DECOMPOSITION FORMULAS OF WEIGHTED ZETA FUNCTIONS OF GRAPHS AND DIGRAPHS

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Abstract

We give decomposition formulas of the multiedge and the multipath zeta function of a regular covering of a graph G with respect to equivalence classes of prime, reduced cycles of G. Furthermore, we give a decomposition formula of the weighted zeta function of a g-cyclic Γ-cover of a symmetric digraph D with respect to equivalence classes of prime cycles of D, for any finite group Γ and $g \in \Gamma$.

Key word: graph covering, digraph covering, zeta function

1 Introduction

Graphs and digraphs treated here are finite and simple. Let $G = (V(G), E(G))$ be a connected graph with vertex $V(G)$ and arc set $E(G)$, and D the symmetric digraph corresponding to G. Note that $E(G) = E(D)$. For $e = (u, v) \in E(G)$, let $o(e) = u$ and $t(e) = v$. The inverse arc of e is denoted by $\bar{e}$. A path P of length n in D(or G) is a sequence $P = (v_0, v_1, \cdots, v_{n-1}, v_n)$ of $n+1$ vertices and n arcs(or edges) such that consecutive vertices share an arc(or edge) (we do not require that all vertices are distinct). Also, P is called a (v_0, v_n)-path. If $e_i = (v_i, v_{i+1})$ for $i = 1, \cdots, n-1$, then we can write $P = (e_1, \cdots, e_{n-1})$. We say that a path has a backtracking if a subsequence of the form $\cdots, x, y, x, \cdots$ appears. A (v, w)-path is called a cycle (or closed path) if $v = w$.

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We introduce an equivalence relation between cycles. Two cycles $C_1 = (v_1, \cdots, v_m)$ and $C_2 = (w_1, \cdots, w_m)$ are called equivalent if $w_j = v_{j+k}$ for all j. Let $[C]$ be the equivalnce class which contains a cycle C. Let B^r be the cycle obtained by going r times around a cycle B. Such a cycle is called a multiple of B. A cycle C is said to be reduced if both C and C^2 have no backtracking. A cycle C is prime if $C \neq B^r$ for some other cycle B and $r \geq 2$. The (Ihara) zeta function of a graph G is defined to be a function of $u \in \mathbf{C}$ with u sufficiently small, by

$$\mathbf{Z}(G,u) = \mathbf{Z}_G(u) = \prod_{[C]} (1 - u^{|C|})^{-1},$$

where $[C]$ runs over all equivalence classes of prime, reduced cycles of G, and $\mid C \mid$ is the length of C.

Zeta functions of graphs started from zeta functions of regular graphs by Ihara [6]. In [6], he showed that their reciprocals are explicit polynomials. Hashimoto [5] treated multivariable zeta functions of bipartite graphs. Bass [1] generalized Ihara's result on the zeta function of a regular graph to an irregular graph, and showed that its reciprocal is a polynomial. Various proof of Bass's Theorem were given by [3,7,13]. Mizuno and Sato [9] obtained a decomposition formula for the zeta function of a regular covering of a graph.

Let G be a connected graph with $2m$ arcs $e_1, \cdots, e_{2m}$, and let the multiedge matrix $\mathbf{W} = \mathbf{W}(G)$ of G be a $2m \times 2m$ matrix with ij entry the complex variable w_{ij} if $t(e_i) = o(e_j)$, $e_j \neq \bar{e}_i$, and $w_{ij} = 0$ otherwise. Furthermore, set $w(e_i, e_j) = w_{ij}$. For a path $C = (e_{i_1}, \cdots, e_{i_l})$ of G, the multiedge norm $\mathbf{N}_E(C)$ of C is defined as follows: $\mathbf{N}_E(C) = w_{i_1 i_2} w_{i_2 i_3} \cdots w_{i_{l-1} i_l}$. The multiedge zeta function of G is defined by

$$\zeta_E(\mathbf{W}, G) = \prod_{[C]} (1 - \mathbf{N}_E(C))^{-1},$$

where $[C]$ runs over all equivalence classes of prime, reduced cycles in G. Stark and Terras [14] showed that the reciprocal of the multiedge zeta function of a graph is a polynomial: $\zeta_E(\mathbf{W}, G)^{-1} = \det(\mathbf{I} - \mathbf{W})$. Furthermore, they obtained a factorization formula for the multiedge zeta function of a regular covering of a graph G as a product of multiedge Artin L-functions.

