Self-dual codes over $\mathbf{F}_\mathrm{p}$ and orthogonal designs

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Abstract

Self-dual codes is an important class of linear codes. Hadamard matrices and weighing matrices have been used widely in the construction of binary and ternary self-dual codes. Recently weighing matrices and orthogonal designs have been used to construct self-dual codes over larger fields. In this paper we further investigate codes over F_p constructed from orthogonal designs. Necessary conditions for these codes to be self-dual are established, and examples are given for lengths up to 40. Self-dual codes of lengths $2n \geq 16$ over $GF(31)$ and $GF(37)$ are investigated here for the first time. We also show that codes obtained from orthogonal designs can generally give better results, with respect to their minimum Hamming distance, than codes obtained from Hadamard matrices, weighing matrices or conference matrices.

Key words and phrases: Self-dual codes, orthogonal designs, weighing matrices.

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1 Introduction and basic definitions

In this section, we recall some basic notions on orthogonal designs and self-dual codes over F_p.

1.1 Orthogonal designs

An *orthogonal design* of order n and type $(s_1, s_2, \dots, s_u)$ $(s_i > 0)$, denoted $OD(n; s_1, s_2, \dots, s_u)$, on the commuting variables $x_1, x_2, \dots, x_u$ is an $n \times n$

matrix D with entries from the set $\{0, \pm x_1, \pm x_2, \ldots, \pm x_u\}$ such that

$$DD^T = (\sum_{i=1}^{u} s_i x_i^2) I_n$$

Alternatively, the rows of D are formally orthogonal and each row has precisely s_i entries of the type $\pm x_i$. In [7], where this was first defined, it was mentioned that

$$D^T D = (\sum_{i=1}^{u} s_i x_i^2) I_n$$

and so our alternative description of D applies equally well to the columns of D. It was also shown in [7] that $u \leq \rho(n)$, where $\rho(n)$ (Radon's function) is defined by $\rho(n) = 8c + 2^d$, when $n = 2^a b$, b odd, $a = 4c + d$, $0 \leq d < 4$. Although orthogonal designs of order n on n variables exist only for $n = 1, 2, 4, 8$, there are many orthogonal designs on u variables, where $u \leq \rho(n)$.

Example 1 D_{8a} is an orthogonal design of order eight on eight variables.

$$D_{8a} = OD(8; 1,1,1,1,1,1,1,1) = \\ = \begin{pmatrix} a & b & c & d & e & f & g & h \\ -b & a & d & -c & f & -e & -h & g \\ -c & -d & a & b & g & h & -e & -f \\ -d & c & -b & a & h & -g & f & -e \\ -e & -f & -g & -h & a & b & c & d \\ -f & e & -h & g & -b & a & -d & c \\ -g & h & e & -f & -c & d & a & -b \\ -h & -g & f & e & -d & -c & b & a \end{pmatrix}.$$

For more details and construction methods for orthogonal designs see [8].

A weighing matrix $W = W(n, k)$ is a square matrix with entries $0, \pm 1$ having k non-zero entries per row and column and inner product of distinct rows zero. Hence W satisfies $WW^T = kI_n$, and W is equivalent to an orthogonal design $OD(n; k)$. The number k is called the *weight* of W. If $k = n$, that is, all the entries of W are ± 1 and $WW^T = nI_n$, then W is called an Hadamard matrix of order n. In this case $n = 1, 2$ or $n \equiv 0 (mod\ 4)$. To make this clear we give the following example.

Example 2 If we replace all variables of D_{8a} by 1 we obtain a Hadamard matrix H of order 8, if we replace one variable by zero (i.e $a = 0$) we have an orthogonal design $D_{8b} = OD(8; 1,1,1,1,1,1,1)$, and if we replace one variable by zero (i.e $a = 0$) and all others by 1 we obtain a weighing matrix W of order 8 and weight 7.

$$H = H_8 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 \\ -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 \\ -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 \\ -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \end{pmatrix}.$$

$$D_{8b} = OD(8; 1,1,1,1,1,1,1) =$$
$$\begin{pmatrix} 0 & b & c & d & e & f & g & h \\ -b & 0 & d & -c & f & -e & -h & g \\ -c & -d & 0 & b & g & h & -e & -f \\ -d & c & -b & 0 & h & -g & f & -e \\ -e & -f & -g & -h & 0 & b & c & d \\ -f & e & -h & g & -b & 0 & -d & c \\ -g & h & e & -f & -c & d & 0 & -b \\ -h & -g & f & e & -d & -c & b & 0 \end{pmatrix}.$$

$$W = W(8,7) =$$
$$= \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & -1 & 1 & -1 & -1 & 1 \\ -1 & -1 & 0 & 1 & 1 & 1 & -1 & -1 \\ -1 & 1 & -1 & 0 & 1 & -1 & 1 & -1 \\ -1 & -1 & -1 & -1 & 0 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 & -1 & 0 & -1 & 1 \\ -1 & 1 & 1 & -1 & -1 & 1 & 0 & -1 \\ -1 & -1 & 1 & 1 & -1 & -1 & 1 & 0 \end{pmatrix}.$$

1.2 Self-dual codes

For our consideration we also need some facts from coding theory. Our terminology and notation follow [10]. Let $F = GF(p)$ be the field with p elements where p is a prime power. An $[n, k]$ linear code C over F is a k-dimensional vector subspace of F^n. The elements of C are called codewords and the weight of the codeword is the number of its non-zero coordinates. The minimum weight is the smallest weight among non-zero codewords. An $[n, k]$ code with minimum weight d is called $[n, k, d]$ code. Two binary codes are equivalent if one can be obtained from the other by a permutation of the coordinates.

The dual code $C^{\perp}$ of C is defined as $C^{\perp} = \{x \in F^n | x \cdot y = 0 \ \ for\ all\ \ y \in C\}$. If $C \subset C^{\perp}$, C is called a self-orthogonal code. C is called self-dual if $C = C^{\perp}$.

A self-dual code C is called extremal if C has the largest possible minimum weight. The known bounds of d for $q = 2, 3, 4$ are given in [12] and [13].

For more details on self-dual codes over finite fields GF(p) where p is a prime power, we refer to [1, 2, 10]. In this paper we only consider self-dual codes over $GF(p)$, for p prime, and thus we will use F_p to denote the $GF(p)$.

Recently Arasu and Gulliver [1] gave some examples of self- dual codes constructed from Hadamard matrices and weighing matrices. The method described in the next section is a generalization of the methods given in [1] since Hadamard matrices and weighing matrices are special cases of orthogonal designs.

2 Construction methods from orthogonal designs

In this section, we give a method for constructing self-dual codes over F_p using orthogonal designs.

The following proposition is a direct method for constructing self-dual codes using orthogonal designs and was given for the first time in [4, 5].

Theorem 1 *Let R be the finite field F_p. Let $D = OD(n; s_1,\ s_2, \ldots, s_u)$ be an orthogonal design of order n and type $(s_1, s_2, \ldots, s_u)$ on the commuting variables $x_1, x_2, \ldots, x_u$. We replace the variables with elements from R in such a way that we obtain*

$$\left(\sum_{i=1}^{u} s_i x_i^2 \right) + z^2 = 0 \ in\ R,\ \ and\ z \in R - \{0\}$$

and denote the derived matrix with A. Suppose that z is a unit of R. Then the matrix

$$G = (\ zI_n\ ,\ A\ )$$

generates a self-dual code C over R of length $2n$.

