On the number of edge-disjoint almost perfect matchings in regular odd order graphs

Lutz Volkmann

Lehrstuhl II für Mathematik, RWTH Aachen, 52056 Aachen, Germany
e-mail: volkm@math2.rwth-aachen.de

Abstract

Let G be a k-regular graph of odd order $n \geq 3$ with $k \geq (n+1)/2$. This implies that k is even. Furthermore, let

$$p = \min\left\{\frac{k}{2}, \left\lceil k - \frac{n}{3}\right\rceil\right\}.$$

If $x_1, x_2, \ldots, x_p$ are arbitray given, pairwise different, vertices of the graph G, then we show in this paper that there exist p pairwise edge-disjoint almost perfect matchings $M_1, M_2, \ldots, M_p$ in G with the property that no edge of M_i is incident with x_i for $i = 1, 2, \ldots, p$.

Keywords: Matching, Regular graph, Disjoint matchings, Perfect matching, Almost perfect matching

We shall assume that the reader is familiar with standard terminology on graphs (see, e.g., Chartrand and Lesniak [1]). In this paper, all graphs are finite and simple. The vertex set and edge set of a graph G are denoted by $V(G)$ and $E(G)$, respectively. The *neighborhood* $N_G(x) = N(x)$ of a vertex x is the set of vertices adjacent with x, and the number $d_G(x) = d(x) = |N(x)|$ is the *degree* of x. The *minimum degree* of G is denoted by $\delta(G)$. If $d \leq d_G(x) \leq d+t$ for each vertex x in a graph G, then we speak of a *$(d, d+t)$-graph.* For a vertex set X of a graph G, we define $G[X]$ as the subgraph induced by X. If M is a matching in a graph G with the property that every vertex (with exactly one exception) is incident with an edge of M, then M is a *perfect matching* (an *almost perfect matching*). If M is a matching and H_1 and H_2 are two disjoint subgraphs of G, then $e_M(H_1, H_2)$ are the number of edges in M with one end in H_1 and the other one in H_2. Analogously, if G is a graph and X_1 and X_2 are two disjoint vertex sets of

JCMCC **50** (2004), pp. 195-205

$V(G)$, then $e_G(X_1, X_2)$ are the number of edges in $E(G)$ with one end in X_1 and the other one in X_2. If G is a graph and $A \subseteq V(G)$, then we denote by $q(G-A)$ the number of odd components in the subgraph $G-A$. The proof of our main theorem is based on the following two well-known results.

Theorem 1 (Dirac [2] 1952). If G is a graph of order $n \geq 3$ and minimum degree $\delta \geq n/2$, then G is Hamiltonian.

Theorem 2 (Tutte [5] 1947). A graph G of even order has a perfect matching if and only if $q(G-S) \leq |S|+1$ for every proper subset S of $V(G)$.

As a generalization of a result by C.Q. Zhang [6] on regular graphs, C. Zhao [7] proved in 1991 the following theorem.

Theorem 3 (Zhao [7] 1991) If G is a $(d, d+1)$-graph of even order $2p \geq 2$ with $d \geq p$, then G contains at least

$$\left\lceil \frac{p+2}{3} + d - p \right\rceil$$

edge-disjoint perfect matchings.

Corollary 4 If G is a k-regular graph of odd order $n = 2p+1 \geq 3$ with $k \geq p+1$, then G contains at least

$$\left\lceil k - \frac{n}{3} \right\rceil$$

edge-disjoint almost perfect matchings.

Proof. Let v be an arbitrary vertex of G and define the graph H by $H = G - v$. Then, H is a $(k-1, k)$-graph of even order $2p$ with $k-1 \geq p$. According to Theorem 3, there are at least

$$\left\lceil \frac{p+2}{3} + k - 1 - p \right\rceil = \left\lceil k - \frac{n}{3} \right\rceil$$

edge-disjoint perfect matchings in H, and the proof is complete. □

All the edge-disjoint almost perfect matchings in the proof of Corollary 4 are not incident with the vertex v. A natural question to ask is how many edge-disjoint almost perfect matchings exist such that all these matchings are not incident with different vertices. In relation to this problem, we will now present the main theorem of this paper.