Cycles, reduced cycles and prime cycles in a simple digraph which is not symmetric are defined similarly to the case of a symmetric digraph. Let D be a connected digraph. Then, the zeta function of D is defined to be a function of $u \in \mathbf{C}$ with u sufficiently small, by

$$\mathbf{Z}(D,u) = \mathbf{Z}_D(u) = \prod_{[C]} (1 - u^{|C|})^{-1},$$

where $[C]$ runs over all equivalence classes of prime cycles of D. Kotani and Sunada [7], Mizuno and Sato [10] gave a determinant expression of the zeta function of a connected digraph D: $\mathbf{Z}(D,u)^{-1} = \det(\mathbf{I} - \mathbf{A}(D)u)$, where $\mathbf{A}(D)$ is the adjacency matrix of D.

Kotani and Sunada [7] stated a connection between zeta functions of graphs and that of strongly connected digraphs. Let $G = (V, E)$ be a connected non-circuit graph. Then the oriented line graph $\vec{L}(G) = (V_L, E_L)$ of G is defined as follows: $V_L = E$; $E_L = \{(e_1, e_2) \in E \times E \mid \bar{e}_1 \neq e_2, t(e_1) = o(e_2)\}$. There exist no reduced cycles in the oriented line graph. Thus, there is a one-to-one correspondence between prime cycles in $\vec{L}(G)$ and prime, reduced cycles in G, and so $\mathbf{Z}(G,u) = \mathbf{Z}(\vec{L}(G), u)$.

Let D be a connected digraph and $V(D) = \{v_1, \cdots, v_n\}$. Then we consider a $n \times n$ matrix $\mathbf{W} = (w_{ij})_{1 \leq i,j \leq n}$ with ij entry the complex variable w_{ij} if $(v_i, v_j) \in E(G)$, and $w_{ij} = 0$ otherwise. The matrix $\mathbf{W}$ is called the weighted matrix of D. For each path $P = (v_{i_1}, \cdots, v_{i_r})$ of D, let $w(P) = w_{i_1 i_2} w_{i_2 i_3} \cdots w_{i_{r-1} i_r}$. Furthermore, let $w(v_i, v_j) = w_{ij}$, $v_i, v_j \in V(D)$. The weighted zeta function of D is defined by

$$\mathbf{Z}(D, w) = \prod_{[C]} (1 - w(C))^{-1},$$

where $[C]$ runs over all equivalence classes of prime cycles of D. Mizuno and Sato [11] obtained a determinant expression of the weighted zeta function of a connected digraph D: $\mathbf{Z}(D,w)^{-1} = \det(\mathbf{I} - \mathbf{W})$.

For a general theory of the representation of groups, the reader is referred to [2].

2 Multiedge zeta functions of regular coverings of graphs

Let G be a connected graph, and let $N(v) = \{w \in V(G) \mid (v,w) \in E(G)\}$ for any vertex v in G. A graph H is called a covering of G with projection $\pi : H \longrightarrow G$ if there is a surjection $\pi : V(H) \longrightarrow V(G)$ such that $\pi|_{N(v')} : N(v') \longrightarrow N(v)$ is a bijection for all vertices $v \in V(G)$ and $v' \in \pi^{-1}(v)$. When a finite group Π acts on a graph G, the quotient graph G/Π is a simple graph whose vertices are the Π-orbits on $V(G)$, with two vertices adjacent in G/Π if and only if some two of their representatives are adjacent in G. A covering $\pi : H \longrightarrow G$ is said to be regular if there is a subgroup B of the automorphism group $AutH$ of H acting freely on H such that the quotient graph H/B is isomorphic to G.