In order to illustrate the above method, consider the following orthogonal design $D = OD(4; 1, 1, 1, 1)$:

$$D = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ -x_2 & x_1 & -x_4 & x_3 \\ -x_3 & x_4 & x_1 & -x_2 \\ -x_4 & -x_3 & x_2 & x_1 \end{pmatrix}.$$

Then by appropriate replacement of the variables we obtain the following matrix.

$$A_1 = \begin{pmatrix} 1 & 3 & 2 & 0 \\ -3 & 1 & 0 & 2 \\ -2 & 0 & 1 & -3 \\ 0 & -2 & 3 & 1 \end{pmatrix}$$

Matrix (I_4 , A_1) generate a self-dual code C_5 over GF(5). It is not difficult to see that the code C_5 is equivalent to the code F_4 in [9].

3 Codes constructed from orthogonal designs

Let $D = OD(n; s_1,\ s_2, \dots, s_u)$ be an orthogonal design of order n and type $(s_1, s_2, \dots, s_u)$ on the commuting variables $x_1, x_2, \dots, x_u$. Using the construction of theorem 1 we can see that each row of the generator matrix has at most

$$d_0 = 1 + \sum_{i=1}^{u} s_i \tag{1}$$

non zero elements. Thus the minimum distance d of the derived code C generated by G as it is shown in theorem 1 is upper bounded by d_0 (i.e. $d \le d_0$). So, it is only necessary to consider orthogonal designs with none (full orthogonal designs) or few zeros. Full orthogonal designs could give codes with larger minimum distances and in this case the following lemma is known.

Lemma 1 *If a full orthogonal design of order n exists, then $n = 1, 2$ or $n \equiv 0 \ (mod\ 4)$.*

For example the bounds on minimum distance (if all variables are replaced by non zero elements) for the codes constructed from orthogonal designs D_{8a} and D_{8b} are 9 and 8 respectively.

In [1] the authors gave in tables the parameters of the codes they have found using Hadamard matrices, weighing matrices and conference matrices. They study examples of self-dual codes in F_p for every p prime up to 23, but in some cases the minimum distance of the codes obtained by their methods are poor. In this section we present the results we found using orthogonal designs and finally we give an update to these tables improving the distances in some cases. We also consider fields F_{29}, F_{31} and F_{37}. These results are also given in tables below.

3.1 [4, 2] codes

There is an orthogonal design of order 2 given by

$$D_2 = OD(2;1,1) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

where a, b are commuting variables. Using this matrix with $a = 1, b = 0$, and $z = p - 1$ we obtain a self dual code over F_p iff $p - 1 \equiv x^2 \ (mod\ p)$, for a $x \in F_p$, i.e. $p = 2, 5, \ldots$. For $p \geq 7$ or $p = 3$ there exist a proper replacement of the variables, as it is shown in Table 1, such that a self-dual code $[4, 2, d_m]$ exists over F_p, for all $p \leq 37$. For $p \leq 23$ same results have been found using conference matrices by Arasu and Gulliver in [1].

3.2 [8, 4] codes

There is an orthogonal design of order 4 given by

$$D_4 = OD(4;1,1,1,1) = \begin{pmatrix} a & b & c & d \\ -b & a & d & -c \\ -c & -d & a & b \\ -d & c & -b & a \end{pmatrix}$$

where a, b, c, d are commuting variables. Using this matrix with a proper replacement of the variables, as it is shown in Table 1, a self-dual code $[8, 4, d_m]$ exists over F_p, for all $p \leq 37$. For $5 \leq p \leq 23$ the same results have been found using conference matrices by Arasu and Gulliver in [1]. A self-dual code $[8, 4, 3]$ over F_3 cannot be constructed using conference matrices (see [1]) but it is easily constructed using orthogonal designs.

Table 1: Self-dual codes from orthogonal designs with $2n = 4, 8$

	$2n = 4$				$2n = 8$					
p	a	b	d_m	d_b	a	b	c	d	d_m	d_b
3	1	1	3	3	0	0	1	1	3	3
5	0	2	2	3	0	1	2	2	4	4
7	2	3	3	3	1	2	2	2	5	5
11	1	3	3	3	1	1	4	5	5	5
13	3	4	3	3	1	2	6	6	5	5
17	1	7	3	3	1	1	1	8	5	5
19	1	6	3	3	1	1	3	8	5	5
23	3	6	3	3	1	1	2	4	5	5
29	2	13	3	3	1	2	4	6	5	5
31	5	6	3	3	1	3	1	9	5	5
37	3	8	3	3	1	1	3	5	5	5

3.3 [12, 6] codes

There is an orthogonal design of order 6 given by

$$D_6 = OD(6;1,4) = \begin{pmatrix} b & a & -b & b & 0 & b \\ -b & b & a & b & b & 0 \\ a & -b & b & 0 & b & b \\ -b & -b & 0 & b & -b & a \\ 0 & -b & -b & a & b & -b \\ -b & 0 & -b & -b & a & b \end{pmatrix}$$

where a, b are commuting variables. Using this matrix and a proper replacement of the variables, as it is shown in Table 2, a self-dual code $[12, 6, d_m]$ exists over F_p, for all $p \leq 37$. For $p = 3, 7$ and 23 the same results have been found, using conference matrices, by Arasu and Gulliver in [1]. The other self-dual codes of length $2n = 12$ over F_p, $p \neq 3, 7, 23$ given in Table 2 cannot be constructed from conference matrices (see [1]) but are easily constructed using orthogonal designs.

Table 2: Self-dual codes from orthogonal designs with $2n = 12, 20$

	$2n = 12$				$2n = 20$			
p	a	b	d_m	d_b	a	b	d_m	d_b
3	1	1	6	6	1	2	6	6
5	0	1	4	6	c = 1		8	8-9
7	3	1	6	6	1	2	6	9-10
11	4	2	6	7	2	2	6	10
13	3	2	6	7	c = 6		8	10-11
17	1	5	6	7	c = 7		8	10-11
19	1	3	6	7	3	9	6	11
23	8	1	6	7	1	4	6	9-11
29	8	7	6	7	c = 4		8	10-11
31	13	2	6	7	2	9	6	10-11
37	3	4	6	7	c = 2		8	10-11

3.4 [16, 8] codes

In this case we are going to use the orthogonal design $D_{8a} = OD(8; 1, 1, 1, 1, 1, 1, 1, 1)$ with a proper replacement of the variables, as it is shown in Table 3, a self-dual code $[16, 8, d_m]$ exists over F_p, for all $p \leq 37$. For $p = 3, 7, 11$ and 13 the same results have been found, using conference matrices, by Arasu and Gulliver in [1]. The other self-dual codes of length $2n = 16$ over

F_p, $p = 7, 17, 19, 23, 29, 31, 37$ given in Table 3 are new since these codes cannot be constructed from conference matrices (see [1]).

