Infinite Order Domination in Graphs

AP Burger[†], EJ Cockayne[†], WR Gründlingh[*], CM Mynhardt[†], JH van Vuuren[*] & W Winterbach[*]

September 11, 2003

Abstract

The previously studied notions of smart and foolproof finite order domination of a simple graph $G = (V, E)$ are generalised in the sense that safe configurations in G are not merely sought after $k \geq 1$ moves, but in the limiting cases where $k \to \infty$. Some general properties of these generalised domination parameters are established, after which the parameter values are found for certain simple graph structures (such as paths, cycles, multipartite graphs and products of complete graphs, cycles and paths).

Keywords: Secure & Weak Roman Domination, Higher Order Domination, Graph Protection.

AMS Subject Classification: 05C69.

1 Introduction

A *guard function* for a graph $G = (V, E)$ is a mapping $f : V \mapsto \{0, 1, 2, \ldots\}$ such that $f(v)$ denotes the number of *guards* stationed at a vertex $v \in V$. A guard function partitions the vertex set of G into subsets $V_i = \{v : f(v) = i\}$, $i = 0, 1, 2, \ldots$ and we (imprecisely) write $f = (V_0, V_1, V_2, \ldots)$. A guard function is called *safe* if each $v \in V_0$ is adjacent to some $u \in V \backslash V_0$ (i.e. if $V \backslash V_0$ is a dominating set of G). The *weight* of a guard function is denoted

$$w(f) = \sum_{v \in V} f(v).$$

[*]Department of Applied Mathematics, Stellenbosch University, Private Bag X1, Matieland, 7602, Republic of South Africa, fax: +27 21 8083778, email: vuuren@sun.ac.za

[†]Department of Mathematics and Statistics, University of Victoria, Box 3045, Victoria, BC, Canada, V8W 3P4, email: cockayne@math.uvic.ca

JCMCC 50 (2004), pp. 179-194

In [1] the following kinds of safe guard functions were considered:

(1) A *smart k–weak Roman dominating function* (k–SWRDF) is a safe guard function $f^{(0)} = (V_0^{(0)}, V_1^{(0)}, V_2^{(0)})$ with the property that, for any sequence of vertices $v_0, v_1, \dots v_{k-1}$, there exists a sequence of vertices $u_i \in V_1^{(i)} \cup V_2^{(i)}$ in the neighbourhood of v_i such that the functions $f^{(i+1)}(s) = \text{move}(f^{(i)}, u_i \to v_i)$ are also safe guard functions for all $i = 0, \dots, k-1$. The minimum weight of a k–SWRDF is denoted
$$\gamma_{r,k}(G) = \min_{k\text{-SWRDFs}} \left(|V_1^{(0)}| + 2|V_2^{(0)}| \right),$$
which is called the *smart k–weak Roman domination number* of G.

(2) Similarly, a *foolproof k–weak Roman dominating function* (k–FWRDF) is a safe guard function $f^{(0)} = (V_0^{(0)}, V_1^{(0)}, V_2^{(0)})$ with the property that, for any sequence of vertices $v_0, v_1, \dots, v_{k-1}$, the functions $f^{(i+1)}(s) = \text{move}(f^{(i)}, u_i \to v_i)$ are also safe guard functions for any sequence of vertices $u_i \in V_1^{(i)} \cup V_2^{(i)}$ in the neighbourhoods of v_i and all $i = 0, \dots, k-1$. The minimum weight of a k–FWRDF is denoted
$$\gamma^*_{r,k}(G) = \min_{k\text{-FWRDFs}} \left(|V_1^{(0)}| + 2|V_2^{(0)}| \right),$$
which is called the *foolproof k–weak Roman domination number* of G.

(3) A *smart k–secure dominating function* (k–SSDF) is a safe guard function $f^{(0)} = (V_0^{(0)}, V_1^{(0)})$ with the property that, for any sequence of vertices $v_0, v_1, \dots, v_{k-1}$, there exists a sequence of vertices $u_i \in V_1^{(i)}$ such that the functions $f^{(i+1)}(s) = \text{move}(f^{(i)}, u_i \to v_i)$ are also safe guard functions for all $i = 0, \dots, k-1$. The minimum weight of a k–SSDF is denoted
$$\gamma_{s,k}(G) = \min_{k\text{-SSDFs}} |V_1^{(0)}|,$$
which is called the *smart k–secure domination number* of G.

(4) Similarly, a *foolproof k–secure dominating function* (k–FSDF) is a safe guard function $f^{(0)} = (V_0^{(0)}, V_1^{(0)})$ with the property that, for any sequence of vertices v_i $(i = 0, \dots, k-1)$, the functions $f^{(i+1)}(s) = \text{move}(f^{(i)}, u_i \to v_i)$ are also safe guard functions for any sequence of vertices $u_i \in V_1^{(i)}$ in the neighbourhoods of v_i and all $i = 0, \dots, k-1$. The minimum weight of a k–SSDF is denoted
$$\gamma^*_{s,k}(G) = \min_{k\text{-FSDFs}} |V_1^{(0)}|,$$
which is called the *foolproof k–secure domination number* of G.

2 Infinite order generalisations

If one is interested in perpetual or eternal security in a graph, then the following generalisations seem natural.

