On Group–magic Eulerian Graphs

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Abstract

Let A be an abelian group. We call a graph $G = (V, E)$ A–**magic** if there exists a labeling $f : E(G) \to A^*$ such that the induced vertex set labeling $f^+ : V(G) \to A$, defined by $f^+(v) = \Sigma f(u, v)$ where $(u, v) \in E(G)$, is a constant map. In this paper, we present some algebraic properties of A–magic graphs. Using them, various results are obtained for group–magic eulerian graphs.

1 Introduction.

Let G be a connected (multi)graph, with no loops. For any abelian group A (written additively), let $A^* = A - \{0\}$. A function $f : E(G) \to A^*$ is called a *labeling* of G. Any such labeling induces a map $f^+ : V(G) \to A$, defined by $f^+(v) = \Sigma f(u, v)$, where $(u, v) \in E(G)$. If there exists a labeling f whose induced map on $V(G)$ is a constant map, we say that f is an *A–magic labeling* and that G is an *A–magic* graph. The *integer–magic spectrum* of a graph G is the set $\{k : G \text{ is } Z_k\text{–magic and } k \geq 2\}$.

In this article, we will use the following notation. Let $[G, A]$ denote the class of distinct A–magic labelings of G. Note that G is A–magic if and only if $[G, A] \neq \emptyset$. For any ring R with unity, $U(R)$ denotes the multiplicative group of units in R.

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Z–magic graphs were considered by Stanley [10,11], where he pointed out that the theory of magic labelings could be studied in the general context of linear homogeneous diophantine equations. Doob [1,2,3] has studied A–magic graphs and Z_k–magic were investigated in [5,6,7].

Within the mathematical literature, various definitions of magic graphs have been introduced. The original concept of an A–magic graph is due to J. Sedlacek [8,9], who defined it to be a graph with real–valued edge labeling such that (i) distinct edges have distinct nonnegative labels, and (ii) the sum of the labels of the edges incident to a particular vertex is the same for all vertices. Previously, Kotzig and Rosa [4] had introduced yet another definition of a magic graph. Over the years, there has been great research interest in graph labeling problems. The interested reader is directed to Wallis' [12] recent monograph on magic graphs.

2 A necessary condition for G to be Z_3–magic.

It is straight–forward to determine a necessary and sufficient condition for G to be Z_2–magic. Clearly, G is Z_2–magic if and only if every vertex of G is of the same parity. It seems that finding a similar condition for G to be Z_3–magic is much more difficult. We establish the following result:

Theorem 1. *Let G be Z_3–magic, with p vertices and q edges. Let $f \in [G, Z_3]$ induce the constant label x on the vertices of G, and $|E_i|$ denote the number of edges labeled i. Then, $px \equiv q + |E_1|$, (mod 3).*

Proof. With any Z_3–magic labeling f of G, we can associate a multigraph $\widehat{G}$ with G. $\widehat{G}$ is formed by replacing every edge in G which was labeled 2, with two edges labeled 1. Note that $\widehat{G}$ is a Z_3–magic multi–graph with p vertices and $2|E_2| + |E_1|$ edges. In any (p, q)–graph, we have that $\Sigma \deg(v_i) = 2q$. Since all of the edges in $\widehat{G}$ are labeled 1, this implies that $px \equiv 4|E_2| + 2|E_1|$, (mod 3). From this, we see that $px \equiv |E_2| + |E_1| + |E_1|$, (mod 3). Thus, $px \equiv q + |E_1|$, (mod 3). □

Several remarks should be made with regard to Theorem 1. First, note that with similar calculations, one can easily derive an analogous result in terms of $|E_2|$ (ie: $px + q + |E_2| \equiv 0$, (mod 3)). Also, Theorem 1 might be used to reduce the number of calculations performed, when trying to find a Z_3–magic labeling of G via computer search. In addition, similar necessary conditions can be established for graphs G to be Z_k–magic.

3 Algebraic properties of A–magic graphs.

After examining a few examples by hand, the observant reader will note a sort of duality appearing in Theorem 1. The next two results give a reason as to why this occurs.