Let G be a graph and Γ a finite group. Then a mapping $\alpha : E(G) \longrightarrow \Gamma$ is called an ordinary voltage assignment if $\alpha(v,u) = \alpha(u,v)^{-1}$ for each

$(u,v) \in E(G)$. The pair (G,α) is called an ordinary voltage graph. The derived graph G^α of the ordinary voltage graph (G,α) is defined as follows: $V(G^\alpha) = V(G) \times \Gamma$ and $((u,h),(v,k)) \in E(G^\alpha)$ if and only if $(u,v) \in E(G)$ and $k = h\alpha(u,v)$. The natural projection $\pi : G^\alpha \longrightarrow G$ is defined by $\pi(u,h) = u, (u,h) \in V(G^\alpha)$. The graph G^α is called a derived graph covering of G with voltages in Γ or a Γ-covering of G. The natural projection π commutes with the right multiplication action of the $\alpha(e), e \in E(G)$ and the left action of $g \in \Gamma$ on the fibers: $g \circ (u,h) = (u,gh), g \in \Gamma$, which is free and transitive. Thus, the Γ-covering G^α is a $\mid \Gamma \mid$-fold regular covering of G with covering transformation group Γ. Futhermore, every regular covering of a graph G is a Γ-covering of G for some group Γ(see [4]).

Let G be a connected graph, Γ a finite group and $\alpha : E(G) \longrightarrow \Gamma$ an ordinary voltage assignment. In the Γ-covering G^α, set $v_g = (v,g)$ and $e_g = (e,g)$, where $v \in V(G), e \in E(G), g \in \Gamma$. For $e = (u,v) \in E(G)$, the arc e_g emanates from u_g and terminates at $v_{g\alpha(e)}$. Note that $\bar{e_g} = (\bar{e})_{g\alpha(e)}$.

Let $\mathbf{W} = \mathbf{W}(G)$ be the multiedge matrix of G. Then we define the multiedge matrix $\tilde{\mathbf{W}} = \mathbf{W}(G^\alpha) = (\tilde{w}(e_g, f_h))$ of G^α derived from $\mathbf{W}$ as follows: $\tilde{w}(e_g,f_h) = w(e,f)$ if $t(e) = o(f)$, $f \neq \bar{e}$,$h = g\alpha(e)$ and $\tilde{w}(e_g,f_h) = 0$ otherwise.

Let G be a connected graph, Γ a finite group and $\alpha : E(G) \longrightarrow \Gamma$ an ordinary voltage assignment. Then we define the *net voltage* $\alpha(P)$ of each path $P = (v_1, \cdots, v_l)$ of G by $\alpha(P) = \alpha(v_1,v_2) \cdot \cdots \cdot \alpha(v_{l-1},v_l)$. We denote the order of $g \in \Gamma$ by $ord(g)$.

Theorem 1 *Let G be a connected graph, Γ a finite group with n elements, and $\alpha : E(G) \longrightarrow \Gamma$ an ordinary voltage assignment. Let $\mathbf{W} = \mathbf{W}(\vec{L}(G))$ be the weighted matrix of G. Suppose that the Γ-covering G^α of G is connected. Then the reciprocal of the multiedge zeta function of G^α is*

$$\zeta_E(\tilde{\mathbf{W}}, G^\alpha)^{-1} = \prod_{[C]} (1 - \mathbf{N}_E(C)^{ord(\alpha(C))})^{n/ord(\alpha(C))},$$

where $[C]$ runs over all equivalence classes of prime, reduced cycles of G.

Proof. Let C be any prime, reduced cycle of G^α and $\pi(C) = C_0^k$, where C_0 is a prime, reduced cycle of G and $\pi : G^\alpha \longrightarrow G$ is the natural projection. Let $m = ord(\alpha(C_0))$. By [4, Theorem 2.1.3], the preimage $\pi^{-1}(C_0)$ of C_0 in G^α is the union of n/m disjoint cycles with length $m \mid C_0 \mid$, and so $k = m$. Therefore, it follows that

$$\zeta_E(\tilde{\mathbf{W}}, G^\alpha)^{-1} = \prod_{[C_0]} (1 - \mathbf{N}_E(C_0)^{ord(\alpha(C_0))})^{n/ord(\alpha(C_0))},$$

where $[C_0]$ runs over all equivalence classes of prime, reduced cycles of G. Q.E.D.