(5) A *smart [foolproof] ∞–weak Roman dominating function* (∞–SWRDF) [(∞–FWRDF)] is a k–SWRDF [k–FWRDF] in the limit as $k \to \infty$. The minimum weight of an ∞–SWRDF [∞–FWRDF] is denoted

$$\gamma_{r,\infty}(G) = \lim_{k\to\infty} \gamma_{r,k}(G) \quad [\gamma^*_{r,\infty}(G) = \lim_{k\to\infty} \gamma^*_{r,k}(G)],$$

which is called the *smart [foolproof] ∞–weak Roman domination number* of G.

(6) A *smart [foolproof] ∞–secure dominating function* (∞–SSDF) [(∞–FSDF)] is a k–SSDF [k–FSDF] in the limit as $k \to \infty$. The minimum weight of an ∞–SSDF [∞–FSDF] is denoted

$$\gamma_{s,\infty}(G) = \lim_{k\to\infty} \gamma_{s,k}(G) \quad [\gamma^*_{s,\infty}(G) = \lim_{k\to\infty} \gamma^*_{s,k}(G)],$$

which is called the *smart [foolproof] ∞–secure domination number* of G.

The question of existence of these infinite–order parameters is settled in the following theorem.

Theorem 1 *For any order n graph G, the limits*

(a) $\gamma_{r,\infty}(G) = \lim_{k\to\infty} \gamma_{r,k}(G)$,

(b) $\gamma^*_{r,\infty}(G) = \lim_{k\to\infty} \gamma^*_{r,k}(G)$,

(c) $\gamma_{s,\infty}(G) = \lim_{k\to\infty} \gamma_{s,k}(G)$ *and*

(d) $\gamma^*_{s,\infty}(G) = \lim_{k\to\infty} \gamma^*_{s,k}(G)$

exist. In fact,

$$1 \le \gamma_{r,\infty}(G), \gamma^*_{r,\infty}(G), \gamma_{s,\infty}(G), \gamma^*_{s,\infty}(G) \le n-1 \tag{1}$$

and both these bounds are attainable for all four parameters.

Proof: The bounds in (1) are trivially true for the four parameters $\gamma_{r,\infty}(G)$, $\gamma^*_{r,\infty}(G)$, $\gamma_{s,\infty}(G)$ and $\gamma^*_{s,\infty}(G)$ if they exist, and the existence of the limits follow from Proposition 3 in [1]. The lower bounds in (1) are attained for all four parameters when G is the complete graph K_n, while the upper bounds in (1) are attained for all four parameters when G is the star $K_{1,n-1}$. ■

We now show that the limiting parameter values $\gamma_{s,\infty}(G)$ and $\gamma_{r,\infty}(G)$ cannot differ, for any graph, G.

Theorem 2 *For any graph G, $\gamma_{s,\infty}(G) = \gamma_{r,\infty}(G)$.*

Proof: Consider an arbitrary problem vertex sequence $v_i \in V(G)$ ($i = 0, 1, 2, \ldots$) for which the move sequence

$$f^{(i+1)} = \text{move}(f^{(i)}, u_i \to v_i) \tag{2}$$

generates safe guard functions $f^{(i)}$ in G $\forall\ i = 0, 1, 2, \ldots$, such that $f^{(0)} = (V_0^{(0)}, V_1^{(0)}, V_2^{(0)})$ is an ∞-SWRDF of weight $w(f^{(0)}) = \gamma_{r,\infty}(G)$. If there exists a $k \in \mathbb{N}$ such that $V_2^{(k)} = \emptyset$, then $f^{(k)}$ is an ∞-SSDF for G, rendering the equality chain

$$w(f^{(0)}) = \gamma_{r,\infty}(G) \leq \gamma_{s,\infty}(G) \leq w(f^{(k)}) = w(f^{(0)}).$$

Now suppose that

$$\gamma_{r,\infty}(G) < \gamma_{s,\infty}(G) \tag{3}$$

for some graph, G. Then it follows by the contrapositive of the above argument that for any sequence (2) originating from a minimum weight ∞-SWRDF, $V_2^{(i)} \neq \emptyset$ for all $i \in \mathbb{N}$. Hence there exists a vertex $v^* \in V_2^{(i)}$ (for all $i \in \mathbb{N}$) which is not included in the move sequence (2), despite the fact that the whole open neighbourhood set $N(v^*)$ may be included in the problem vertex sequence $v_0, v_1, v_2, \ldots$. This means that the value of $f^{(0)}(v^*)$ is not minimal, contradicting the fact that $f^{(0)}$ is a minimum weight ∞-SWRDF. We conclude that the strict inequality (3) is impossible, for any graph G. ■

Not only is it possible to prove a result similar to the above for the parameters $\gamma^*_{r,\infty}(G)$ and $\gamma^*_{s,\infty}(G)$; it is, in fact, possible to find an exact value for $\gamma^*_{r,\infty}(G) = \gamma^*_{s,\infty}(G)$, for any graph G.

