From Equi-graphical Sets to Graphical Permutations

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Abstract

A set $\{a_1, a_2, \ldots, a_n\}$ of positive integers with $a_1 < a_2 < \cdots < a_n$ is said to be equi-graphical if there exists a graph with exactly a_i vertices of degree a_i for each i with $1 \leq i \leq n$. It is known that such a set is equi-graphical if and only if $\sum_{i=1}^{n} a_i$ is even and $a_n \leq \sum_{i=1}^{n-1} a_i^2$. This concept is generalized to the following problem: Given a set S of positive integers and a permutation π on S, determine when there exists a graph containing exactly a_i vertices of degree $\pi(a_i)$ for each i $(1 \leq i \leq n)$. If such a graph exists, then π is called a graphical permutation. In this paper, the graphical permutations on sets of size four are characterized and using a criterion of Fulkerson, Hoffman, and McAndrew, we show that a permutation π of $S = \{a_1, a_2, \ldots, a_n\}$, where $1 \leq a_1 < a_2 < \cdots < a_n$ and such that $\pi(a_n) = a_n$, is graphical if and only if $\sum_{i=1}^{n} a_i\pi(a_i)$ is even and $a_n \leq \sum_{i=1}^{n-1} a_i\pi(a_i)$.

1 Introduction

For a graph G with vertices $v_1, v_2, \ldots, v_n$ and degrees $d_i = \deg v_i$ $(i = 1, 2, \ldots, n)$, we call $d_1, d_2, \ldots, d_n$ a degree sequence for G. A sequence $s : d_1, d_2, \ldots, d_n$ of nonnegative integers is a *graphical sequence* if there exists a graph having degree sequence s. If $s : d_1, d_2, \ldots, d_n$ is a graphical sequence, then $d_i \leq n - 1$ for each i $(1 \leq i \leq n)$ and the sum $\sum_{i=1}^{n} d_i$ is even. Of course, these two conditions are necessary but not sufficient for a

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sequence s to be graphical. Many characterizations of graphical sequences can be found in the literature. Perhaps the most well known are those by Havel [5] and Hakimi [3] and by Erdös and Gallai [1].

Theorem 1 *(Havel, Hakimi) A sequence $s : d_1, d_2, \ldots, d_n$ of nonnegative integers with $d_1 \geq d_2 \geq \cdots \geq d_n$, where $n \geq 2$ and $d_1 \geq 1$, is graphical if and only if the sequence $s_1 : d_2 - 1, d_3 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n$ is graphical.*

Theorem 2 *(Erdös and Gallai) A sequence $s : d_1, d_2, \ldots, d_n$ of nonnegative integers with $d_1 \geq d_2 \geq \cdots \geq d_n$, where $n \geq 2$ and $d_1 \geq 1$, is graphical if and only if $\sum_{i=1}^{n} d_i$ is even and for each integer k $(1 \leq k \leq n-1)$, the following holds:*

$$\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{n} min\{k, d_i\}.$$

In [4] a set $S = \{a_1, a_2, \ldots, a_n\}$ of n distinct positive integers is said to be *equi-graphical* if there exists a graph of order $\sum_{i=1}^{n} a_i$ that contains exactly a_i vertices of degree a_i for every i $(1 \leq i \leq n)$. A simple condition given in [4] determines whether a set is equi-graphical.

Theorem 3 *Let $S = \{a_1, a_2, \ldots, a_n\}$ with $n \geq 2$ be a set of n positive integers such that $a_1 < a_2 < \cdots < a_n$ and $\sum_{i=1}^{n} a_i$ is even. then S is equi-graphical if and only if $a_n \leq a_1^2 + a_2^2 + \cdots + a_{n-1}^2$.*