Table 3: Self-dual codes from orthogonal designs with $2n = 16$

	$2n = 16$									
p	a	b	c	d	e	f	g	h	d_m	d_b
3	1	1	1	1	1	1	1	1	6	6
5	1	2	2	2	2	2	2	2	7	$7-8$
7	1	1	1	1	2	1	3	3	6	$7-8$
11	1	1	1	1	1	1	5	1	7	$7-8$
13	1	1	1	1	2	1	5	2	7	$7-9$
17	1	2	3	3	7	5	6	6	8	$8-9$
19	1	1	1	1	4	2	8	5	8	$8-9$
23	1	1	1	1	3	1	8	6	8	$8-9$
29	1	1	1	1	2	1	12	7	8	$8-9$
31	1	1	1	1	2	1	9	8	8	$8-9$
37	1	1	1	1	2	1	14	4	8	$8-9$

3.5 [20, 10] codes

There is an orthogonal design D_{10a} of order 10 given by

$$D_{10a} = OD(10; 4, 4) = \\ = \begin{pmatrix} a & b & a & -b & 0 & a & b & -a & b & 0 \\ 0 & a & b & a & -b & 0 & a & b & -a & b \\ -b & 0 & a & b & a & b & 0 & a & b & -a \\ a & -b & 0 & a & b & -a & b & 0 & a & b \\ b & a & -b & 0 & a & b & -a & b & 0 & a \\ -a & 0 & -b & a & -b & a & 0 & -b & a & b \\ -b & -a & 0 & -b & a & b & a & 0 & -b & a \\ a & -b & -a & 0 & -b & a & b & a & 0 & -b \\ -b & a & -b & -a & 0 & -b & a & b & a & 0 \\ 0 & -b & a & -b & -a & 0 & -b & a & b & a \end{pmatrix}$$

where a, b are commuting variables and an orthogonal design D_{10b} given in [6]. Using these matrices and a proper replacement of the variables, as it is shown in Table 2, a self-dual code $[20, 10, d_m]$ exists over F_p, for all $p \leq 37$. For $p = 5$ the same result has been found, using conference matrices, by

Arasu and Gulliver in [1].

$$D_{10b} = OD(10;9) =$$
$$= \begin{pmatrix} c & c & c & c & -c & c & c & -c & 0 & -c \\ -c & c & c & c & c & -c & c & c & -c & 0 \\ c & -c & c & c & c & 0 & -c & c & c & -c \\ c & c & -c & c & c & -c & 0 & -c & c & c \\ c & c & c & -c & c & c & -c & 0 & -c & c \\ -c & c & 0 & c & -c & c & -c & c & c & c \\ -c & -c & c & 0 & c & c & c & -c & c & c \\ c & -c & -c & c & 0 & c & c & c & -c & c \\ 0 & c & -c & -c & c & c & c & c & c & -c \\ c & 0 & c & -c & -c & -c & c & c & c & c \end{pmatrix}$$

The other self-dual codes of length $2n = 20$ over F_p, $p \neq 5$ given in Table 2 cannot be constructed from conference matrices (see [1]) but are easily constructed using orthogonal designs.

3.6 [24, 12] codes

There are 12 different cases of orthogonal designs, given in [8, p. 348], of order 12 on four commuting variables a, b, c, d. We shall use $D_{12-1}, D_{12-2}, \ldots,$ D_{12-12} to denote these designs in the order that are given in [8, p. 348]. Using these matrices and a proper replacement of the variables, as it is shown in Table 4, a self-dual code $[24, 12, d_m]$ exists over F_p, for all $p \leq 37$. For $p = 3, 5$ and 7 the same results have been found, using conference matrices, by Arasu and Gulliver in [1]. The other self-dual codes of length 24 over F_p, $p \neq 3, 5, 7$ given in Table 4 cannot be constructed from conference matrices (see [1]) but are easily constructed using orthogonal designs. These codes improve the distance bounds as these were given by Arasu and Gulliver in [1]. Note that for $p = 23$ a $[24, 12, 13]$ code exists (see [10, Theorem 9, p.323]).

We give explicitly the two designs that give the best self-dual codes we found. From now on we denote $-x$ by $\bar{x}$.

$$D_{12-3} = OD(12;1,1,1,9) =$$

$$= \begin{pmatrix}
a & d & \bar{d} & \bar{d} & d & b & \bar{d} & d & c & d & d & d \\
\bar{d} & a & d & d & b & \bar{d} & d & c & \bar{d} & d & d & d \\
d & \bar{d} & a & b & \bar{d} & d & c & \bar{d} & d & d & d & d \\
d & \bar{d} & \bar{b} & a & d & \bar{d} & d & d & d & \bar{d} & d & \bar{c} \\
\bar{d} & \bar{b} & d & \bar{d} & a & d & d & d & d & d & \bar{c} & \bar{d} \\
\bar{b} & d & \bar{d} & d & \bar{d} & a & d & d & d & \bar{c} & \bar{d} & d \\
d & \bar{d} & \bar{c} & \bar{d} & \bar{d} & \bar{d} & a & d & \bar{d} & d & \bar{d} & b \\
\bar{d} & \bar{c} & d & \bar{d} & \bar{d} & \bar{d} & \bar{d} & a & d & \bar{d} & b & d \\
\bar{c} & d & \bar{d} & \bar{d} & \bar{d} & \bar{d} & d & \bar{d} & a & b & d & \bar{d} \\
\bar{d} & \bar{d} & \bar{d} & d & \bar{d} & c & \bar{d} & d & \bar{b} & a & d & \bar{d} \\
\bar{d} & \bar{d} & \bar{d} & \bar{d} & c & d & d & \bar{b} & \bar{d} & \bar{d} & a & d \\
\bar{d} & \bar{d} & \bar{d} & c & d & \bar{d} & \bar{b} & \bar{d} & d & d & \bar{d} & a
\end{pmatrix}$$

$$D_{12-5} = OD(12;1,1,2,8) =$$

$$= \begin{pmatrix}
a & d & \bar{d} & \bar{d} & d & b & d & d & c & d & d & \bar{c} \\
\bar{d} & a & d & d & b & \bar{d} & d & c & d & d & \bar{c} & d \\
d & \bar{d} & a & b & \bar{d} & d & c & d & d & \bar{c} & d & d \\
d & \bar{d} & \bar{b} & a & d & \bar{d} & d & d & \bar{c} & \bar{d} & \bar{d} & \bar{c} \\
\bar{d} & \bar{b} & d & \bar{d} & a & d & d & \bar{c} & d & \bar{d} & \bar{c} & \bar{d} \\
\bar{b} & d & \bar{d} & d & \bar{d} & a & \bar{c} & d & d & \bar{c} & \bar{d} & \bar{d} \\
\bar{d} & \bar{d} & \bar{c} & \bar{d} & \bar{d} & c & a & d & \bar{d} & d & \bar{d} & b \\
\bar{d} & \bar{c} & \bar{d} & \bar{d} & c & \bar{d} & \bar{d} & a & d & \bar{d} & b & d \\
\bar{c} & \bar{d} & \bar{d} & c & \bar{d} & \bar{d} & d & \bar{d} & a & b & d & \bar{d} \\
\bar{d} & \bar{d} & c & d & d & c & \bar{d} & d & \bar{b} & a & d & \bar{d} \\
\bar{d} & c & \bar{d} & d & c & d & d & \bar{b} & \bar{d} & \bar{d} & a & d \\
c & \bar{d} & \bar{d} & c & d & d & \bar{b} & \bar{d} & d & d & \bar{d} & a
\end{pmatrix}$$

3.7 [28, 14] codes

There are many different cases of orthogonal designs, given in [11], of order 14 on one or two commuting variables a or a, b respectively. We shall use some of these matrices with a proper replacement of the variables, as it is shown in Table 5, to construct self-dual codes $[28, 14, d_m]$ over F_p, for all $p \leq 37$. For $p = 7$ and 11 the same results have been found, using conference matrices, by Arasu and Gulliver in [1]. The other self-dual codes of length $2n = 28$ over F_p, $p \neq 7, 11$ given in Table 5 cannot be constructed from conference matrices (see [1]) but are easily constructed using orthogonal designs. These codes improve the distance bounds as these were given by Arasu and Gulliver in [1].