Theorem 3 *For any order n graph G with minimal degree δ,*

$$\gamma^*_{r,\infty}(G) = \gamma^*_{s,\infty}(G) = n - \delta.$$

Proof: If $n - \delta \leq |V_1| \leq n$, then $f = (V_0, V_1)$ is an ∞-FSDF for G, since if there were an undominated vertex of G after a sequence of k moves from the safe guard function f (for any $k \geq 1$), then this would imply that $|V_0| \geq \delta + 1$. Therefore

$$\gamma^*_{s,\infty}(G) \leq n - \delta. \tag{4}$$

Now suppose that $\gamma^*_{s,\infty}(G) < n - \delta$. Because any sequence of k moves of the form $f^{(i+1)} = \text{move}(f^{(i)}, u_i \to v_i)$ may follow a given sequence of

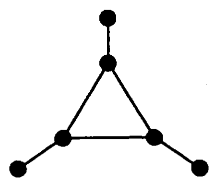


Figure 2.1: This graph may be partitioned into three subcliques of order two.

problem vertices $v_i \in V_0^{(i)}$ for $i = 0, \dots, k-1$, it may be possible that the entire closed neighbourhood $N[v^*]$ of a vertex v^* of minimal degree in G is included in the set $V_0^{(k)}$, resulting in v^* being undominated. This contradiction shows that

$$\gamma_{s,\infty}^*(G) \geq n - \delta. \tag{5}$$

The desired result in the case of $\gamma_{s,\infty}^*(G)$ follows by a combination of (4) and (5). The proof of the corresponding result for $\gamma_{r,\infty}^*(G)$ is similar. ■

It follows by Theorems 2 and 3 that the r and s subscripts are superfluous in the case of the infinite order parameters. Hence we shall simply denote the values of $\gamma_{r,\infty}(G) = \gamma_{s,\infty}(G)$ by $\gamma_\infty(G)$, and that of $\gamma_{r,\infty}^*(G) = \gamma_{s,\infty}^*(G)$ by $\gamma_\infty^*(G)$. In a similar vein we shall henceforth refer to ∞–SSDFs and ∞–SWRDFs simply as ∞-smart dominating functions (∞–SDFs), while referring to ∞–FSDFs and ∞–FWRDFs simply as ∞–foolproof dominating functions (∞–FDFs).

We next prove that $\gamma_\infty(G)$ is bounded from above by the minimum number of subcliques into which G may be partitioned (somewhat imprecisely speaking).

Theorem 4 *If the vertex set of G may be partitioned into c subsets $S_1, \dots, S_c$ such that S_i induces a clique in G for all $i = 1, \dots, c$, then $\gamma_\infty(G) \leq c$.*

Proof: Let G be an order n graph and let $\mathcal{S} = \{S_1, \dots, S_c\}$ be a partition of $V(G)$ such that, for any $S_i \in \mathcal{S}$, $\langle S_i \rangle$ is complete. Also, let $\{v_1, v_2, \dots, v_c\} \subseteq V(G)$ be vertices such that $v_i \in S_i$ for $i = 1, 2, \dots, c$. Then $f = (V_0^{(0)}, V_1^{(0)})$ is an ∞–SDF for G, where $V_1^{(0)} = \cup_{i=1}^{c}\{v_i\}$ and $V_0^{(0)} = V(G)\backslash V_1^{(0)}$. This is true, because $f^{(0)}$ is certainly a safe guard function. Furthermore, given an arbitrary problem sequence $v_{i_j} \in V_0^{(j)}$ $(j = 0, \dots, k-1)$ where $v_{i_j} \in S_{i_j}$, the moves $f^{(j+1)} = \text{move}(f^{(j)}, u_j \to v_{i_j})$ render a sequence of safe guard functions $f^{(j+1)}$, where $u_j \in V(S_{i_j})\backslash\{v_{i_j}\}$, for all $j = 0, \dots, k-1$. The weight of $f^{(0)}$ is $|V_1^{(0)}| = c$, yielding the desired upper bound on $\gamma_\infty(G)$. ■

Good bound realisations of the above result for general graphs are hard to achieve, since determining the minimum value of c in Theorem 4 is a known hard problem, called the minimum clique partition problem. In fact, solving the minimum clique partition problem is equivalent to solving the minimum vertex colouring problem for the complement of the graph, for which no known m-optimal algorithm exists (m being a constant). For example, optimal values of c are not necessarily obtained via a greedy approach whereby one starts the partition with the largest clique in G, then chooses the next largest clique, and continues in this fashion until all vertices are accommodated in some clique. Figure 2.1 shows an example where this approach yields a value of $c = 4$, while $c = 3$ is in fact the minimum number of subcliques. To complicate matters further, it is not certain when the minimum value of c in Theorem 4 (which is the vertex chromatic number of the graph complement, $\chi(\overline{G})$) is equal to the value of $\gamma_\infty(G)$. Graphs exist for which $\gamma_\infty(G) \neq \chi(\overline{G})$. However, it is possible to show that equality does, in fact, hold between $\gamma_\infty(G)$ and $\chi(\overline{G})$ for certain graphs, G, as we now show.