Following the conclusion of [4], we consider a generalization of equi-graphical sets. Given a set $S = \{a_1, a_2, \ldots, a_n\}$ of n distinct positive integers and a permutation π on S, we say that π is a *graphical permutation* if there exists a graph G of order $\sum_{i=1}^{n} a_i$ containing exactly a_i vertices of degree $\pi(a_i)$ for each $i = 1, 2, \ldots, n$. Observe that a set is equi-graphical if and only if the identity mapping is a graphical permutation. Characterizations for graphical permutations on sets with cardinalities two and three are witnessed in [4]. A condition is missing for the permutation $\pi = (a\ b\ c)$, where $1 \leq a < b < c$. In [4] the result states that π is graphical if and only if at most one of a, b, and c is odd. However, the inequality $bc \leq ab + ac + b(b-1)$ is also needed, for consider a graph G with a vertices of degree b, b vertices of degree c, and c vertices of degree a. Let A be the set of a vertices of degree b, B be the set of b vertices of degree c, and C be the set of c vertices of degree a. Further, let x denote the number of edges that join vertices of B to vertices of $A \cup C$. Since the degree of each vertex of A is b and the degree of each vertex of C is a, we know at most $ab + ac$ edges leave $A \cup C$. Thus $x \leq ab + ac$. Further, a vertex of B has degree at most $b - 1$

in $\langle B \rangle$. So at least $c - b + 1$ edges for each vertex of B must leave the set B. Hence $x \geq b(c - b + 1)$. Therefore $b(c - b + 1) \leq ab + ac$ or equivalently $bc \leq ab + ac + b(b - 1)$. The details for the construction of a graph with a vertices of degree b, b vertices of degree c, and c vertices of degree a when at most one of a, b, and c is odd and $bc \leq ab + ac + b(b - 1)$ can be found in [7].

The following lemma was provided in [4] and is extremely useful.

Lemma 4 *Let x, y, and r be nonnegative integers such that $x + y > 0$ and $r < x + y - 1$. Then there exists a graph G of order $x + y$ containing x vertices of degree r and y vertices of degree $r + 1$ if and only if $rx + (r + 1)y$ is even.*

In [4] the following conjecture is made.

Conjecture 5 *Let $S = \{a_1, a_2, \ldots, a_n\}$ be a set of integers ($n \geq 2$) with $1 \leq a_1 < a_2 < \cdots < a_n$ and π a permutation on S such that $\sum_{i=1}^{n} a_i \pi(a_i)$ is even. Then*

1. *if $\pi(a_n) = a_n$, then π is graphical if and only if $a_n \leq \sum_{i=1}^{n-1} a_i \pi(a_i)$ and*

2. *if $\pi(a_n) \neq a_n$, then π is graphical.*

Of particular importance is that by providing the necessary inequality for $\pi = (a\ b\ c)$, we have disproved part 2 of Conjecture 5. Next we consider graphical permutations on sets of cardinality four. We return to this conjecture in the final section, where a proof of part 1 is given as well as a modification of part 2.

2 Graphical permutations on sets of cardinality four

This section contains the characterization for graphical permutations on a set of cardinality four. We provide only the proof for $\pi_{24} = (a\ c\ b\ d)$ as it is the most interesting case. Additional details for the other cases may be found in [7], and of course, some of these also follow as special cases of the general results given in the next section.

Theorem 6 *Let $S = \{a, b, c, d\}$ with $1 \leq a < b < c < d$. Then*

1. *$\pi_1 = (a)$ is graphical if and only if $d \leq a^2 + b^2 + c^2$ and $a + b + c + d$ is even,*

2. $\pi_2 = (cd)$ *is graphical if and only if* a *and* b *are of the same parity,*

3. $\pi_3 = (bd)$ *is graphical if and only if* a *and* c *are of the same parity,*

4. $\pi_4 = (ad)$ *is graphical if and only if* b *and* c *are of the same parity,*

5. $\pi_5 = (bc)$ *is graphical if and only if* a *and* d *are of the same parity and* $d \leq a^2 + 2bc$,

6. $\pi_6 = (ac)$ *is graphical if and only if* c *and* d *are of the same parity and* $d \leq b^2 + 2ac$,