Table 4: Self-dual codes from orthogonal designs with $2n = 24$

		$2n = 24$					
p	Design	a	b	c	d	d_m	d_b
3	D_{12-3}	0	1	1	1	9	9
5	D_{12-3}	0	2	2	2	9	$9-10$
7	D_{12-3}	0	1	2	2	9	$10-12$
11	D_{12-5}	0	1	1	4	9	$9-12$
13	D_{12-3}	0	1	6	4	10	$10-13$
17	D_{12-3}	0	1	4	7	10	$10-13$
19	D_{12-3}	0	1	4	6	10	$10-13$
23	D_{12-3}	0	6	7	4	10	13
29	D_{12-3}	0	1	2	3	10	$10-13$
31	D_{12-3}	0	1	2	12	10	$10-13$
37	D_{12-3}	0	1	3	9	10	$10-13$

We give explicitly the three designs that give the best self-dual codes we found.

$$D_{14-1} = OD(14;1,4) =$$

$$= \begin{pmatrix}
0 & 0 & 0 & b & 0 & b & 0 & 0 & 0 & 0 & b & \bar{a} & \bar{b} & 0 \\
0 & 0 & 0 & 0 & b & 0 & b & 0 & 0 & 0 & 0 & b & \bar{a} & \bar{b} \\
b & 0 & 0 & 0 & 0 & b & 0 & \bar{b} & 0 & 0 & 0 & 0 & b & \bar{a} \\
0 & b & 0 & 0 & 0 & 0 & b & \bar{a} & \bar{b} & 0 & 0 & 0 & 0 & b \\
b & 0 & b & 0 & 0 & 0 & 0 & b & \bar{a} & \bar{b} & 0 & 0 & 0 & 0 \\
0 & b & 0 & b & 0 & 0 & 0 & 0 & b & \bar{a} & \bar{b} & 0 & 0 & 0 \\
0 & 0 & b & 0 & b & 0 & 0 & 0 & 0 & b & \bar{a} & \bar{b} & 0 & 0 \\
0 & 0 & b & a & \bar{b} & 0 & 0 & 0 & 0 & b & 0 & b & 0 & 0 \\
0 & 0 & 0 & b & a & \bar{b} & 0 & 0 & 0 & 0 & b & 0 & b & 0 \\
0 & 0 & 0 & 0 & b & a & \bar{b} & 0 & 0 & 0 & 0 & b & 0 & b \\
\bar{b} & 0 & 0 & 0 & 0 & b & a & b & 0 & 0 & 0 & 0 & b & 0 \\
a & \bar{b} & 0 & 0 & 0 & 0 & b & 0 & b & 0 & 0 & 0 & 0 & b \\
b & a & \bar{b} & 0 & 0 & 0 & 0 & b & 0 & b & 0 & 0 & 0 & 0 \\
0 & b & a & \bar{b} & 0 & 0 & 0 & 0 & b & 0 & b & 0 & 0 & 0
\end{pmatrix}$$

$$D_{14-2} = OD(14;2,8) =$$

$$= \begin{pmatrix}
b & a & \bar{b} & b & 0 & b & 0 & b & a & \bar{b} & \bar{b} & 0 & \bar{b} & 0 \\
0 & b & a & \bar{b} & b & 0 & b & 0 & b & a & \bar{b} & \bar{b} & 0 & \bar{b} \\
b & 0 & b & a & \bar{b} & b & 0 & \bar{b} & 0 & b & a & \bar{b} & \bar{b} & 0 \\
0 & b & 0 & b & a & \bar{b} & b & 0 & \bar{b} & 0 & b & a & \bar{b} & \bar{b} \\
b & 0 & b & 0 & b & a & \bar{b} & \bar{b} & 0 & \bar{b} & 0 & b & a & \bar{b} \\
\bar{b} & b & 0 & b & 0 & b & a & \bar{b} & \bar{b} & 0 & \bar{b} & 0 & b & a \\
a & \bar{b} & b & 0 & b & 0 & b & a & \bar{b} & \bar{b} & 0 & \bar{b} & 0 & b \\
\bar{b} & 0 & b & 0 & b & b & \bar{a} & b & 0 & b & 0 & b & \bar{b} & a \\
\bar{a} & \bar{b} & 0 & b & 0 & b & b & a & b & 0 & b & 0 & b & \bar{b} \\
b & \bar{a} & \bar{b} & 0 & b & 0 & b & \bar{b} & a & b & 0 & b & 0 & b \\
b & b & \bar{a} & \bar{b} & 0 & b & 0 & b & \bar{b} & a & b & 0 & b & 0 \\
0 & b & b & \bar{a} & \bar{b} & 0 & b & 0 & b & \bar{b} & a & b & 0 & b \\
b & 0 & b & b & \bar{a} & \bar{b} & 0 & b & 0 & b & \bar{b} & a & b & 0 \\
0 & b & 0 & b & b & \bar{a} & \bar{b} & 0 & b & 0 & b & \bar{b} & a & b
\end{pmatrix}$$

$$D_{14-3} = OD(14;1,9) =$$

$$= \begin{pmatrix}
a & b & b & \bar{b} & b & \bar{b} & \bar{b} & 0 & b & b & 0 & b & 0 & 0 \\
\bar{b} & a & b & b & \bar{b} & b & \bar{b} & 0 & 0 & b & b & 0 & b & 0 \\
\bar{b} & \bar{b} & a & b & b & \bar{b} & b & 0 & 0 & 0 & b & b & 0 & b \\
b & \bar{b} & \bar{b} & a & b & b & \bar{b} & b & 0 & 0 & 0 & b & b & 0 \\
\bar{b} & b & \bar{b} & \bar{b} & a & b & b & 0 & b & 0 & 0 & 0 & b & b \\
b & \bar{b} & b & \bar{b} & \bar{b} & a & b & b & 0 & b & 0 & 0 & 0 & b \\
b & b & \bar{b} & b & \bar{b} & \bar{b} & a & b & b & 0 & b & 0 & 0 & 0 \\
0 & 0 & 0 & \bar{b} & 0 & \bar{b} & \bar{b} & a & \bar{b} & \bar{b} & b & \bar{b} & b & b \\
\bar{b} & 0 & 0 & 0 & \bar{b} & 0 & \bar{b} & b & a & \bar{b} & \bar{b} & b & \bar{b} & b \\
\bar{b} & \bar{b} & 0 & 0 & 0 & \bar{b} & 0 & b & b & a & \bar{b} & \bar{b} & b & \bar{b} \\
0 & \bar{b} & \bar{b} & 0 & 0 & 0 & \bar{b} & \bar{b} & b & b & a & \bar{b} & \bar{b} & b \\
\bar{b} & 0 & \bar{b} & \bar{b} & 0 & 0 & 0 & b & \bar{b} & b & b & a & \bar{b} & \bar{b} \\
0 & \bar{b} & 0 & \bar{b} & \bar{b} & 0 & 0 & \bar{b} & b & \bar{b} & b & b & a & \bar{b} \\
0 & 0 & \bar{b} & 0 & \bar{b} & \bar{b} & 0 & \bar{b} & \bar{b} & b & \bar{b} & b & b & a
\end{pmatrix}$$