Theorem 5 Let G be a k-regular graph of odd order $n \geq 3$ with $k \geq (n+1)/2$, and let

$$p = \min\left\{\frac{k}{2}, \left\lceil k - \frac{n}{3} \right\rceil\right\}.$$

If the vertices $x_1, x_2, \ldots, x_p$ of the graph G are arbitray given and pairwise different, then there exist p pairwise edge-disjoint almost perfect matchings $M_1, M_2, \ldots, M_p$ in G with the property that no edge of M_i is incident with x_i for $i = 1, 2, \ldots, p$.

Proof. Let p be chosen maximal such that for p arbitrary given and pairwise different vertices $x_1, x_2, \ldots, x_p$ in G, there exist p pairwise edge-disjoint almost perfect matchings $M_1, M_2, \ldots, M_p$ in G with the property that no edge of M_i is incident with x_i for $i = 1, 2, \ldots, p$. Since $k \geq (n+1)/2$, Theorem 1 immediately shows that $p \geq 1$. If we define

$$G_s = G - \bigcup_{i=1}^{s} M_i$$

for $s \in \{1, 2, \ldots, p\}$, then G_s is a $(k-s, k-s+1)$-graph with the property that exactly the vertices $x_1, x_2, \ldots, x_s$ are of degree $k-s+1$. In the case that $s \leq k-(n+1)/2$, we have $\delta(G_s) = k - s \geq (n+1)/2$ and thus, according to Theorem 1, G_s contains a matching M_{s+1} with the property that no edge of M_{s+1} is incident with x_{s+1}. This implies $p \geq k-(n+1)/2+1 = k-(n-1)/2$.

Define next the graph $H = G_{k-(n+1)/2}$ with $G_0 = G$. If $k = (n+1)/2$, then H is the k-regular graph G, and if $k > (n+1)/2$, then H is an $((n+1)/2, (n+3)/2)$-graph with the property that exactly the vertices $x_1, x_2, \ldots, x_{k-(n+1)/2}$ are of degree $(n+3)/2$.

Now let t be chosen maximal such that for t arbitrary given, pairwise different, vertices $y_1, y_2, \ldots, y_t$ in H with $d(y_i, H) = (n+1)/2$, there exist t pairwise edge-disjoint almost perfect matchings $N_1, N_2, \ldots, N_t$ in H with the property that no edge of N_i is incident with y_i for $i = 1, 2, \ldots, t$. Because of $d(y_i, H) = (n+1)/2$ for $i = 1, 2, \ldots, t$, the vertices $y_1, y_2, \ldots, y_t$ are different from the vertices $x_1, x_2, \ldots, x_{k-(n+1)/2}$, if $k > (n+1)/2$. Therefore, it remains to show that

$$t \geq \min\left\{\frac{k}{2}, k - \frac{n}{3}\right\} - \left(k - \frac{n+1}{2}\right) = \min\left\{\frac{n+1-k}{2}, \frac{n+3}{6}\right\}. \quad (1)$$

We proceed by contradiction. Suppose, to the contrary that

$$t < \min\left\{\frac{n+1-k}{2}, \frac{n+3}{6}\right\}. \quad (2)$$

If we define

$$H_t = H - \bigcup_{i=1}^{t} N_i,$$

then H_t is an $((n+1)/2-t,(n+3)/2-t)$-graph with the property that exactly the vertices $x_1, x_2, \dots, x_{k-(n+1)/2}$ (if $k > (n+1)/2$) and $y_1, y_2, \dots, y_t$, are of degree $(n+3)/2-t$.