Proposition 1 *For any graph G, $\gamma_\infty(G) \geq \beta(G)$.*

Proof: Let $\mathcal{I} = \{v_1, v_2, \ldots, v_\beta\}$ be an independent set in G. Suppose, to the contrary, that $\gamma_\infty(G) < \beta(G)$, and let $f^{(0)} = (V_0^{(0)}, V_1^{(0)})$ be any minimum weight ∞-SDF for G. Then the guard function $f^{(\beta(G))}$ obtained from the move sequence $f^{(i+1)} = \text{move}(f^{(i)}, u_i \to v_{i+1})$ is not a safe guard function for G, no matter how the vertex sequence $u_i \in V(G)$ is chosen for $i = 0, \ldots, \beta(G) - 1$. This contradiction shows that $\gamma_\infty(G) \geq \beta(G)$. ∎

Note that, for any graph G, $\beta(G) \leq \chi(\overline{G})$. Thus, if $\beta(G) = \chi(\overline{G})$, then $\gamma_\infty(G) = \chi(\overline{G})$ by Theorem 4. Furthermore, equality also holds between $\gamma_\infty(G)$ and $\chi(\overline{G})$ for graphs whose complements are vertex colourable with relatively few colours.

Theorem 5 *If $\chi(\overline{G}) \leq 3$, then $\gamma_\infty(G) = \chi(\overline{G})$.*

Proof: If $\chi(\overline{G}) \leq 2$ the theorem holds trivially. Therefore let $\chi(\overline{G}) = 3$, but suppose, to the contrary, that $\gamma_\infty(G) \leq 2$. By Proposition 1, $\overline{G}$ cannot contain a 3-cycle. Consider the situation where $\overline{G}$ contains an odd cycle C : $v_1 v_2 \cdots v_{2k+1}$ ($k \geq 2$), and define the problem sequence $v_1, v_2, v_4, v_6, \ldots, v_{2k}$ in $V(G)$. Because v_1 and v_2 are independent in G, it follows that $\gamma_\infty(G) \geq 2$. Moreover, $v_1, v_2 \in V_1^{(2)}$, starting out from any minimum weight ∞-SDF, $f^{(0)} = (V_0^{(0)}, V_1^{(0)})$. Hence any safe guard functions $f^{(3)}, f^{(4)}, \ldots, f^{(k)}$ must emanate from the move sequence $f^{(i+1)} = \text{move}(f^{(i)}, v_{2i-2} \to v_{2i})$, $i = 2, 3, \ldots, k-1$. But then there exists no safe guard function $f^{(k+1)}$, because any move resulting from the safe guard function $f^{(k)}$ will result in either v_{2k-1} or v_{2k+1} not being dominated. Therefore we conclude that $\overline{G}$ has no odd cycles. But then $\overline{G}$ is bipartite, and hence $\chi(\overline{G}) = 2$, which is a contradiction. We conclude that $\gamma_\infty(G) = 3$. ■

For specific graph classes the result of Theorem 4 is certainly of practical value (as will be demonstrated in §3) and the more concrete corollary below follows immediately from the theorem.

Corollary 1 *If G is an order n graph such that, for some subset of vertices $S = \{v_1, \ldots, v_m\} \subseteq V(G)$, the graph $G - S$ possesses a perfect matching, then*

$$\gamma_\infty(G) \leq \frac{n-m}{2} + m = \frac{m+n}{2}.$$

This result corresponds to the worst case upper bound on γ_∞ in (1) for the case where $G \cong K_{1,n-1}$. If the vertices of $K_{1,n-1}$ are labelled such that $(\{v_1, v_2, \ldots, v_{n-1}\}, \{v_n\})$ are the partite sets, then $K_{1,n-1} - \{v_1, v_2, \ldots, v_{n-2}\}$ possesses the perfect matching $\langle v_{n-1}, v_n\rangle \cong K_2$, yielding the upper bound

$$\gamma_\infty(G) \leq \frac{n + (n-2)}{2} = n - 1,$$

as stated in (1). Finally, we summarise the relationships between the six new parameters considered in [1] and in this paper, for further reference.

Theorem 6 *The relationships*

$$\left.\begin{array}{ccccccccc} \gamma(G) & \leq & \gamma_{r,k}(G) & \leq & \gamma_{s,k}(G) & \leq & \gamma_\infty(G) & \leq & \chi(\overline{G}) \\ & & \text{\rotatebox{90}{$\geq$}} & & \text{\rotatebox{90}{$\geq$}} & & \text{\rotatebox{90}{$\geq$}} & & \\ \gamma(G) & \leq & \gamma^*_{r,k}(G) & \leq & \gamma^*_{s,k}(G) & \leq & \gamma^*_\infty(G) & = & n-\delta \end{array}\right\} \quad (6)$$

hold for all $k \in \mathbb{N}$ and any order n graph G with minimum degree δ.

3 Parameters values for special graphs

In this section we consider a number of simple graph classes and find values for the two new infinite order domination parameters considered in this paper.

3.1 Paths

The following corollary is a direct result of Theorem 3 in [1] and Theorem 3 in this paper.

Corollary 2 *For any path P_n,*

(a) $\gamma_\infty(P_n) = \left\lceil \frac{n}{2} \right\rceil$,

(b) $\gamma^*_\infty(P_n) = n - 1$.

The limiting value in Corollary 2(a) is obtained by $\gamma_{r,k}(P_n)$ and $\gamma_{s,k}(P_n)$ when $k = \lceil \frac{n-5}{8} \rceil$ if n is odd and when $k = \lceil \frac{n-2}{4} \rceil$ if n is even, while that in Corollary 2(b) is obtained by $\gamma^*_{s,k}(P_n)$ when $k = n - 2$.

3.2 Cycles

The following corollary is a direct result of Theorem 4 in [1] and Theorem 3 in this paper.