3.8 [32, 16] codes

Since $16 = 2^4$ there are many different cases of orthogonal designs, given in [8], of order 16 on up to 9 variables. We shall use one of these matrices with a proper replacement of the variables, as it is shown in Table 6, to construct self-dual codes $[32, 16, d_m]$ over F_p, for all $p \leq 37$. For $p = 5, 7, 11$ and 13 the same results have been found, using conference matrices, by Arasu and Gulliver in [1]. The other self-dual codes of length 32 over F_p, $p \neq$

Table 5: Self-dual codes from orthogonal designs with $2n = 28$

		$2n = 28$			
p	Design	a	b	d_m	d_b
3	D_{14-1}	1	1	6	$8-9$
5	D_{14-2}	1	2	8	$8-13$
7	D_{14-3}	2	1	10	$11-13$
11	D_{14-3}	3	4	10	$10-14$
13	D_{14-3}	4	1	11	$11-15$
17	D_{14-3}	1	8	11	$11-15$
19	D_{14-3}	1	2	11	$11-15$
23	D_{14-3}	2	5	11	$11-15$
29	D_{14-3}	2	14	11	$11-15$
31	D_{14-3}	4	6	11	$11-15$
37	D_{14-3}	3	15	11	$11-15$

$5, 7, 11, 13$ given in Table 6 cannot be constructed from conference matrices (see [1]) but are easily constructed using orthogonal designs. These codes improve the distance bounds as these were given by Arasu and Gulliver in [1]. Note that for $p = 31$ a $[32, 16, 17]$ code exists (see [10, Theorem 9, p.323]).

Orthogonal design $OD(16; 1, 1, 1, 1, 2, 2, 2, 2, 2)$ that gives the best self-dual codes we have found is given explicitly in [8, p. 361].

3.9 $[36, 18]$ codes

Since $18 = 2{\cdot}9$ there are only few different cases of orthogonal designs, given in [11], of order 18 on up to 2 variables. So the results obtained here are not so good, as expected. We shall use three of these matrices with a proper replacement of the variables, as it is shown in Table 7, to construct self-dual codes $[36, 18, d_m]$ over F_p, for all $p \leq 37$. For $p = 3, 7, 11, 13$ and 23 the same results have been found, using conference matrices, by Arasu and Gulliver in [1]. The other self-dual codes of length 36 over F_p, $p \neq 3, 7, 11, 13, 23$ given in Table 7 cannot be constructed from conference matrices (see [1]) but are easily constructed using orthogonal designs. Unfortunately, these codes have low distance. We give explicitly the three designs that give the best self-dual codes we found.

Table 6: Self-dual codes from orthogonal designs with $2n = 32$

	$2n = 32$										
p	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	d_m	d_b
3	1	1	0	0	1	1	1	0	0	9	9
5	2	1	0	0	1	1	1	0	2	10	10 – 13
7	3	3	3	0	1	3	1	3	1	11	11 – 14
11	5	5	5	5	3	5	2	2	1	11	11 – 16
13	6	6	0	3	4	6	1	3	1	11	11 – 16
17	6	3	1	1	8	8	5	1	3	12	12 – 17
19	15	15	7	0	14	8	1	9	15	12	12 – 17
23	10	0	5	6	10	9	9	6	9	12	12 – 17
29	14	13	2	11	13	6	1	7	6	12	12 – 17
31	21	25	16	7	3	14	14	20	1	12	17
37	18	18	11	2	3	18	0	17	8	12	12 – 17

$$D_{18-1} = OD(18;17) =$$

$$= \begin{pmatrix}
a & \bar{a} & \bar{a} & a & a & a & a & \bar{a} & \bar{a} & 0 & \bar{a} & a & \bar{a} & \bar{a} & \bar{a} & \bar{a} & a & \bar{a} \\
\bar{a} & a & \bar{a} & \bar{a} & a & a & a & a & \bar{a} & \bar{a} & 0 & \bar{a} & a & \bar{a} & \bar{a} & \bar{a} & \bar{a} & a \\
\bar{a} & \bar{a} & a & \bar{a} & \bar{a} & a & a & a & a & a & \bar{a} & 0 & \bar{a} & a & \bar{a} & \bar{a} & \bar{a} & \bar{a} \\
a & \bar{a} & \bar{a} & a & \bar{a} & \bar{a} & a & a & a & \bar{a} & a & \bar{a} & 0 & \bar{a} & a & \bar{a} & \bar{a} & \bar{a} \\
a & a & \bar{a} & \bar{a} & a & \bar{a} & \bar{a} & a & a & \bar{a} & \bar{a} & a & \bar{a} & 0 & \bar{a} & a & \bar{a} & \bar{a} \\
a & a & a & \bar{a} & \bar{a} & a & \bar{a} & \bar{a} & a & \bar{a} & \bar{a} & \bar{a} & a & \bar{a} & 0 & \bar{a} & a & \bar{a} \\
a & a & a & a & \bar{a} & \bar{a} & a & \bar{a} & \bar{a} & \bar{a} & \bar{a} & \bar{a} & \bar{a} & a & \bar{a} & 0 & \bar{a} & a \\
\bar{a} & a & a & a & a & \bar{a} & \bar{a} & a & \bar{a} & a & \bar{a} & \bar{a} & \bar{a} & \bar{a} & a & \bar{a} & 0 & \bar{a} \\
\bar{a} & \bar{a} & a & a & a & a & \bar{a} & \bar{a} & a & \bar{a} & a & \bar{a} & \bar{a} & \bar{a} & \bar{a} & a & \bar{a} & 0 \\
0 & a & \bar{a} & a & a & a & a & \bar{a} & a & a & \bar{a} & \bar{a} & a & a & a & a & \bar{a} & \bar{a} \\
a & 0 & a & \bar{a} & a & a & a & a & \bar{a} & \bar{a} & a & \bar{a} & \bar{a} & a & a & a & a & \bar{a} \\
\bar{a} & a & 0 & a & \bar{a} & a & a & a & a & \bar{a} & \bar{a} & a & \bar{a} & \bar{a} & a & a & a & a \\
a & \bar{a} & a & 0 & a & \bar{a} & a & a & a & a & \bar{a} & \bar{a} & a & \bar{a} & \bar{a} & a & a & a \\
a & a & \bar{a} & a & 0 & a & \bar{a} & a & a & a & a & \bar{a} & \bar{a} & a & \bar{a} & \bar{a} & a & a \\
a & a & a & \bar{a} & a & 0 & a & \bar{a} & a & a & a & a & \bar{a} & \bar{a} & a & \bar{a} & \bar{a} & a \\
a & a & a & a & \bar{a} & a & 0 & a & \bar{a} & a & a & a & a & \bar{a} & \bar{a} & a & \bar{a} & \bar{a} \\
\bar{a} & a & a & a & a & \bar{a} & a & 0 & a & \bar{a} & a & a & a & a & \bar{a} & \bar{a} & a & \bar{a} \\
a & \bar{a} & a & a & a & a & \bar{a} & a & 0 & \bar{a} & \bar{a} & a & a & a & a & \bar{a} & \bar{a} & a
\end{pmatrix}$$