Let $w_{ij} = u$ unless $w_{ij} = 0$. Then we obtain Theorem 1 in [12].

Corollary 1 (Sato) *Let G be a connected graph, Γ a finite group with n elements, and $\alpha : E(G) \longrightarrow \Gamma$ an ordinary voltage assignment. Suppose that the Γ-covering G^{α} of G is connected. Then the reciprocal of the zeta function of G^{α} is*

$$Z(G^{\alpha}, u)^{-1} = \prod_{[C]} (1 - u^{|C| ord(\alpha(C))})^{n/ord(\alpha(C))},$$

where $[C]$ runs over all equivalence classes of prime, reduced cycles of G.

3 Multipath zeta functions of regular coverings of graphs

Let $G = (V, E)$ be a connected graph and T a spanning tree of G. Then there exist $r = \frac{1}{2} \mid E \mid - \mid V \mid +1$ undirected edges of G not contained in T. Let $e_1, \cdots, e_r, \bar{e}_1, \cdots, \bar{e}_r$ denote the arcs left out of T. Then the fundamental group of G can be identified with the group generated by $\{e_1, \cdots, e_r, \bar{e}_1, \cdots, \bar{e}_r\}$.

Set $e_{r+1} = \bar{e}_1, \cdots, e_{2r} = \bar{e}_r$. Let the multipath matrix $\mathbf{Z} = \mathbf{Z}(G)$ of G be a $2r \times 2r$ matrix with ij entry the complex variable z_{ij} if $e_j \neq \bar{e}_i$, and $z_{ij} = 0$ otherwise. Furthermore, set $z(e_i, e_j) = z_{ij}$.

Let G' be the graph obtained from G by contracting T to a vertex. Note that G' is the bouquet with one vertex and r undirected edges. We consider a prime, reduced cycles $C = (a_1, \cdots, a_s)$ of G', where $a_j \in \{e_1, \cdots, e_r, e_{r+1}, \cdots, e_{2r}\}$. Note that C is a "reduced " product in the generators of the fundamental group of G. Then the multipath norm $\mathbf{N}_P(C)$ of C is defined as follows: $\mathbf{N}_P(C) = z(a_1, a_2) z(a_2, a_3) \cdots z(a_s, a_1)$. The multipath zeta function of G is defined by

$$\zeta_P(\mathbf{Z}, G) = \prod_{[C]} (1 - \mathbf{N}_P(C))^{-1},$$

where $[C]$ runs over all equivalence classes of prime, reduced cycles in G'. Stark and Terras [13] showed that the reciprocal of the multipath zeta function of a graph is a polynomial: $\zeta_P(\mathbf{Z}, G)^{-1} = \det(\mathbf{I} - \mathbf{Z})$.

Let $G = (V, E)$ be a connected graph, T a spanning tree of G, Γ a finite group and $\alpha : E(G) \longrightarrow \Gamma$ an ordinary voltage assignment. Furthermore, let G' be the graph obtained from G by contracting T to a vertex. In $(G')^{\alpha}$, $\mid \Gamma \mid -1$ of the lifted edges from G' must be used to complete a spanning tree of $(G')^\alpha$. The remaining $|\,\Gamma\,|\,r-(|\,\Gamma\,|-1)=|\,\Gamma\,|\,(r-1)+1$ edges of the contracted $(G^\alpha)'=((G')^\alpha)'$ give rise to the generating paths of the fundamental group of G^α.

Let $\mathbf{Z}$ be the multipath matrix of G. Then we define the multipath matrix $\tilde{\mathbf{Z}}=\mathbf{Z}(G^\alpha)=(\tilde{z}(e_g,f_h))$ of G^α derived from $\mathbf{Z}$ as follows: $\tilde{z}(e_g,f_h)=z(e,f)$ if $f\neq\bar{e}$, and $\tilde{z}(e_g,f_h)=0$ otherwise. Here the edges $(e_g,f_h)\in E(G^\alpha)$ are restricted to those of $(G^\alpha)'$, and so on.

