4.6, if $a_1 \geq 3$, then*

$$IM(G) = \bigcup_{i=1}^{k} p_i\mathbb{N} - \{\ d \mid d \ \textit{is a divisor of}\ a_i - 2\ (1 \leq i \leq n)\ \}.$$

Proof. We observe that if $a_1 \geq 3$, then $\sigma = 1 - 2n + \sum_{i=1}^{n} a_i \geq 1 - 2n + 3(n - 1) + a_i > a_i - 2$. Therefore, $\sigma \neq 0$ and it can not be a divisor of $a_i - 2$. □

Corollary 4.8. *In $TF(n; a_1, a_2, \cdots, a_n)$ let $a_1 = a_2 = \cdots = a_n = m \geq 3$; that is choose n copies of $ST(m)$ and identify one of their end vertices. Also let $mn - 2n + 1 = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ be the prime factorization of $mn - 2n + 1$. Then*

$$IM(\,TF(n;\, m, m, \cdots, m)\,) = \{\, h \mid \gcd(mn - 2n + 1, h) > 1 \,\} = \bigcup_{i=1}^{k} p_i \mathbb{N}.$$

Examples 4.9.

(a) $IM(\,TF(2;\, 4, 4)\,) = 5\mathbb{N}$; here $2m - 3 = 5$.

(b) $IM(\,TF(2;\, 6, 17)\,) = 2\mathbb{N} \cup 3\mathbb{N} - \{2, 4, 5, 8, 15, 16\}$; here $a_1 + a_2 - 3 = 20$, and we need to exclude the divisors of $a_i - 1$, $a_i - 2$.

(c) $IM(\,TF(2;\, 9, 24)\,) = 2\mathbb{N} \cup 3\mathbb{N} \cup 5\mathbb{N} - \{2, 4, 8, 22\}$; here $a_1 + a_2 - 3 = 30$.

(d) $IM(\,TF(3;\, 5, 5, 5)\,) = 2\mathbb{N} \cup 5\mathbb{N}$; here $3m - 5 = 10 = 2 \times 5$.

(e) $IM(\,TF(3;\, 3, 5, 9)\,) = 2\mathbb{N} \cup 3\mathbb{N} - \{3\}$; here $a_1 + a_2 + a_3 - 5 = 12$ and 3 is a divisor of $a_2 - 2$.

(f) $IM(\,TF(4;\, 1, 1, 3, 5)\,) = \emptyset$; here $\sigma = a_1 + a_2 + a_3 + a_4 - 7 = 3$ and $\sigma = 3$ is a divisor of $a_4 - 2$.

(g) $IM(\,TF(6;\, 1, 1, 1, 3, 3, 4)\,) = \emptyset$; here $\sigma = a_1 + a_2 + a_3 + a_4 + a_5 + a_6 - 11 = 2$, which is a divisor of $a_6 - 2$.

References

[1] C. Berge, Regularisable graphs, *Annals of Discrete Mathematics* **3** (1978), 11-19.

[2] M. Doob, On the construction of magic graphs, *Proceedings of the Fifth Southeast Conference on Combinatorics, Graph Theory, and Computing* (1974), 361-374.

[3] M. Doob, Generalizations of magic graphs, *Journal of Combinatorial Theory, Series B* **17** (1974), 205-217.

[4] M. Doob, Characterizations of regular magic graphs, *Journal of Combinatorial Theory, Series B* **25** (1978), 94-104.

[5] F. Gobel and C. Hoede, Magic labelings of graphs, *Ars Combinatoria* **51** (1999), 3-19.

[6] Y.S. Ho and Sin-Min Lee, An initial results of supermagicness of regular complete k-partite graphs, *Journal of Combinatorial Mathematics and Combinatorial Computing* **39** (2001), 3-17.

[7] N. Hartsfield and G. Ringel, Supermagic and antimagic graphs, *Journal of Recreational Mathematics* **21** (1989), 107-115.

[8] R.H. Jeurissen, The incidence matrix and labelings of a graph, *Journal of Combinatorial Theory, Series B* **30** (1981), 290-301.

[9] R.H. Jeurissen, Disconnected graphs with magic labelings, *Discrete Mathematics* **43** (1983), 47-53.

[10] R.H. Jeurissen, Pseudo-magic graphs, *Discrete Mathematics* **43** (1983), 207-214.

[11] S. Jezny and M. Trenkler, Characterization of magic graphs, *Czechoslovak Mathematical Journal* **33** (1983), 435-438.

[12] M.C.Kong, S-M Lee, and Hugo Sun, On magic strength of graphs, *Ars Combinatoria* **45** (1997), 193-200.

[13] A. Kotzig and A. Rosa, Magic valuations of finite graphs, *Canadian Mathematics Bulletin* **13** (1970), 451-461.

[14] S-M Lee, Alexander Nien-Tsu Lee, Hugo Sun, and Ixin Wen, On integer-magic spectra of graphs, JCMCC, **42** (2002), 177-185.

[15] S-M Lee, F. Saba, Ebrahim Salehi, Hugo Sun, On the V_4-magic graphs, *Congressus Numerantium* **156** (2002), 59-67.

[16] S-M Lee, F. Saba, and G. C. Sun, Magic strength of the k-th power of paths, *Congressus Numeruntium*, **92** (1993), 177-184.

[17] S-M Lee, Ebrahim Salehi, Integer-magic spectra of amalgamations of stars and cycles, *ARS Combiinatoria* **67** (2003), 199-212.

[18] S-M Lee, Hugo Sun, and Ixin Wen, On group magic graphs, *Journal of Combinatorial Mathematics and Combinatorial Computing* **38** (2001), 197-207.

[19] S-M Lee, L. Valdes, and Yong-Song Ho, On group magic trees, double trees, and abbreviated trees, to appear in the JCMCC.

[20] L. Sandorova and M. Trenkler, On a generalization of magic graphs, *in "Combinatorics 1987", Proc. 7th Hungary Colloq. Eger/Hung. Colloquia Mathematica Societatis Janos Bolyai,* **52** (1988), 447-452.

[21] J. Sedlacek, On magic graphs, *Math. Slov.* **26** (1976),329-335.

[22] J. Sedlacek, Some properties of magic graphs, in Graphs, Hypergraph, *Bloc Syst. 1976, Proc. Symp. Comb. Anal., Zielona Gora* (1976), 247-253.

[23] R.P. Stanley, Linear homogeneous diophantine equations and magic labelings of graphs, *Duke Mathematics Journal,* **40** (1973), 607-632.

[24] R.P. Stanley, Magic labeling of graphs, symmetric magic squares, systems of parameters and Cohen-Macaulay rings, *Duke Mathematics Journal* **40** (1976), 511-531.

[25] B.M. Stewart, Magic graphs, *Canadian Journal of Mathematics*,**18** (1966), 1031- 1059.

[26] B.M. Stewart, Supermagic complete graphs, *Canadian Journal of Mathematics*, **19** (1967), 427-438.

[27] G.C. Sun, J. Guan, and Sin-Min Lee, A labeling algorithm for magic graph, *Congressus Numerantium* **102** (1994), 129-137.

[28] G.C. Sun and Sin-Min Lee, Constructions of magic graphs, *Congressus Numerantium* **103** (1994), 243-251.

[29] M. Trenkler, Some results on magic graphs, in "Graphs and other Combinatorial Topics", *Proceeding Of the third Czechoslovak Symposium Prague,* 1983, edited by M. Fieldler, Teubner-Texte zur Mathematik Band, Leipzig **59** (1983), 328-332.