$$D_{18-2} = OD(18;13) =$$

$$= \begin{pmatrix}
a & 0 & \bar{a} & a & \bar{a} & 0 & 0 & a & a & \bar{a} & 0 & 0 & a & a & a & \bar{a} & a & a \\
a & a & 0 & \bar{a} & a & \bar{a} & 0 & 0 & a & a & \bar{a} & 0 & 0 & a & a & a & \bar{a} & a \\
a & a & a & 0 & \bar{a} & a & \bar{a} & 0 & 0 & a & a & \bar{a} & 0 & 0 & a & a & a & \bar{a} \\
0 & a & a & a & 0 & \bar{a} & a & \bar{a} & 0 & \bar{a} & a & a & \bar{a} & 0 & 0 & a & a & a \\
0 & 0 & a & a & a & 0 & \bar{a} & a & \bar{a} & a & \bar{a} & a & a & \bar{a} & 0 & 0 & a & a \\
\bar{a} & 0 & 0 & a & a & a & 0 & \bar{a} & a & a & a & \bar{a} & a & a & \bar{a} & 0 & 0 & a \\
a & \bar{a} & 0 & 0 & a & a & a & 0 & \bar{a} & a & a & a & \bar{a} & a & a & \bar{a} & 0 & 0 \\
\bar{a} & a & \bar{a} & 0 & 0 & a & a & a & 0 & 0 & a & a & a & \bar{a} & a & a & \bar{a} & 0 \\
0 & \bar{a} & a & \bar{a} & 0 & 0 & a & a & a & 0 & 0 & a & a & a & \bar{a} & a & a & \bar{a} \\
a & \bar{a} & \bar{a} & a & \bar{a} & \bar{a} & \bar{a} & 0 & 0 & a & a & a & 0 & 0 & \bar{a} & a & \bar{a} & 0 \\
0 & a & \bar{a} & \bar{a} & a & \bar{a} & \bar{a} & \bar{a} & 0 & 0 & a & a & a & 0 & 0 & \bar{a} & a & \bar{a} \\
0 & 0 & a & \bar{a} & \bar{a} & a & \bar{a} & \bar{a} & \bar{a} & \bar{a} & 0 & a & a & a & 0 & 0 & \bar{a} & a \\
\bar{a} & 0 & 0 & a & \bar{a} & \bar{a} & a & \bar{a} & \bar{a} & a & \bar{a} & 0 & a & a & a & 0 & 0 & \bar{a} \\
\bar{a} & \bar{a} & 0 & 0 & a & \bar{a} & \bar{a} & a & \bar{a} & \bar{a} & a & \bar{a} & 0 & a & a & a & 0 & 0 \\
\bar{a} & \bar{a} & \bar{a} & 0 & 0 & a & \bar{a} & \bar{a} & a & 0 & \bar{a} & a & \bar{a} & 0 & a & a & a & 0 \\
a & \bar{a} & \bar{a} & \bar{a} & 0 & 0 & a & \bar{a} & \bar{a} & 0 & 0 & \bar{a} & a & \bar{a} & 0 & a & a & a \\
\bar{a} & a & \bar{a} & \bar{a} & \bar{a} & 0 & 0 & a & \bar{a} & a & 0 & 0 & \bar{a} & a & \bar{a} & 0 & a & a \\
\bar{a} & \bar{a} & a & \bar{a} & \bar{a} & \bar{a} & 0 & 0 & a & a & a & 0 & 0 & \bar{a} & a & \bar{a} & 0 & a
\end{pmatrix}$$

$$D_{18-3} = OD(18;8,8) =$$

$$= \begin{pmatrix}
a & a & \bar{a} & a & b & b & b & \bar{b} & 0 & a & \bar{a} & \bar{a} & \bar{a} & b & \bar{b} & b & b & 0 \\
0 & a & a & \bar{a} & a & b & b & b & \bar{b} & 0 & a & \bar{a} & \bar{a} & \bar{a} & b & \bar{b} & b & b \\
\bar{b} & 0 & a & a & \bar{a} & a & b & b & b & b & 0 & a & \bar{a} & \bar{a} & \bar{a} & b & \bar{b} & b \\
b & \bar{b} & 0 & a & a & \bar{a} & a & b & b & b & b & 0 & a & \bar{a} & \bar{a} & \bar{a} & b & \bar{b} \\
b & b & \bar{b} & 0 & a & a & \bar{a} & a & b & \bar{b} & b & b & 0 & a & \bar{a} & \bar{a} & \bar{a} & b \\
b & b & b & \bar{b} & 0 & a & a & \bar{a} & a & b & \bar{b} & b & b & 0 & a & \bar{a} & \bar{a} & \bar{a} \\
a & b & b & b & \bar{b} & 0 & a & a & \bar{a} & \bar{a} & b & \bar{b} & b & b & 0 & a & \bar{a} & \bar{a} \\
\bar{a} & a & b & b & b & \bar{b} & 0 & a & a & \bar{a} & \bar{a} & b & \bar{b} & b & b & 0 & a & \bar{a} \\
a & \bar{a} & a & b & b & b & \bar{b} & 0 & a & \bar{a} & \bar{a} & \bar{a} & b & \bar{b} & b & b & 0 & a \\
\bar{a} & 0 & \bar{b} & \bar{b} & b & \bar{b} & a & a & a & a & 0 & \bar{b} & b & b & b & a & \bar{a} & a \\
a & \bar{a} & 0 & \bar{b} & \bar{b} & b & \bar{b} & a & a & a & a & 0 & \bar{b} & b & b & b & a & \bar{a} \\
a & a & \bar{a} & 0 & \bar{b} & \bar{b} & b & \bar{b} & a & \bar{a} & a & a & 0 & \bar{b} & b & b & b & a \\
a & a & a & \bar{a} & 0 & \bar{b} & \bar{b} & b & \bar{b} & a & \bar{a} & a & a & 0 & \bar{b} & b & b & b \\
\bar{b} & a & a & a & \bar{a} & 0 & \bar{b} & \bar{b} & b & b & a & \bar{a} & a & a & 0 & \bar{b} & b & b \\
b & \bar{b} & a & a & a & \bar{a} & 0 & \bar{b} & \bar{b} & b & b & a & \bar{a} & a & a & 0 & \bar{b} & b \\
\bar{b} & b & \bar{b} & a & a & a & \bar{a} & 0 & \bar{b} & b & b & b & a & \bar{a} & a & a & 0 & \bar{b} \\
\bar{b} & \bar{b} & b & \bar{b} & a & a & a & \bar{a} & 0 & \bar{b} & b & b & b & a & \bar{a} & a & a & 0 \\
0 & \bar{b} & \bar{b} & b & \bar{b} & a & a & a & \bar{a} & 0 & \bar{b} & b & b & b & a & \bar{a} & a & a
\end{pmatrix}$$