7. $\pi_7 = (ab)$ *is graphical if and only if* c *and* d *are of the same parity and* $d \leq c^2 + 2ab$,

8. $\pi_8 = (bcd)$ *is graphical if and only if* $a^2 + bc + cd + bd$ *is even and* $cd \leq a^2 + bc + bd + c(c-1)$,

9. $\pi_9 = (acd)$ *is graphical if and only if* $ac + b^2 + cd + ad$ *is even and* $cd \leq ac + b^2 + ad + c(c-1)$,

10. $\pi_{10} = (bdc)$ *is graphical if and only if* $a^2 + bd + bc + cd$ *is even,*

11. $\pi_{11} = (adc)$ *is graphical if and only if* $ad + b^2 + ac + cd$ *is even,*

12. $\pi_{12} = (adb)$ *is graphical if and only if* $ad + ab + c^2 + bd$ *is even,*

13. $\pi_{13} = (acb)$ *is graphical if and only if* $ac + ab + bc + d^2$ *is even and* $d \leq ab + bc + ac$,

14. $\pi_{14} = (abd)$ *is graphical if and only if* $ab + bd + c^2 + ad$ *is even and* $bd \leq ab + bc + ad + b(b-1)$,

15. $\pi_{15} = (abc)$ *is graphical if and only if* $ab + bc + ac + d^2$ *is even and* $d \leq ab + bc + ac$,

16. $\pi_{16} = (ab)(cd)$ *is graphical,*

17. $\pi_{17} = (ac)(bd)$ *is graphical,*

18. $\pi_{18} = (ad)(bc)$ *is graphical,*

19. $\pi_{19} = (abdc)$ *is graphical if and only if* a *and* d *or* b *and* c *have the same parity,*

20. $\pi_{20} = (adbc)$ *is graphical if and only if* a *and* b *or* c *and* d *have the same parity,*

21. $\pi_{21} = (adcb)$ *is graphical if and only if* a *and* c *or* b *and* d *have the same parity,*

22. *$\pi_{22} = (abcd)$ is graphical if and only if a and c or b and d have the same parity and $cd \leq ab + bc + ad + c(c-1)$,*

23. *$\pi_{23} = (acdb)$ is graphical if and only if a and d or b and c have the same parity and $cd \leq ac + ab + bd + c(c-1)$, and*

24. *$\pi_{24} = (acbd)$ is graphical if and only if a and b or c and d have the same parity and either (1) if $c \leq a + b - 1$, then $bd \leq ab + bc + ad + b(b-1)$ or (2) if $c \geq a+b-1$, then $ac+bd \leq bc+ad+(a+b)(a+b-1)$.*

Proof Assume that π_{24} is graphical. Then there exists a graph G of order $a + b + c + d$ containing a vertices of degree c, b vertices of degree d, c vertices of degree b, and d vertices of degree a. Since the sum of the degrees of the vertices of G is even, we have $ac+bd+bc+ad = (a+b)(c+d)$ is even. So either a and b have the same parity or c and d have the same parity. Define $V(G) = A \cup B \cup C \cup D$, where A contains a vertices of degree c, B contains b vertices of degree d, C contains c vertices of degree b, and D contains d vertices of degree a.

Suppose, first, that $c \leq a+b-1$ and let x denote the number of edges that join vertices of B to vertices of $A \cup C \cup D$. Since the degree of each vertex of A is c, but each vertex of a can be adjacent to only b vertices of B, the degree of each vertex of C is b, and the degree of each vertex of d is a, we know at most $ab + bc + ad$ edges leave $A \cup B \cup D$. Thus $x \leq ab + bc + ad$. Further, a vertex of B has degree at most $b - 1$ in $\langle B \rangle$. So at least $d - b + 1$ edges for each vertex of B must leave the set B. Hence $x \geq b(d - b + 1)$. Therefore $b(d - b + 1) \leq ac + bc + ad$, or equivalently $bd \leq ac + bc + ad + b(b-1)$.