Corollary 3 *For any cycle C_n,*

(a) $\gamma_\infty(C_n) = \left\lceil \frac{n}{2} \right\rceil$,

(b) $\gamma^*_\infty(C_n) = n - 2$.

The limiting value in Corollary 3(a) is obtained by $\gamma_{r,k}(C_n)$ and $\gamma_{s,k}(C_n)$ when $k = \lceil \frac{n-5}{8} \rceil$ if n is odd and when $k = \lceil \frac{n-2}{4} \rceil$ if n is even, while that in Corollary 3(b) is obtained by $\gamma^*_{s,k}(C_n)$ when $k = n - 3$.

3.3 Complete multipartite graphs

Although exact values for the four finite order domination parameters considered in [1] are not known for the class of complete multipartite graphs, these values may, in fact, be found in the case of the two infinite order domination parameters.

Theorem 7 *For the complete multipartite graph $K_{p_1,p_2,\ldots,p_t}$, with $p_1 \leq p_2 \leq \ldots \leq p_t$,*

$$\gamma_\infty(K_{p_1,p_2,\ldots,p_t}) = \gamma^*_\infty(K_{p_1,p_2,\ldots,p_t}) = p_t,$$

for all $t \geq 2$.

Proof: For $K_{p_1,p_2,\ldots,p_t}$, with $p_1 \leq p_2 \leq \ldots \leq p_t$, the minimum degree is given by $\delta = \sum_{i=1}^{t-1} p_i$. Hence it follows, by Theorem 6, that

$$\gamma_\infty(K_{p_1,p_2,\ldots,p_t}) \leq \gamma^*_\infty(K_{p_1,p_2,\ldots,p_t}) = n - \delta = \sum_{i=1}^{t} p_i - \sum_{i=1}^{t-1} p_i = p_t. \quad (7)$$

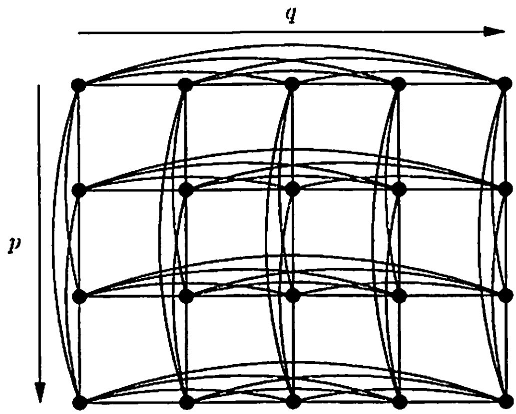


Figure 3.2: Grid–representation of $K_p \times K_q$, with $p = 4$ and $q = 5$.

Now suppose that $\gamma_\infty(K_{p_1,p_2,\ldots,p_t}) < p_t$. Let $S_1, \ldots, S_t$ be the partite sets of $K_{p_1,p_2,\ldots,p_t}$, with $|S_i| = p_i$ for all $i = 1, \ldots, t$. Let $f^{(0)}$ be a safe guard function of weight $\gamma_{s,\infty}(K_{p_1,p_2,\ldots,p_t})$, and let $\{v_1, \ldots, v_m\}$ denote the set of vertices in $S_t \cap V_0^{(0)}$. Then clearly $0 < p_t - \gamma_\infty(K_{p_1,p_2,\ldots,p_t}) \le m \le p_t$. Now consider the sequence of problem vertices $v_1, \ldots, v_m$. It is clear that there exists no sequence of moves $f^{(i+1)} = \text{move}(f^{(i)}, u_i \to v_i)$ $(i = 0, \ldots, m-1)$ such that $f^{(m)}$ is a safe guard function of $K_{p_1,p_2,\ldots,p_t}$. This contradiction shows that

$$\gamma_\infty(K_{p_1,p_2,\ldots,p_t}) \ge p_t. \tag{8}$$

The desired result therefore follows by a combination of (7) and (8). ∎

3.4 Products of complete graphs, paths and cycles

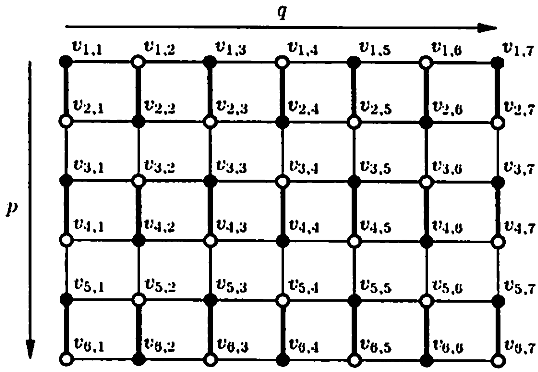
In this section we consider Cartesian products of complete graphs, Cartesian products of paths and Cartesian products of cycles, and find values for or bounds on the two infinite order parameters for these simple graph classes.

Proposition 2 *For the complete graphs K_p and K_q, with $p \le q$,*

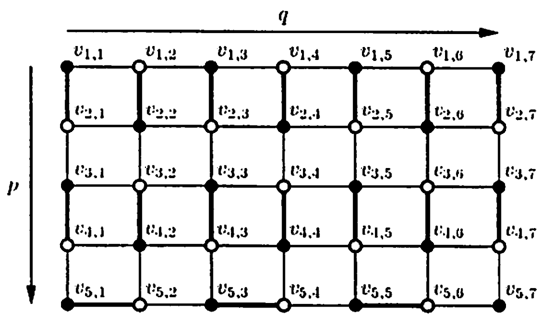
(a) $\gamma_\infty(K_p \times K_q) = p$,

(b) $\gamma^*_\infty(K_p \times K_q) = pq - (p + q) + 2$.