Let x be a base vertex of T. For an ordinary voltage assignment $\alpha: E(G)\longrightarrow\Gamma$, the T-voltage α_T of α is defined by $\alpha_T(u,v)=\alpha(P_u)\alpha(u,v)\,\alpha(P_v)^{-1}$ for each $(u,v)\in E(G)$, where P_u is the unique path from x to u in T, and so on. Note that α_T is an ordinary voltage assignment, and $\alpha_T(C)=\alpha(C)$ for any cycle C of G. Furthermore, we have $\alpha_T(u,v)=1$ for each $(u,v)\in E(T)$.

For $h\in\Gamma$, the *permutation matrix* $\mathbf{P}_h=(p_{ij})$ of h in Γ is the square matrix of order n such that $p_{ij}=1$ if $g_ih=g_j$, and $p_{ij}=0$ otherwise, where $n=|\,\Gamma\,|$ and $\Gamma=\{g_1=1,g_2,\cdots,g_n\}$. A cyclic permutation $(h_1\ h_2\cdots\ h_m)$ is the permutation such that $h_1\to h_2\to\cdots\to h_m\to h_1$.

Theorem 2 *Let G be a connected graph, Γ a finite group with n elements and $\alpha: E(G)\longrightarrow\Gamma$ an ordinary voltage assignment. Furthermore, let $\tilde{\mathbf{Z}}$ be the multipath matrix of G^α derived from the multipath matrix $\mathbf{Z}$ of G. Then the reciprocal of the multipath zeta function of G^α is*

$$\zeta_P(\tilde{\mathbf{Z}},G^\alpha)^{-1}=\prod_{[C]}(1-\mathbf{N}_P(C)^{ord(\alpha(C))})^{n/ord(\alpha(C))},$$

where $[C]$ runs over all equivalence classes of prime, reduced cycles of G.

Proof. Let T be a spanning tree of G. By Corollary of Theorem 11 in [13] and the fact that $\alpha_T(C)=\alpha(C)$ for any cycle C, we have

$$\zeta_P(\tilde{\mathbf{Z}},G^\alpha)^{-1}=\prod_{[C]}\prod_{\rho}\det(\mathbf{I}-\rho(\alpha_T(C))\mathbf{N}_P(C))^f$$

$$=\prod_{[C]}\prod_{\rho}\det(\mathbf{I}-\rho(\alpha(C))\mathbf{N}_P(C))^f,$$

where ρ runs over all inequivalent irreducible representations of Γ and $f=\deg\rho$. The property of the right regular representation of a finite group implies that

$$\prod_{\rho}\det(\mathbf{I}-\rho(\alpha(C))\mathbf{N}_P(C))^f=\det(\mathbf{I}_n-\sigma(\alpha(C))\mathbf{N}_P(C))$$

$$=\det(\mathbf{I}_n-\mathbf{P}_{\alpha(C)}\mathbf{N}_P(C)),$$

where $\mathbf{P}_h$ is the permutation matrix of h in Γ, and σ is the right regular representation of Γ(see [2]).

Let $\gamma = \alpha(C)$, $H =< \gamma >$ the subgroup of Γ generated by γ, $m = ord(\gamma)$ and $k = n/m$. Furthermore, Let $\{h_1 = 1, h_2, \cdots, h_k\}$ be a set of all representatives of Γ/H. Then the disjoint cycle decomposition of $\sigma(\gamma)$ is

$$\sigma(\gamma) = (1\ \gamma \cdots\ \gamma^{m-1})(h_2\ h_2\gamma \cdots\ h_2\gamma^{m-1}) \cdots (h_k\ h_k\gamma \cdots\ h_k\gamma^{m-1}).$$

Thus,

$$\det(\mathbf{I}_n - \mathbf{P}_\gamma \mathbf{N}_P(C)) = \det(\mathbf{I}_m - \mathbf{P}'_\gamma \mathbf{N}_P(C))^k = (1 - \mathbf{N}_P(C)^m)^k,$$

where $\mathbf{P}'_\gamma$ is the permutation matrix of γ in H. Therefore, the result follows. Q.E.D.