Now suppose that $c > a + b - 1$ and let x denote the number of edges that join vertices of $A \cup B$ to vertices of $C \cup D$. Since the degree of each vertex of C is b and the degree of each vertex of D is a, we know at most $bc + ad$ edges leave $C \cup D$. Thus $x \leq bc + ad$. Further, a vertex of $A \cup B$ has degree at most $a + b - 1$ in $\langle A \cup B \rangle$. So at least $d - a - b + 1$ edges for each vertex of B and at least $c - a - b + 1$ edges for each vertex of A must leave the set $A \cup B$. Hence $x \geq b(d - a - b + 1) + a(c - a - b + 1)$. Therefore $b(d - a - b + 1) + a(c - a - b + 1) \leq bc + ad$, or equivalently $ac + bd \leq bc + ad + (a + b)(a + b - 1)$.

For the converse, assume that a and b or c and d have the same parity and let $V(G) = A \cup B \cup C \cup D$, where $|A| = a$, $|B| = b$, $|C| = c$, and $|D| = d$. Begin by placing the complete bipartite graph $K_{b,c}$ on the sets B and C, using B and C as the partite sets, so that each vertex of C has degree b and each vertex of B needs its degree increased by $d - c$. The remainder of the construction is now divided into three cases depending whether $bd \leq ab+bc$, $ab + bc < bd \leq ab + ad + bc$, or $ab + ad + bc < bd \leq ab + ad + bc + b(b-1)$.

Case 1 *Suppose that* $bd \leq ab + bc$. Since $b(d-c) \leq ab$, we distribute $b(d-c)$ edges from B to A. Define the integers q and r by

$$b(d-c) = aq + r,$$

where $0 \leq r \leq a-1$. Denote this graph by G'. In G' we see that the vertices of B have degree d and the vertices of C have degree b and in A we have r vertices that have degree $q+1$ in G' and $a-r$ that have degree q in G'. We now consider two possibilities.

Subcase 1.1 *Assume that* $c-q-1 < a-1$. It remains to construct a graph with r vertices of degree $c-q-1$, $a-r$ vertices of degree $c-q$, and d vertices of degree a. Place the complete bipartite graph $K_{a,c-q-1}$ on the vertices of A and $c-q-1$ vertices of D. In A, r vertices now have degree c and $a-r$ vertices have degree $c-1$ while in D, $d-c+q+1$ vertices need degree a. Now if a and r have the same parity, we can use Lemma 4 to raise the remaining vertices of D to degree a, while if a and r are of opposite parity, we add an edge between a vertex of D needing degree a and a vertex of A needing its degree increased by one and then use Lemma 4. The remaining $a-r$ or $a-r-1$ vertices can be paired and their degree increased by one using edges from $\langle B \cup C\rangle$.

Subcase 1.2 *Assume that* $c-q-1 \geq a-1$. Proceed by placing the complete graph K_a on the set A so that r vertices need their degree increased by $c-q-a$ and $a-r$ need their degree increased by $c-q-a+1$. Place $K_{a,c-q-a}$ on the vertices of A and $c-q-a$ vertices of D so that $a-r$ vertices of A need their degree increased by one and $d-c+q+a$ vertices of D need their degree increased by a. Notice that $d-c+q+a > a-r$, so we add $a-r$ edges between the vertices in A that need their degree increased by one and $a-r$ vertices of the $d-c+q+a$ in D that need their degree increased by a. Thus it remains to show that there exists a graph H of order $d-c+q+a$ such that $a-r$ vertices have degree $a-1$ and $d-c+q+r$ have degree a. By Lemma 4, this is possible if and only if $(a-r)(a-1)+a(d-c+q+r)$ is even and $a < d-c+q+a-1$. Clearly $a < d-c+q+a-1$. Also $(a-r)(a-1)+a(d-c+q+r) = a(a-1)+ad-ac+bd-bc$ is even and such a graph H exists.