Proof: (a) The vertex set $V(K_p \times K_q)$ may be partitioned into p subsets, each inducing a q–clique in $K_p \times K_q$. Hence it follows, by Theorem 4, that $\gamma_\infty(K_p \times K_q) \le p$. But it follows by Proposition 1 that $\gamma_\infty(K_p \times K_q) \ge$



(a) $p = 6$ and $q = 7$



(b) $p = 5$ and $q = 7$

Figure 3.3: SDFs $f^{(0)} = (V_0^{(0)}, V_1^{(0)})$ for $P_p \times P_q$ showing that $\gamma_{s,\infty}(P_p \times P_q) \leq \lceil \frac{pq}{2} \rceil$. Dark vertices denote elements of $V_1^{(0)}$, while dark edges delimit matching pairs.

$\beta(K_p \times K_q) = p$ (as may be seen in Figure 3.2 for the special case where $p = 4$ and $q = 5$).

(b) The graph $K_p \times K_q$ is $(p + q - 2)$–regular. Hence the result follows directly from Theorem 3 with $n = pq$ and $\delta = p + q - 2$. ■

Theorem 8 *For any paths P_p and P_q, with $p, q \geq 2$*

(a) $\gamma_\infty(P_p \times P_q) = \left\lceil \frac{pq}{2} \right\rceil$,

(b) $\gamma^*_\infty(P_p \times P_q) = pq - 2$.

Proof: (a) Label the vertex in row i and column j of the grid graph $P_p \times P_q$ as $v_{i,j}$ for all $i = 1, \ldots, p$ and $j = 1, \ldots, q$. Consider first the case where p or q is even (or both). In this case there exists a perfect matching of $P_p \times P_q$, as illustrated in Figure 3.3(a) for the special case where $p = 6$ and $q = 7$. Hence Corollary 1 with $m = 0$ renders the upper bound

$$\gamma_\infty(P_p \times P_q) \leq \frac{pq}{2} \text{ if } p \text{ and/or } q \text{ are even.} \tag{9}$$

This upper bound is indeed the exact value of $\gamma_\infty(P_p \times P_q)$, by Proposition 1, since $V(P_p \times P_q)$ cannot be partitioned into fewer than $\frac{pq}{2}$ sets, each inducing a clique in $P_p \times P_q$ (because if this were true, we would need at least one triangle as subgraph of $P_p \times P_q$). Furthermore it is possible to define an ∞–SDF $f^{(0)} = (V_0^{(0)}, V_1^{(0)})$ for which $V_1^{(0)}$ is an independent set in $P_p \times P_q$ and which contains exactly one vertex from each matching pair (as depicted by the dark vertices in Figure 3.3(a)).

In the case where both p and q are odd, any maximum cardinality matching of the graph $P_p \times P_q - v_{p,q}$ leaves exactly one unmatched vertex, as illustrated in Figure 3.3(b) for the special case where $p = 5$ and $q = 7$. Hence Corollary 1 with $m = 1$ renders the upper bound

$$\gamma_\infty(P_p \times P_q) \leq \frac{pq-1}{2} + 1 \text{ if } p \text{ and } q \text{ are odd.} \tag{10}$$

Again this bound is in fact the exact value of $\gamma_\infty(P_p \times P_q)$, by Proposition 1, using the same arguments as above. A combination of the exact values of $\gamma_\infty(P_p \times P_q)$ in the upper bound formulas in (9) and (10) renders the desired result.

(b) For the grid graph $P_p \times P_q$, the minimum degree is $\delta = 2$. Hence the result follows directly from Theorem 3 with $n = pq$ and $\delta = 2$. ■

The next result follows immediately from Theorems 3 and 8.

Corollary 4 *For any cycles C_p and C_q, with $p, q \geq 4$,*

(a) $\frac{7pq}{23} \leq \gamma_\infty(C_p \times C_q) \leq \lceil \frac{pq}{2} \rceil$,

(b) $\gamma_\infty^*(C_p \times C_q) = pq - 4$.

Proof: (a) The lower bound follows, by Theorem 6, from the bound

$$\gamma_{s,1}(G) \geq \frac{n(2\Delta - 2t + 5)}{(\Delta+1)^2 - (t-1)(t-2)} \tag{11}$$

in [3] for any K_t-free graph G of order n and with maximum degree Δ. Here we may take $n = pq$, $t = 3$ and $\Delta = 4$. Furthermore, $\gamma_{s,\infty}(C_p \times C_q) \leq$

$\gamma_{s,\infty}(P_p \times P_q) \leq \lceil \frac{pq}{2} \rceil$ for all $p, q \in \mathbb{N}$, since $P_p \times P_q$ is a spanning subgraph of $C_p \times C_q$.