4 Weighted zeta functions of cyclic Γ-covers

Let D be a symmetric digraph and Γ a finite group. A function $\alpha : E(D) \longrightarrow \Gamma$ is called alternating if $\alpha(y,x) = \alpha(x,y)^{-1}$ for each $(x,y) \in E(D)$. For $g \in \Gamma$, a g-cyclic Γ-cover $D_g(\alpha)$ of D is the digraph defined as follows(see [8]): $V(D_g(\alpha)) = V(D) \times \Gamma$, and $((v,h),(w,k)) \in E(D_g(\alpha))$ if and only if $(v,w) \in E(D)$ and $k^{-1}h\alpha(v,w) = g$. The natural projection $\pi : D_g(\alpha) \longrightarrow D$ is a function from $V(D_g(\alpha))$ onto $V(D)$ which erases the second coordinates. A digraph D' is called a cyclic Γ-cover of D if D' is a g-cyclic Γ-cover of D for some $g \in \Gamma$. The pair (D,α) of D and α can be considered as the ordinary voltage graph $(\tilde{D},\alpha)$ of the underlying graph $\tilde{D}$ of D. Thus the 1-cyclic Γ-cover $D_1(\alpha)$ corresponds to the Γ-covering $\tilde{D}^\alpha$, where 1 is the unit of Γ.

Let $\mathbf{W} = \mathbf{W}(D)$ be the weighted matrix of D. Then we define the weighted matrix $\mathbf{W}(D_g(\alpha)) = (\tilde{w}(u_h, v_k))$ of $D_g(\alpha)$ derived from $\mathbf{W}$ as follows: $\tilde{w}(u_h, v_k) = w(u,v)$ if $(u,v) \in E(D)$, $k = h\alpha(u,v)g^{-1}$, and $\tilde{w}(u_h, v_k) = 0$ otherwise.

Let D be a connected symmetric digraph, Γ a finite group and $\alpha : E(D) \longrightarrow \Gamma$ an alternating function. Furhthermore, let $g \in \Gamma$. Then we define the function $\alpha_g : E(D) \longrightarrow \Gamma$ as follows: $\alpha_g(v,w) = \alpha(v,w)g^{-1}$,$(v,w) \in E(D)$. For each path $P = (v_1, \cdots, v_l)$ of D, let $\alpha_g(P) = \alpha(v_1,v_2)g^{-1} \cdots \cdot \alpha(v_{l-1},v_l)g^{-1}$. Note that, if $g^2 \neq 1$, then α_g is not alternating, and so $D_g(\alpha)$ is not a Γ-covering of the underlying graph of D.

Theorem 3 *Let D be a connected symmetric digraph, Γ a finite group with n elements, $g \in \Gamma$ and $\alpha : E(D) \longrightarrow \Gamma$ an alternating function. Let $\mathbf{W} = \mathbf{W}(D)$ be the weighted matrix of D. Then the reciprocal of the* *weighted zeta function of* $D_g(\alpha)$ *is*

$$\mathbf{Z}(D_g(\alpha), \tilde{w})^{-1} = \prod_{[C]} (1 - w(C)^{ord(\alpha_g(C))})^{n/ord(\alpha_g(C))},$$

where $[C]$ *runs over all equivalence classes of prime cycles of* D.

Proof. By Corollary 5 in [11], we have

$$\mathbf{Z}(D_g(\alpha), \tilde{w})^{-1} = \prod_{[C]} \prod_{\rho} \det(\mathbf{I} - \rho(\alpha_g(C)) w(C))^f,$$

where ρ runs over all inequivalent irreducible representations of Γ and $f = \deg \rho$. Similarly to the proof of Theorem 2, the result follows. Q.E.D.

Let $w_{ij} = u$ unless $w_{ij} = 0$. Then we obtain Theorem 2 in [12].

Corollary 2 (Sato) *Let* D *be a connected symmetric digraph,* Γ *a finite group with* n *elements,* $g \in \Gamma$ *and* $\alpha : E(D) \longrightarrow \Gamma$ *an alternating function. Then the reciprocal of the zeta function of* $D_g(\alpha)$ *is*

$$\mathbf{Z}(D_g(\alpha), u)^{-1} = \prod_{[C]} (1 - u^{|C| ord(\alpha_g(C))})^{n/ord(\alpha_g(C))},$$

where $[C]$ *runs over all equivalence classes of prime cycles of* D.

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