Let now $y_{t+1} \in V(H_t)$ be an arbitrary vertex with $d(y_{t+1}, H_t) = (n+1)/2-t$ and define the graph

$$F = H_t - y_{t+1}.$$

It follows that F is an $((n-1)/2-t,(n+3)/2-t)$-graph. If λ is the number of neighbors of y_{t+1} of degree $(n+3)/2-t$ in H_t, then F has exactly

$$\frac{n+1}{2} - t - \lambda \text{ vertices of degree } \frac{n-1}{2} - t, \tag{3}$$

$$n - k - 1 + 2\lambda \text{ vertices of degree } \frac{n+1}{2} - t, \tag{4}$$

$$k - \frac{n+1}{2} + t - \lambda \text{ vertices of degree } \frac{n+3}{2} - t. \tag{5}$$

By the choice of t, we conclude that F doesn't contain a perfect matching. Applying Theorem 2, we deduce that there exists a subset $S \subset V(F)$ such that $q(F-S) \geq |S| + 2$. In the following we distinguish two cases.

Case 1. Let $S = \emptyset$.

In this case, F has at least two odd components Q_1 and Q_2. On the one hand, the fact that $\delta(F) = (n-1)/2 - t$ implies that each odd component of F has at least $(n+1)/2-t$ vertices. On the other hand, our assumption (2), shows that $t < (n+3)/6$ and hence, F consists exactly of the two odd components Q_1 and Q_2. If we assume, without loss of generality, that $|V(Q_1)| \leq |V(Q_2)|$, then (2) implies $0 < (n+3)/2 - 3t$, and we obtain

$$\begin{aligned} |V(Q_2)| &= |V(F)| - |V(Q_1)| \leq n - 1 - \frac{n+1}{2} + t \\ &= \frac{n-3}{2} + t < \frac{n-3}{2} + t + \frac{n+3}{2} - 3t = n - 2t. \end{aligned}$$

Since $|V(Q_2)|$ and n are odd integers, this leads to $|V(Q_2)| \leq n - 2 - 2t$. Therefore, we see that

$$\frac{n+1}{2} - t \leq |V(Q_1)| \leq \frac{n-1}{2}, \tag{6}$$

$$\frac{n-1}{2} \leq |V(Q_2)| \leq n - 2 - 2t. \tag{7}$$

Subcase 1.1. Assume that $e_{N_i}(Q_1, Q_2) \leq 1$ for each almost perfect matching $N_i \in \{N_1, N_2, \ldots, N_t\}$. It follows from (2) and (6)

$$\begin{aligned}\sum_{i=1}^{t} e_{N_i}(Q_1, Q_2) &\leq t < \frac{n+3}{6} \leq \frac{n}{3} = \frac{n+1}{2} - \frac{n+3}{6} \\ &< \frac{n+1}{2} - t \leq |V(Q_1)|.\end{aligned}$$

Thus, there exists a vertex w in Q_1 such that $N_H(w) \subseteq V(Q_1) \cup \{y_{t+1}\}$. This leads to $d_H(w) \leq (n-1)/2$, a contradiction to $\delta(H) = (n+1)/2$.

Subcase 1.2. There is an index $j \in \{1, 2, \ldots, t\}$ such that $e_{N_j}(Q_1, Q_2) \geq 2$. Define the graph $H_t^* = H_t \cup N_j$, and let u_1u_2 and v_1v_2 be two edges of N_j such that $u_i, v_i \in V(Q_i)$ for $i = 1, 2$. The inequalities (6) and (7) lead to

$$|V(Q_i) - \{v_i\}| \leq n - 2t - 3 \tag{8}$$

for $i = 1, 2$. Since $\delta(H_t) = (n+1)/2 - t$, we conclude that

$$\delta(H_t^*[V(Q_i) - \{v_i\}]) \geq \delta(H_t[V(Q_i) - \{v_i\}]) \geq \frac{n-3}{2} - t,$$

and hence, (8) yields for $i = 1, 2$ that

$$2\delta(H_t^*[V(Q_i) - \{v_i\}]) \geq n - 3 - 2t \geq |V(Q_i) - \{v_i\}|.$$

As a consequence of Theorem 1, we see that the graph $H_t^*[V(Q_i) - \{v_i\}]$ is Hamiltonian for $i = 1, 2$. Thus, the even order graph $H_t^*[V(Q_i) - \{v_i\}]$ has a perfect matching M_i^* for $i = 1, 2$. If we define the matching $N_j^* = M_1^* \cup M_2^* \cup \{v_1v_2\}$, then N_j^* is a perfect matching of $H_t^* - y_{t+1}$. Furthermore, we define the graphs