(b) For $C_p \times C_q$, the minimum degree is $\delta = 4$. Hence the result follows directly from Theorem 3 with $n = pq$ and $\delta = 4$. ∎

Note that if both p and q are even, then the same independent set structure as depicted in Figure 3.3(a) may be used as $V_1^{(0)}$ for an ∞–SDF $f^{(0)} = (V_0^{(0)}, V_1^{(0)})$ for the graph $C_p \times C_q$, and hence, in this case, $\gamma_\infty(C_p \times C_q) = \lceil pq/2 \rceil$, by Proposition 1. We conjecture that this value for $\gamma_\infty(C_p \times C_q)$ is always the case if $p, q \geq 4$. However, the lower bound in Corollary 4(a) is sharp if $p = 3$ and q is small enough (for example, if $4 \leq p \leq 11$).

Conjecture 1 *For any $p, q \in \mathbb{N}$ satisfying $p, q \geq 4$, $\gamma_\infty(C_p \times C_q) = \lceil pq/2 \rceil$.*

3.5 Hexagonal graphs

The notion of higher order domination in graphs strongly resembles a game of strategy, where a player attempts to prepare against a worst case scenario of a sequence of opposition moves. This resemblance suggests that one consider higher order domination strategies on hexagonal graphs, since war games are typically played on boards consisting of hexagonal cells, where pieces may move from a cell to any of its 6 adjacent cells. We therefore define a hexagonal graph $\mathcal{H}_{p,q}$ consisting of pq vertices, numbered according to the edge set structure shown in Figure 3.4(a).

Theorem 9 *For any integers $p, q \geq 2$,*

$$(a)\ \gamma_\infty(\mathcal{H}_{p,q}) = \begin{cases} \lceil \frac{2q}{3} \rceil \frac{p}{2} & \text{if } p \text{ is even,} \\ \lceil \frac{2q}{3} \rceil \frac{p-3}{2} + q + 1 & \text{if } p \text{ is odd.} \end{cases}$$

$$(b)\ \gamma_\infty^*(\mathcal{H}_{p,q}) = pq - 2.$$

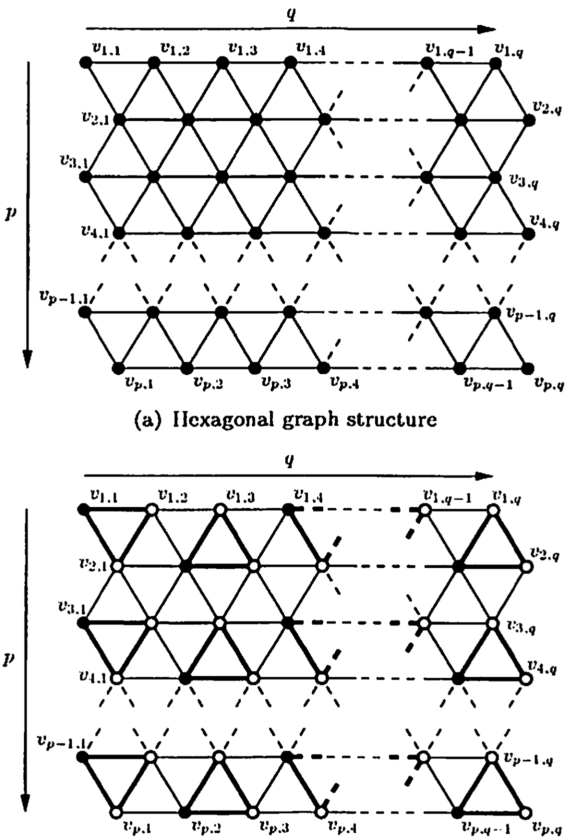
Proof: Define the triangular subgraphs

$$\overline{T}_{i,j} = \langle v_{3i+1,2j+1}, v_{3i+2,2j+1}, v_{3i+1,2(j+1)} \rangle$$

and

$$\underline{T}_{i,j} = \langle v_{3(i+1),2j+1}, v_{3i+2,2(j+1)}, v_{3(i+1),2(j+1)} \rangle$$

of $\mathcal{H}_{p,q}$ for all $i = 0, \ldots, \lceil 2q/3 \rceil$ and $j = 0, \ldots, \lfloor p/2 \rfloor$ with the convention that these triangles are pruned to subgraphs isomorphic to K_1 or K_2 if respectively one or two subscripts of vertices of $\overline{T}_{i,j}$ or $\underline{T}_{i,j}$ are out of range with respect to the vertex numbering of $\mathcal{H}_{p,q}$.



(a) Hexagonal graph structure

(b) p even

Figure 3.4: ∞–SDF for $\mathcal{H}_{p,q}$.