Table 7: Self-dual codes from orthogonal designs with $2n = 36$

	$2n = 36$				
p	Design	a	b	d_m	d_b
3	D_{18-1}	1	-	12	12
5	D_{18-3}	2	2	8	$10 - 16$
7	D_{18-1}	3	-	12	$12 - 16$
11	D_{18-1}	8	-	12	$12 - 18$
13	D_{18-1}	9	-	12	$12 - 18$
17	D_{18-2}	8	-	10	$10 - 19$
19	D_{18-2}	4	-	10	$10 - 19$
23	D_{18-1}	2	-	12	$12 - 19$
29	D_{18-2}	7	-	10	$10 - 19$
31	D_{18-1}	12	-	12	$12 - 19$
37	D_{18-3}	17	17	8	$8 - 19$

3.10 [40, 20] codes

There are many different cases of orthogonal designs, given in [8, p. 348], of order 20 on four commuting variables a, b, c, d. Using these matrices with a proper replacement of the variables, as it is shown in Table 8, a self-dual code $[40, 20, d_m]$ exists over F_p, for all $p \leq 37$.

For $p \leq 17$ the same results have been found, using conference matrices, by Arasu and Gulliver in [1]. The other self-dual codes of length 40 over F_p, $19 \leq p \leq 37$ given in Table 8 cannot be constructed from conference matrices (see [1]) but are easily constructed using orthogonal designs. These codes improve the distance bounds as these were given by Arasu and Gulliver in [1].

The computational time increases rapidly with the length n. So, length 40 is the computational limit of our search. We give explicitly the two designs that give us the best self-dual codes we found. We search as many designs as possible but it was impossible to search the totality of the orthogonal designs. It took us in about 400 hours of CPU time, on a Pentium IV 1700Hz PC, to achieve distance $d = 14$ of [40, 20] code over F_{37}.

We present the two orthogonal designs we used to obtain the codes.

Table 8: Self-dual codes from orthogonal designs with $2n = 40$

	$2n = 40$						
p	Design	a	b	c	d	d_m	d_b
3	D_{20-1}	1	1	1	1	12	12
5	D_{20-2}	0	1	1	1	13	13 – 16
7	D_{20-2}	0	2	2	2	13	13 – 18
11	D_{20-1}	8	5	9	9	13	13 – 20
13	D_{20-1}	3	7	3	9	13	13 – 20
17	D_{20-1}	11	0	4	11	13	13 – 20
19	D_{20-1}	9	9	6	3	13	13 – 20
23	D_{20-1}	11	11	9	3	14	14 – 21
29	D_{20-1}	14	14	13	7	14	14 – 21
31	D_{20-1}	15	15	8	1	14	14 – 21
37	D_{20-1}	18	18	17	10	14	14 – 21

$$D_{20-1} = OD(20; 2, 3, 6, 9) =$$

$$\begin{pmatrix}
b & d & c & a & \bar{d} & \bar{c} & \bar{b} & \bar{d} & d & a & c & c & \bar{d} & d & \bar{b} & d & c & \bar{c} & d & d \\
\bar{d} & b & d & c & a & \bar{b} & \bar{d} & d & a & \bar{c} & c & \bar{d} & d & \bar{b} & c & c & \bar{c} & d & d & d \\
a & \bar{d} & b & d & c & \bar{d} & d & a & \bar{c} & \bar{b} & \bar{d} & d & \bar{b} & c & c & \bar{c} & d & d & d & c \\
c & a & \bar{d} & b & d & d & a & \bar{c} & \bar{b} & \bar{d} & d & \bar{b} & c & c & \bar{d} & d & d & d & c & \bar{c} \\
d & c & a & \bar{d} & b & a & \bar{c} & \bar{b} & \bar{d} & d & \bar{b} & c & c & \bar{d} & d & d & d & c & \bar{c} & d \\
c & b & d & \bar{d} & \bar{a} & b & d & c & a & \bar{d} & d & \bar{c} & c & d & d & \bar{d} & d & \bar{c} & \bar{c} & b \\
b & d & \bar{d} & \bar{a} & c & \bar{d} & b & d & c & a & \bar{c} & c & d & d & d & d & \bar{c} & \bar{c} & b & \bar{d} \\
d & \bar{d} & \bar{a} & c & b & a & \bar{d} & b & d & c & c & d & d & d & \bar{c} & \bar{c} & \bar{c} & b & \bar{d} & d \\
\bar{d} & \bar{a} & c & b & d & c & a & \bar{d} & b & d & d & d & d & \bar{c} & c & \bar{c} & b & \bar{d} & d & \bar{c} \\
\bar{a} & c & b & d & \bar{d} & d & c & a & \bar{d} & b & d & d & \bar{c} & c & d & b & \bar{d} & d & \bar{c} & \bar{c} \\
\bar{c} & \bar{c} & d & \bar{d} & b & \bar{d} & c & \bar{c} & \bar{d} & \bar{d} & b & d & c & a & \bar{d} & d & \bar{d} & \bar{b} & \bar{c} & a \\
\bar{c} & d & \bar{d} & b & \bar{c} & c & \bar{c} & \bar{d} & \bar{d} & \bar{d} & \bar{d} & b & d & c & a & \bar{d} & \bar{b} & \bar{c} & a & d \\
d & \bar{d} & b & \bar{c} & \bar{c} & \bar{c} & \bar{d} & \bar{d} & \bar{d} & c & a & \bar{d} & b & d & c & \bar{b} & \bar{c} & a & d & \bar{d} \\
\bar{d} & b & \bar{c} & \bar{c} & d & \bar{d} & \bar{d} & \bar{d} & c & \bar{c} & c & a & \bar{d} & b & d & \bar{c} & a & d & \bar{d} & \bar{b} \\
b & \bar{c} & \bar{c} & d & \bar{d} & \bar{d} & \bar{d} & c & \bar{c} & \bar{d} & d & c & a & \bar{d} & b & a & d & \bar{d} & \bar{b} & \bar{c} \\
\bar{d} & \bar{c} & c & \bar{d} & \bar{d} & d & \bar{d} & c & c & \bar{b} & \bar{d} & d & b & c & \bar{a} & b & d & c & a & \bar{d} \\
\bar{c} & c & \bar{d} & \bar{d} & \bar{d} & \bar{d} & c & c & \bar{b} & d & d & b & c & \bar{a} & \bar{d} & \bar{d} & b & d & c & a \\
c & \bar{d} & \bar{d} & \bar{d} & \bar{c} & c & c & \bar{b} & d & \bar{d} & b & c & \bar{a} & \bar{d} & d & a & \bar{d} & b & d & c \\
\bar{d} & \bar{d} & \bar{d} & \bar{c} & c & c & \bar{b} & d & \bar{d} & c & c & \bar{a} & \bar{d} & d & b & c & a & \bar{d} & b & d \\
\bar{d} & \bar{d} & \bar{c} & c & \bar{d} & \bar{b} & d & \bar{d} & c & c & \bar{a} & \bar{d} & d & b & c & d & c & a & \bar{d} & b
\end{pmatrix}$$