$$F^* = (V(H_t^*) - \{y_j\}, E(H_t^*) - N_j^*)$$

and for $i = 1, 2$

$$Q_i^* = Q_i - y_j, \text{ if } y_j \in V(Q_i),$$
$$Q_i^* = Q_i, \text{ if } y_j \notin V(Q_i).$$

Subcase 1.2.1. Let $y_j \in V(Q_i)$ for $i = 1$ or $i = 2$, then, in view of (6) and (7), it follows that

$$|V(Q_i^*)| = |V(Q_i)| - 1 \leq n - 2t - 3. \tag{9}$$

In addition, we observe that

$$\begin{aligned}\delta(F^*[V(Q_i^*)]) &\geq \delta(H_t^*[V(Q_i^*)]) - 1 \geq \delta(H_t^*[V(Q_i)]) - 2 \\ &\geq \delta(F[V(Q_i)]) - 2 \geq \frac{n-1}{2} - t - 2 = \frac{n-5}{2} - t.\end{aligned}$$

Because of $v_1v_2 \in N_j \cap N_j^*$, we deduce from (9) that

$$\begin{aligned} d(v_i, F^*[V(Q_i^*)]) &= d(v_i, F[V(Q_i^*)]) \geq d(v_i, F[V(Q_i)]) - 1 \\ &\geq \frac{n-1}{2} - t - 1 = \frac{n-3}{2} - t. \end{aligned}$$

If we assume that $F^*[V(Q_i^*]$ is disconnected, then, since v_i is contained in one of the components, we obtain by the last two inequality chains the following contradiction to (9)

$$|V(Q_i^*)| \geq \frac{n-1}{2} - t + \frac{n-3}{2} - t = n - 2 - 2t.$$

Consequently, the graph $F^*[V(Q_i^*)]$ is connected.

Subcase 1.2.2. Let $y_j \notin V(Q_i)$ for $i = 1$ or $i = 2$, then, in view of (6) and (7), it follows that

$$|V(Q_i^*)| = |V(Q_i)| \leq n - 2t - 2. \tag{10}$$

In addition, we have

$$\begin{aligned} \delta(F^*[V(Q_i^*)]) &\geq \delta(H_t^*[V(Q_i^*)]) - 1 = \delta(H_t^*[V(Q_i)]) - 1 \\ &\geq \delta(F[V(Q_i)]) - 1 \geq \frac{n-1}{2} - t - 1 = \frac{n-3}{2} - t. \end{aligned}$$

If we assume that $F^*[V(Q_i^*]$ is disconnected, then, the last inequality chain yields the following contradiction to (10)

$$|V(Q_i^*)| \geq \frac{n-1}{2} - t + \frac{n-1}{2} - t = n - 1 - 2t.$$

Consequently, the graph $F^*[V(Q_i^*)]$ is connected.

Combining the Subcases 1.2.1 and 1.2.2, we find that $F^*[Q_1^*)]$ and $F^*[Q_2^*]$ are connected graphs. Therefore, also the graph F^* is connected, because u_1u_2 is an edge joining $F^*[Q_1^*)]$ and $F^*[Q_2^*]$ and since $d(y_{t+1}, F^*) \geq (n-1)/2 - t > 0$. Now we replace F by F^* and y_{t+1} by y_j and discuss the next case.

Case 2. Let $S \neq \emptyset$.