Theorem 2. *Let A be a non-trivial denumerable abelian group, underlying some ring R with unity. If $d \in U(A)$ and $f \in [G, A]$, then $df \in [G, A]$.*

Proof. Suppose that f induces a constant label x on all of the vertices of G. Consider an arbitrary vertex v and let $|E_i|$ denote the number of edges labeled a_i, which are adjacent to v. Then, $x = \Sigma(a_i|E_i|)$; where a_i varies through all the elements of A^*. Let us examine what effect df has on the labeling of v. By multiplying every edge adjacent to v by d, we get the following relationship: $dx = d\Sigma(a_i|E_i|)$. The new induced labeling on v is dx. Also, since $d \in U(A)$, each edge adjacent to v in this new labeling is not equal to 0_A. Thus, df is an A-magic labeling of G. □

The following result is an immediate consequence of Theorem 2.

Corollary 1. *If $d \in U(Z_k)$ and $f \in [G, Z_k]$, then $df \in [G, Z_k]$.*

Proof. Let $A = Z_n$, the group of integers, modulo n. Now, apply Theorem 2. □

It should be noted that in Theorem 2 and Corollary 1, f and df might yield the same group-magic labeling on G. Also, a natural question to ask is the following: If G is an A-magic graph, does there exist some labeling of G for which all other possible A-magic labelings arise, by applying this group action? In general, the answer is no. Consider the following two labelings for a Z_9-magic graph G:

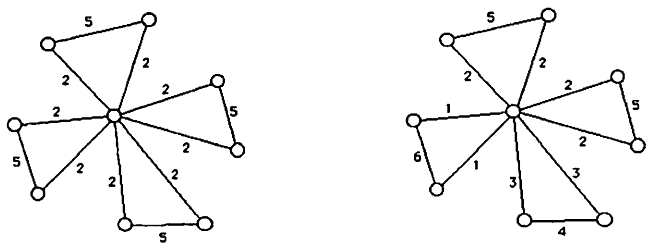

Figure 1.

Let $f \in [G, Z_9]$ and d_1, $d_2 \in U(Z_9)$ where $d_i f$ gives the i^{th} labeling. By multiplying each edge in the first labeling by $d_2 d_1^{-1}$, we obtain the second labeling. However, this is impossible.

We wish to continue to develop an algebraic framework from which group-magic graphs can be analyzed. Some of the following results will give us additional tools for studying A-magic eulerian graphs.

Theorem 3. *Let A_1 be an abelian group which contains a subgroup isomorphic to A_2. If graph G is A_2-magic, then G is A_1-magic.*

Proof. Let $H \leq A_1$. Suppose that $f \in [G, A_2]$ and that $\phi : A_2 \to H$ is a group isomorphism. Now, let f induce a constant label x on all of the vertices of G. Consider an arbitrary vertex v and let $|E_i|$ denote the number of edges labeled a_i, which are adjacent to v. Then, $x = \Sigma(a_i|E_i|)$; where a_i varies through all the elements of $A_2{}^*$. Now, apply ϕ to the edges which are adjacent to v. Under this new labeling, we get the following relationship: $\phi(x) = \phi[\Sigma(a_i|E_i|)] = \Sigma\phi(a_i)|E_i|$. Since $a_i \neq 0_{A_2}$ and ϕ is a group isomorphism, no edge is labeled 0_{A_1}. The new induced labeling on v is $\phi(x)$. Hence, we have an A_1–magic labeling of G. □

Although the next result is an immediate corollary of Theorem 3, for the sake of clarity, a detailed proof has been given.

Corollary 2. *Let G be a Z_k–magic graph, with $k|n$. Then, G is a Z_n–magic graph.*

Proof. Suppose that we have a Z_k–magic labeling on G. Let x be the constant label on the vertices of G and suppose that $kd = n$. Now, consider an arbitrary vertex v and let $|E_i|$ denote the number of edges labeled i, which are adjacent to v. Then, $x \equiv \Sigma(i|E_i|)$, mod k; where i varies from 1 to $k-1$. By multiplying every edge adjacent to v by d, we get the following relationship: $dx \equiv d\Sigma(i|E_i|)$, mod kd and hence $dx \equiv d\Sigma(i|E_i|)$, mod n. The new induced labeling on v is dx. Also, since $1 \leq i \leq k-1$ and $d \neq 0$, we have that $0 < di < n$. In particular, di is not congruent to 0, mod n. Thus, in this new labeling, no edge is labeled 0. Since v was taken to be an arbitrary vertex, we have shown that G is Z_n–magic. □