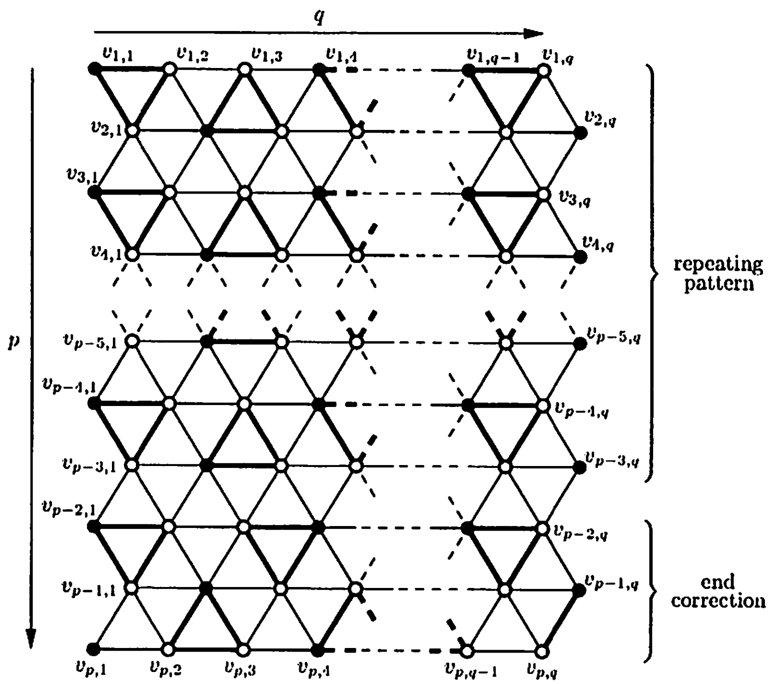


Figure 3.5: ∞–SDF for $\mathcal{H}_{p,q}$ (continued).

(a) If p is even (as shown in Figure 3.4(b)), then these (possibly pruned) triangles constitute a partition of the vertex set $V(\mathcal{H}_{p,q})$ into $\lceil\frac{2q}{3}\rceil\frac{p}{2}$ independent c–cliques (where $c \in \{1,2,3\}$), and hence $\gamma_{s,\infty}(\mathcal{H}_{p,q}) \le \lceil\frac{2q}{3}\rceil\frac{p}{2}$ by Theorem 4. But there exists an independent set of cardinality $\lceil\frac{2q}{3}\rceil\frac{p}{2}$ as indicated by the dark vertices in Figure 3.4(b), and hence $\gamma_{s,\infty}(\mathcal{H}_{p,q}) \ge \lceil\frac{2q}{3}\rceil\frac{p}{2}$ by Proposition 1, thereby proving the first equality.

For the second equality (i.e. if p is odd) the pattern for the last three rows may be altered to improve the upper bound $\gamma_\infty(\mathcal{H}_{p,q}) \le \lceil\frac{2q}{3}\rceil\frac{p}{2}$ slightly. It is easy to see that the last three rows of the graph contain $q+1$ independent cliques. Hence $\gamma_\infty(\mathcal{H}_{p,q}) \le \lceil\frac{2q}{3}\rceil\frac{p-3}{2}+q+1$. But there exists an independent set of cardinality $\lceil\frac{2q}{3}\rceil\frac{p-3}{2}+q+1$ as indicated by the dark vertices in Figure 3.5 when q is even, and hence $\gamma_{s,\infty}(\mathcal{H}_{p,q}) \ge \lceil\frac{2q}{3}\rceil\frac{p-3}{2}+q+1$, again by Proposition 1. For the case where both p and q are odd, an independent set can be found by selecting all vertices of the graph whose indices are both odd.

(b) This result follows directly from Theorem 3. ■

4 Conclusion

In this paper the previously studied notions of smart [foolproof] k-weak Roman and of smart [foolproof] k-secure domination [1] were generalised in the sense that safe guard functions in a simple graph were not merely sought after $k \geq 1$ moves, but in the limiting case where $k \to \infty$ instead, called perpetually or eternally smart [foolproof] domination. Some general properties of these generalised domination parameters were established in §2, after which the parameter values were found for certain simple graph structures in §3.

Further work may involve generalising the parameters in [1] and in this paper to the situation where an arbitrary number of guards may be stationed at any vertex of the graph. Also, examples of graphs exist for which $\gamma_\infty(G) < \chi(\overline{G})$. A characterision of exactly when strict inequality occurs in $\gamma_\infty(G) \leq \chi(\overline{G})$, which holds for all graphs, will certainly be of interest.

Acknowledgements

A.P. Burger, E.C. Cockayne & C.M. Mynhardt were visiting the Department of Applied Mathematics at Stellenbosch University when research towards this paper was done. This paper is based upon work supported by the South African National Research Foundation under grant number GUN 2053755 and Research Sub-Committee B at the University of Stellenbosch. Any opinions, findings and conclusions or recommendations expressed in this paper are those of the authors and do not necessarily reflect the views of the South African National Research Foundation.

References

[1] A.P. Burger, E.J. Cockayne, W.R. Gründlingh, C.M. Mynhardt, J.H. van Vuuren & W. Winterbach, *Finite Order Domination in Graphs*, submitted.

[2] E.J. Cockayne, P.A. Dreyer, S.M. Hedetniemi and S.T. Hedetniemi, *Roman domination in graphs*, to appear in *Discrete Math.*

[3] E.J. Cockayne, O. Favaron and C.M. Mynhardt, *Secure domination, weak Roman domination and forbidden subgraphs*, to appear in *Bull. Inst. Combin. Appl.*

[4] E.J. Cockayne, P.J.P. Grobler, W.R. Gründlingh, J. Munganga and J.H. van Vuuren, *Protection of a graph*, to appear in *Utilitas Math.*

[5] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, New York, 1998.

[6] M.A. Henning and S.M. Hedetniemi, *Defending the Roman Empire – A new strategy*, to appear in *Discrete Math.*

[7] I. Stewart, *Defend the Roman Empire!*, Scientific American, December 1999, 136–138.