Let Q be an odd component of $F - S$, and let $v \in V(Q)$. Since $\delta(F) = (n-1)/2 - t$ and $N(v, F) \subseteq V(Q) \cup S$, we observe that

$$|V(Q)| + |S| \geq \frac{n+1}{2} - t. \tag{11}$$

We call an an odd component of $F - S$ large if it has at least $(n+1)/2 - t$ vertices and small otherwise. If we denote by α and β the number of small and large components, respectively, then Theorem 2 implies

$$\alpha + \beta \geq |S| + 2. \tag{12}$$

Subcase 2.1. Let $\beta \geq 3$. Then, the assumption (2) leads to the following contradiction

$$n-1=|V(F)|>3\left(\frac{n+1}{2}-t\right)>3\left(\frac{n+1}{2}-\frac{n+3}{6}\right)=n.$$

Subcase 2.2. Let $\beta = 2$. Since $S \neq \emptyset$, inequality (12) yields $\alpha \geq 1$. If Q is an odd component of $F-S$ with $|V(Q)| \leq (n-1)/2-t$, then we obtain by (11) and (2) the contradiction

$$n-1=|V(F)|\geq 2\left(\frac{n+1}{2}-t\right)+|V(Q)|+|S|\geq 3\left(\frac{n+1}{2}-t\right)>n.$$

Subcase 2.3. Let $\beta = 1$. In view of (12), we have

$$\alpha \geq |S|+1. \tag{13}$$

It follows from (2) and (13) that

$$\begin{aligned} n-1 &= |V(F)|\geq \alpha+\frac{n+1}{2}-t+|S| \\ &> 2|S|+1+\frac{n+1}{2}-\frac{n+3}{6}=2|S|+1+\frac{n}{3} \end{aligned}$$

and hence,

$$1\leq |S|<\frac{n}{3}-1. \tag{14}$$

Applying (11), we see that the graph $F-S$ has α odd components with at least $(n+1)/2-t-|S|$ vertices and one odd component with at least $(n+1)/2-t$ vertices. Thus, we conclude from (2) and (14) that

$$\begin{aligned} n-1 &= |V(F)|\geq \alpha\left(\frac{n+1}{2}-t-|S|\right)+\frac{n+1}{2}-t+|S| \\ &> (|S|+1)\left(\frac{n+1}{2}-\frac{n+3}{6}-|S|\right)+\frac{n+1}{2}-\frac{n+3}{6}+|S| \\ &= (|S|+1)\left(\frac{n}{3}-|S|\right)+\frac{n}{3}+|S| \\ &= \frac{n}{3}|S|-|S|^2+\frac{2}{3}n \end{aligned}$$

and so, we arrive at

$$|S|^2-\frac{n}{3}|S|+\frac{n}{3}-1>0. \tag{15}$$

If we define the function

$$g(x)=x^2-\frac{n}{3}x+\frac{n}{3}-1$$

in the interval $I : 1 \le x \le (n/3) - 1$, then it is a simple matter to verify that

$$\max_{x \in I}\{g(x)\} = g(1) = g((n/3) - 1) = 0.$$

However, this is a contradiction to (14) and (15).

Subcase 2.4. Let $\beta = 0$. In view of (12), we have at least $|S| + 2$ small components of order at most $(n-1)/2 - t$. If Q is such a component of order $q \le (n-1)/2 - t$, then, every vertex of Q is joint with S by at least $\min_{x \in V(Q)}\{d(x, F)\} - (q-1)$ edges and we obtain

$$e_F(V(Q), S) \ge q\left(\min_{x \in V(Q)}\{d(x, F)\} - (q-1)\right).$$

Because of $1 \le q \le \min_{x \in V(Q)}\{d(x, F)\}$, this leads easily to

$$e_F(V(Q), S) \ge \min_{x \in V(Q)}\{d(x, F)\}. \qquad (16)$$

If $|S| + 2 \le (n+1)/2 - t - \lambda$, then it follows from $\delta(F) = (n-1)/2 - t$ and (16)

$$\begin{aligned} e_F(S, V(F) - S) &\ge (|S| + 2)\left(\frac{n-1}{2} - t\right) \\ &= (|S| + 2)\left(\frac{n+1}{2} - t\right) - (|S| + 2) \\ &\ge (|S| + 2)\left(\frac{n+1}{2} - t\right) - \frac{n+1}{2} + t + \lambda. \end{aligned}$$