The reader should observe that the converse of Corollary 2 is not true. (ie. If G is Z_n–magic, with $k|n$, it does not follow that G is Z_k–magic.) For example, let G be the eulerian graph consisting of a C_4 block and a C_3 block and sharing one common vertex ($p = 6$, $q = 7$). Now, G is Z_2–magic. By Corollary 2, G is Z_6–magic. However, it is straight–forward to verify that G is not Z_3–magic.

Also, Corollary 2 allows us to obtain information about the integer–magic spectrum of G. For example, if G is Z_p–magic for all primes p, then G is Z_n–magic for all $n \geq 2$.

4 Results on eulerian graphs.

There are still many open questions with regard to the characterization of A–magic eulerian graphs. In this section, an assortment of results is given.

Corollary 3. *Every eulerian graph G is Z_k–magic, for k even.*

Proof. This follows immediately from Corollary 2. □

Corollary 4. *Let A be an abelian group containing an element of order 2. Then, every eulerian graph G is A–magic.*

Proof. Suppose that x is an involution in A. Label every edge of G with x. Then, every vertex of G has an induced labeling of 0. Hence, G is A–magic. □

Note that Corollary 4 also follows from Theorem 3.

Theorem 4. *Let A be any non-trivial abelian group. Then, every eulerian graph G with an even number of edges must be A-magic.*

Proof. Suppose that $a \in A$, with $a \neq 0$. Let $e_1e_2e_3 \cdots e_{2n}$ be an eulerian circuit, starting and ending at vertex v. The following labeling scheme will give an A–magic labeling of G:

$$f(e_i) = \begin{cases} a, & \text{if } i \text{ is odd.} \\ -a, & \text{if } i \text{ is even.} \end{cases}$$

Note that every vertex has an induced labeling of 0. □

We have already seen an example of an eulerian graph with an odd number of edges, which is not A–magic (e.g. G, consisting of a C_4 block and a C_3 block and sharing one common vertex). In contrast to this, there are also eulerian graphs with an odd number of edges, which are A–magic. For example, any odd cycle is A–magic.

Let us now focus our attention on eulerian graphs with an odd number of edges. First, we begin with a definition.

Definition 1. Given a connected graph $G = (V, E)$, let $T(G)$ denote the graph which is obtained from G by adding a disjoint uv–path of length 2 between every adjacent pair of vertices u, v in $V(G)$.

Note that $T(G)$ has $|V(G)| + |E(G)|$ vertices and $|E(G)| + 2|E(G)|$ edges. Also, it is straight–forward to show that $T(G)$ is eulerian. We investigate the following question: For which G with $|E(G)|$ odd, is $T(G)$ A–magic for all non-trivial, finite abelian groups A?

Theorem 5. *$T(P_{2k})$ is A–magic, where P_{2k} is a path of order $2k$ and A is any non–trivial, finite abelian group.*

Proof. Let $a \in A$, with $a \neq 0$. The following diagram gives an A–magic labeling of $T(P_{2k})$.

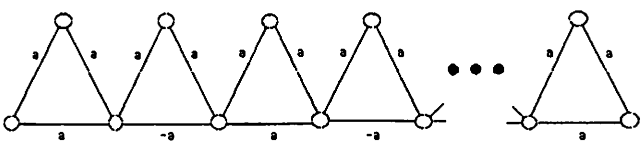

Figure 2.

□

Theorem 6. *$T(K_{1,2n+1})$ is A-magic, for all non-trivial, finite abelian groups A.*

Proof. First, we show that $T(K_{1,2n+1})$ is Z_p–magic, for all prime p. Since $T(K_{1,2n+1})$ is eulerian (and thus is Z_2–magic), let $p \geq 3$. Furthermore, let v denote the vertex of degree $2(2n+1)$ and E_v be the set of edges incident to v in $T(K_{1,2n+1})$. There are two cases to consider: (i). $2(2n+1) \equiv 1$, mod p and (ii). $2(2n+1) \not\equiv 1$, mod p.