$$D_{20-2} = OD(20;1,5,5,9) =$$

$$\begin{pmatrix}
a & b & c & \bar{c} & \bar{b} & b & \bar{d} & d & b & \bar{c} & \bar{d} & c & c & d & b & d & d & d & d & \bar{d} \\
\bar{b} & a & b & c & \bar{c} & \bar{d} & d & b & \bar{c} & b & c & c & d & b & \bar{d} & d & d & d & \bar{d} & d \\
\bar{c} & \bar{b} & a & b & c & d & b & \bar{c} & b & \bar{d} & c & d & b & \bar{d} & c & d & d & \bar{d} & d & d \\
c & \bar{c} & \bar{b} & a & b & b & \bar{c} & b & \bar{d} & d & d & b & \bar{d} & c & c & d & \bar{d} & d & d & d \\
b & c & \bar{c} & \bar{b} & a & \bar{c} & b & \bar{d} & d & b & b & \bar{d} & c & c & d & \bar{d} & d & d & d & d \\
\bar{b} & d & \bar{d} & \bar{b} & c & a & b & c & \bar{c} & \bar{b} & d & d & d & d & \bar{d} & \bar{d} & \bar{c} & \bar{c} & d & \bar{b} \\
d & \bar{d} & \bar{b} & c & \bar{b} & \bar{b} & a & b & c & \bar{c} & d & d & d & \bar{d} & d & \bar{c} & \bar{c} & d & \bar{b} & \bar{d} \\
\bar{d} & \bar{b} & c & \bar{b} & d & \bar{c} & \bar{b} & a & b & c & d & d & \bar{d} & d & d & \bar{c} & d & \bar{b} & \bar{d} & \bar{c} \\
\bar{b} & c & \bar{b} & d & \bar{d} & c & \bar{c} & \bar{b} & a & b & d & \bar{d} & d & d & d & d & \bar{b} & \bar{d} & \bar{c} & \bar{c} \\
c & \bar{b} & d & \bar{d} & \bar{b} & b & c & \bar{c} & \bar{b} & a & \bar{d} & d & d & d & d & \bar{b} & \bar{d} & \bar{c} & \bar{c} & d \\
d & \bar{c} & \bar{c} & \bar{d} & \bar{b} & \bar{d} & \bar{d} & \bar{d} & \bar{d} & d & a & b & c & \bar{c} & \bar{b} & b & d & \bar{d} & b & \bar{c} \\
\bar{c} & \bar{c} & \bar{d} & \bar{b} & d & \bar{d} & \bar{d} & \bar{d} & d & \bar{d} & \bar{b} & a & b & c & \bar{c} & d & \bar{d} & b & \bar{c} & b \\
\bar{c} & \bar{d} & \bar{b} & d & \bar{c} & \bar{d} & \bar{d} & d & \bar{d} & \bar{d} & \bar{c} & \bar{b} & a & b & c & \bar{d} & b & \bar{c} & b & d \\
\bar{d} & \bar{b} & d & \bar{c} & \bar{c} & \bar{d} & d & \bar{d} & \bar{d} & \bar{d} & c & \bar{c} & \bar{b} & a & b & b & \bar{c} & b & d & \bar{d} \\
\bar{b} & d & \bar{c} & \bar{c} & \bar{d} & d & \bar{d} & \bar{d} & \bar{d} & \bar{d} & b & c & \bar{c} & \bar{b} & a & \bar{c} & b & d & \bar{d} & b \\
\bar{d} & \bar{d} & \bar{d} & \bar{d} & d & d & c & c & \bar{d} & b & \bar{b} & \bar{d} & d & \bar{b} & c & a & b & c & \bar{c} & \bar{b} \\
\bar{d} & \bar{d} & \bar{d} & d & \bar{d} & c & c & \bar{d} & b & d & \bar{d} & d & \bar{b} & c & \bar{b} & \bar{b} & a & b & c & \bar{c} \\
\bar{d} & \bar{d} & d & \bar{d} & \bar{d} & c & \bar{d} & b & d & c & d & \bar{b} & c & \bar{b} & \bar{d} & \bar{c} & \bar{b} & a & b & c \\
\bar{d} & d & \bar{d} & \bar{d} & \bar{d} & \bar{d} & b & d & c & c & \bar{b} & c & \bar{b} & \bar{d} & d & c & \bar{c} & \bar{b} & a & b \\
d & \bar{d} & \bar{d} & \bar{d} & \bar{d} & b & d & c & c & \bar{d} & c & \bar{b} & \bar{d} & d & \bar{b} & b & c & \bar{c} & \bar{b} & a
\end{pmatrix}$$

References

[1] K.T. Arasu and T. Aaron Gulliver, Self-dual codes over F_p and weighing matrices, *IEEE Trans. Inform. Theory*, 47 (2001), 2051-2055.

[2] K. Betsumiya, S. Georgiou, T.A. Gulliver, M. Harada and C. Koukouvinos, On self-dual codes over some prime fields, *Discrete Mathematics*, 262 (2003), 37-58.

[3] J.H. Conway and N.J.A. Sloane, A new upper bound on the minimal distance of self-dual codes, *IEEE Trans. Inform. Theory*, 36 (1990), 1319-1333.

[4] S. Georgiou and C. Koukouvinos, New self-dual codes over $GF(5)$, in *Cryptography and Coding*, M. Walker (Ed.), Lecture Notes in Computer Science, vol. 1746, Springer-Verlag, Heidelberg, 1999, 63-69.

[5] S. Georgiou and C. Koukouvinos, Self-dual codes over $GF(7)$ and orthogonal designs, *Utilitas Math.*, 60 (2001), 79-89.

[6] S. Georgiou and C. Koukouvinos, New infinite classes of weighing matrices, *Sankhya Ser. B*, 64 (2002), 26-36.

[7] A.V. Geramita, J.M. Geramita, and J. Seberry Wallis, Orthogonal designs, *Linear and Multilinear Algebra*, 3 (1976), 281-306.

[8] A.V. Geramita and J. Seberry, *Orthogonal Designs: Quadratic Forms and Hadamard Matrices*, Marcel Dekker, New York-Basel, 1979.

[9] J.S. Leon, V. Pless and N.J.A. Sloane, Self-dual codes over $GF(5)$, *J. Combin. Theory Ser. A*, (1982), 178–194.

[10] F.J. MacWilliams and N.J.A. Sloane, *The Theory of Error-Correcting Codes*, North-Holland, Amsterdam, 1977.

[11] C. Koukouvinos, and J. Seberry, New weighing matrices and orthogonal designs constructed using two sequences with zero autocorrelation function - a review, *J. Statist. Plann. Inference*, 81 (1999), 153–182.

[12] E. Rains and N.J.A. Sloane, Self-dual codes, in *Handbook of Coding Theory*, eds. V. Pless et al., Elsevier, Amsterdam, 1998.

[13] V.D. Tonchev, Codes, in *The CRC Handbook of Combinatorial Designs*, ed. C.J. Colbourn and J.H. Dinitz, CRC Press, Boca Raton, Fla., 1996, 517-543.