If $|S| + 2 > (n+1)/2 - t - \lambda$, then it follows from (3), (4), (5), and (16) that there are at least $|S| + 2 - ((n+1)/2 - t - \lambda)$ odd components Q such that

$$e_F(V(Q), S) \ge \min_{x \in V(Q)}\{d(x, F)\} \ge \frac{n+1}{2} - t.$$

Thus, it follows from (3), (4), and (5) that

$$\begin{aligned} e_F(S, V(F) - S) &\ge \left(\frac{n+1}{2} - t - \lambda\right)\left(\frac{n-1}{2} - t\right) \\ &+ \left(|S| + 2 - \frac{n+1}{2} + t + \lambda\right)\left(\frac{n+1}{2} - t\right) \\ &= (|S| + 2)\left(\frac{n+1}{2} - t\right) - \frac{n+1}{2} + t + \lambda. \end{aligned}$$

So, in every case, we arrive at

$$e_F(S, V(F) - S) \ge (|S| + 2)\left(\frac{n+1}{2} - t\right) - \frac{n+1}{2} + t + \lambda. \qquad (17)$$

According to (5), there are at most $k - (n+1)/2 + t - \lambda$ vertices in S of degree $(n+3)/2 - t$. Hence, if $|S| > k - (n+1)/2 + t - \lambda$, then (3), (4), and (5) imply

$$\begin{aligned} e_F(S, V(F) - S) &\le \left(k - \frac{n+1}{2} + t - \lambda\right)\left(\frac{n+3}{2} - t\right) \\ &+ \left(|S| - k + \frac{n+1}{2} - t + \lambda\right)\left(\frac{n+1}{2} - t\right) \\ &= |S|\left(\frac{n+1}{2} - t\right) + k - \frac{n+1}{2} + t - \lambda. \end{aligned}$$

If $|S| \le k - (n+1)/2 + t - \lambda$, then we conclude from (3), (4), and (5) that

$$\begin{aligned} e_F(S, V(F) - S) &\le |S|\left(\frac{n+3}{2} - t\right) = |S|\left(\frac{n+1}{2} - t\right) + |S| \\ &\le |S|\left(\frac{n+1}{2} - t\right) + k - \frac{n+1}{2} + t - \lambda \end{aligned}$$

and hence, we have in every case that

$$e_F(S, V(F) - S) \le |S|\left(\frac{n+1}{2} - t\right) + k - \frac{n+1}{2} + t - \lambda. \qquad (18)$$

Combining (17) and (18), we find on the one hand that

$$(|S|+2)\left(\frac{n+1}{2} - t\right) - \frac{n+1}{2} + t + \lambda \le |S|\left(\frac{n+1}{2} - t\right) + k - \frac{n+1}{2} + t - \lambda,$$

and this is equivalent with

$$2\left(\frac{n+1}{2} - t\right) - k + 2\lambda \le 0. \qquad (19)$$

On the other hand, our assumption (2) implies

$$2\left(\frac{n+1}{2} - t\right) - k + 2\lambda > 2\left(\frac{n+1}{2} - \frac{n+1-k}{2}\right) - k = 0,$$

a contradiction to (19), and the proof of Theorem 5 is complete. □

If G is of order $n = 4s + 1$ and $k = (n-1)/2 = 2s$ in Theorem 5, then, by a theorem of Nash-Williams [4] or Jackson [3] (cf. [1], p. 108), the graph G is Hamiltonian. Using this fact, one can show similarly to the proof of Theorem 5 the following supplement to the main theorem.

Theorem 6 Let G be a $2s$-regular graph of odd order $n = 4s + 1$ and let $p = \lceil (n-3)/6 \rceil$. If $x_1, x_2, \ldots, x_p$ are arbitray given, pairwise different, vertices of the graph G, then there exist p pairwise edge-disjoint almost perfect matchings $M_1, M_2, \ldots, M_p$ with the property that no edge of M_i is incident with x_i for $i = 1, 2, \ldots, p$.