(i). $2(2n+1) \equiv 1$, mod p. From E_v, label $2(2n+1) - 2$ edges with 1 and the remaining two edges, with $p - 1$. Now, v has an induced labeling of $2(2n+1) - 2 + 2(p-1) \equiv p - 3$, mod p. We label the remaining edges of $T(K_{1,2n+1})$ in the following manner: If the edge is adjacent to two edges labeled 1, then label it $p-4$, mod p; otherwise, label the edge $p-2$, mod p. This yields a Z_p–magic labeling of $T(K_{1,2n+1})$.

(ii). $2(2n+1) \not\equiv 1$, mod p. Label every edge in E_v with 1. Note that v has an induced labeling of $2(2n+1) \equiv 4n+2$, mod p. We label the remaining edges of $T(K_{1,2n+1})$ with $4n+1$, mod p. Since $4n+2 \not\equiv 1$, mod p, we have that $4n+1 \not\equiv 0$, mod p. Hence, we have a Z_p–magic labeling of $T(K_{1,2n+1})$.

Therefore, $T(K_{1,2n+1})$ is Z_p–magic, for all prime p. Now, every finite abelian group A can be written as a direct sum of cyclic groups, each of order a power of a prime. Also, every finite p–group has Z_p as a subgroup. Hence, by Theorem 3, $T(K_{1,2n+1})$ is A–magic.

□

Here is another construction which yields eulerian graphs with an odd number of edges.

Definition 2. A graph H is *homeomorphic from G* if either H is isomorphic to G or H is isomorphic to a graph obtained by subdividing some sequence of edges of G.

Definition 3. A *cycle-snake* is any graph which is homeomorphic from $T(P_k)$.

The reader will note that by making subdivisions on the edges of $T(P_k)$, for $k \geq 2$, many eulerian graphs having an odd number of edges can be created. We will show that certain types of cycle–snakes are A–magic, for all non–trivial, abelian groups A. Before we do that, the following lemma is needed.

Lemma 1. *Let graph G have an A–magic labeling with a vertex–induced label x. Furthermore, let $a \in A^*$, where $2a = x$, and $E_a(G)$ denote the edges of G which are labeled a. Then, any graph obtained from G by subdividing edges in $E_a(G)$ is A–magic.*

Proof. Since edges in $E_a(G)$ are subdivided, they are replaced with paths of length 2 in the new graph. Label the edges of the paths with a. The new graph obtained will be A–magic, having the same vertex–induced label as G. □

Theorem 7. *Let $E_a[T(P_{2k})]$ denote the set of edges corresponding to the A–magic labeling found in Figure 2. Then, any cycle-snake obtained from $T(P_{2k})$ by subdividing edges in $E_a[T(P_{2k})]$ is A–magic.*

Proof. This follows immediately from Theorem 5 and Lemma 1. □

Not all cycle–snakes are A–magic. For example, it is straight–forward to show that the graph consisting of a C_4 block and a C_3 block and sharing one common vertex, has integer–magic spectrum equaling $2N$.

Up to this point, the integer–magic spectrum of the eulerian graphs that we have come across has either been $2N$ or $N - \{1\}$. This might lead the reader to believe that this is the case for all eulerian graphs. However, there are eulerian graphs, whose integer–magic spectrum is neither $2N$ nor $N - \{1\}$. For example, the following diagram gives a Z_7–magic labeling of an eulerian graph. It is straight–forward to show that this graph is not Z_3–magic.

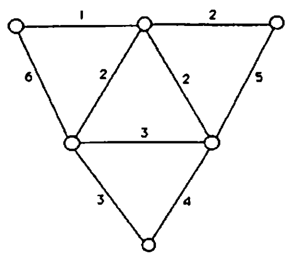

Figure 3.

5 Directions for further research.

Open Problem 1. *Characterize the A–magic eulerian graphs with an odd number of edges.*

Open Problem 2. *Find necessary and sufficient conditions for a graph G to be Z_3–magic.*

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