The next examples will show that the conditions $k \geq (n+1)/2$ (cf. Theorem 5) in the case $n = 4s+3$ and $k \geq (n-1)/2$ (cf. Theorems 5 and 6) in the case $n = 4s+1$ cannot be weakened.

Example 7 Let $n = 4s+1$ and let $k = 2s-2$ with $s \geq 2$.

a) Let G_1 be the complete graph K_{2s-1}, and let $G_2 = K_{2s+2} - (M_1 \cup M_2 \cup M_3)$, where M_1, M_2, and M_3 are three edge-disjoint perfect matchings of the complete graph K_{2s+2}. Now the disjoint union of G_1 and G_2 is a k-regular graph G of order $n = 4s+1$. However, if x is an arbitrary vertex of G_2, then there doesn't exist an almost perfect matching M in G with the property that no edge of M is incident with x.

b) For $s \geq 3$, let H_1 be a the complete graphs K_{2s-1}, and let $H_2 = K_{2s+1} - E(C)$, where C is a Hamiltonian cycle of the complete graph K_{2s+1}. In addition, let $M_1 = \{x_1y_1, x_2y_2, \ldots, x_{s-2}y_{s-2}\}$ be a matching in H_1, and let $M_2 = \{u_1v_1\}$ be a matching in H_2. Now let G be the disjoint union of $H_1 - M_1$, $H_2 - M_2$, and a further vertex w, together with the edges wu_1, wv_1, wx_i, and wy_i for $i = 1, 2, \ldots, s-2$. Obviously, G is a connected $(2s-2)$-regular graph of order $n = 4s+1$. However, there doesn't exist an almost perfect matching M in G with the property that no edge of M is incident with w.

Example 8 Let $n = 4s+3$ and $k \leq (n-1)/2$ with $s \geq 2$. Since k is even, it follows that $k \leq (n-3)/2 = 2s$.

a) Let G_1 be the complete graph K_{2s+1}, and let G_2 be the graph $K_{2s+2} - M_1$, where M_1 is a perfect matching of the complete graph K_{2s+2}. Now the disjoint union of G_1 and G_2 is a $2s$-regular graph G of order $n = 4s+3$. However, if x is an arbitrary vertex of G_2, then there doesn't exist an almost perfect matching M in G with the property that no edge of M is incident with x.

b) Let H_1 and H_2 be two copies of the complete graph K_{2s+1} and let w be a further vertex. In addition, let $M_1 = \{x_1y_1, x_2y_2, \ldots, x_{s-1}y_{s-1}\}$ a matching in H_1, and let $M_2 = \{u_1v_1\}$ be a matching in H_2. Now let G be the disjoint union of $H_1 - M_1$ and $H_2 - M_2$ together with the edges wu_1, wv_1, wx_i, and wy_i for $i = 1, 2, \ldots, s-1$. Obviously, G is a connected $2s$-regular graph of order $n = 4s+3$. However, there doesn't exist an almost perfect matching M in G with the property that no edge of M is incident with w.

References

[1] G. Chartrand, L. Lesniak, Graphs and Digraphs, 3rd Edition, Chapman and Hall, London, 1996.

[2] G.A. Dirac, Some theorems on abstract graphs, *Proc. London Math. Soc.* (3) **2** (1952), 69-81.

[3] B. Jackson, Hamilton cycles in regular 2-connected graphs, *J. Combin. Theory* **29 B** (1980), 27-46.

[4] C.St.J.A. Nash-Williams, Valency sequences which force a graph to have hamiltonian circuits, *Report of the Univ. of Waterloo* (1972).

[5] W.T. Tutte, The factorizations of linear graphs, *J. London Math. Soc.* **22** (1947), 459-474.

[6] C.Q. Zhang, On a theorem of Hilton, *Ars Combin.* **27** (1989), 66-68.

[7] C. Zhao, The disjoint 1-factors of $(d, d+1)$-graphs, *J. Combin. Math. Combin. Comput.* **9** (1991), 195-198.