Case 2 *Suppose that* $ab+bc < bd \leq ab+bc+ad$. Place $K_{a,b}$ on the partite sets A and B and notice that the vertices of A need their degree increased by $c-b$ while the vertices of B need their degree increased by $d-c-a$. The construction is further divided into two subcases.

Subcase 2.1 *Suppose that* $c \leq a+b-1$. We proceed by distributing $b(d-c-a)$ edges among the vertices of D. Define the integers q and r by $b(d-c-a) = dq+r$, where $0 \leq r \leq d-1$. Distributing $b(d-c-a)$ edges among the vertices of D we have r vertices of D that have degree $q+1$ and $d-r$ that have degree q. Thus it remains to show that there exists a graph H of order d such that r vertices have degree $a-q-1$ and $d-r$ have degree $a-q$. By Lemma 4, this is possible if and only if $(a-q-1)r+(a-q)(d-r)$ is even and $a-q-1 < d-1$. Clearly $a-q-1 < d-1$. Also $(a-q-1)r+(a-q)(d-r) = -r+ad-dq = bc-bd+ad+ab$, and $bc-bd+ad+ab = bc-bd+ad+ac-a(c+b)$, which is even if and only if $a(c+b)$ is even. When $a(c+b)$ is even the graph on D may be finished and we obtain the desired degrees for the vertices in A using Lemma 4. If $a(c+b)$ is odd, we add an edge from a vertex in A to a vertex in D needing degree $a-q$ so that the parity condition now holds and we may finish A and D both using Lemma 4.

Subcase 2.2 *Suppose that* $c > a+b-1$. We proceed by placing the graph K_{a+b} on $A \cup B$. Now each vertex of A needs its degree increased by $c-a-b+1$ while each vertex of B needs its degree increased by $d-c-a-b+1$. We accomplish this by distributing $a(c-a-b+1)+b(d-c-a-b+1)$ edges from $A \cup B$ to the set D. Notice that $a(c-a-b+1)+b(d-c-a-b+1) \leq ad$ since $c > a+b-1$. Define the integers q and r by $a(c-a-b+1)+b(d-c-a-b+1) = qd+r$, where $0 \leq r \leq d-1$. Distributing these edges among the vertices of D we have r vertices that need their degree increased by $a-q-1$ and $d-r$ vertices that need their degree increased by $a-q$. By Lemma 4, this is possible if and only if $(a-q-1)r+(a-q)(d-r)$ is even and $a-q-1 < d-1$. Clearly $a-q-1 < d-1$. Also $(a-q-1)r+(a-q)(d-r) = -r+ad-dq = ad-ac-bd+bc+2ab+(a+b)(a+b-1)$, which is even.

Case 3 *Suppose that* $ab+ad+bc < bd \leq ab+ad+bc+b(b-1)$. The construction is once again divided into two subcases.

Subcase 3.1 *Suppose that* $c \leq a+b-1$. We proceed by placing the graph $K_{a,b}$ on the partite sets A and B and distribute ad edges from the set D to the vertices of B. Define the integers q and r by $ad = bq+r$, where $0 \leq r \leq b-1$. Thus it remains to show that there exists a graph H of order b such that r vertices have degree $d-c-a-q-1$ and $b-r$ have degree $d-c-a-q$. By Lemma 4, this is possible if and only if $(d-c-a-q-1)r+(d-c-a-q)(b-r)$ is even and $d-c-a-q-1 < b-1$. By definition $q = (ad-r)/b$ and the desired inequality is equivalent to $bd < ab+bc+ad+b(b-1)+b-r$. By assumption $bd \leq ab+bc+ad+b(b-1)$ and since $b-r > 0$, the inequality holds. Also, $(d-c-a-q-1)r+(d-c-a-q)(b-r) = -r+bd-bc-ab-bq = -ad+bd-bc+ac-a(b+c)$. Thus, if $a(b+c)$ is even then such a graph H is possible on B and also we can use Lemma 4 to achieve the desired degrees for the vertices in A. If $a(b+c)$ is odd, we remove an edge from the complete bipartite graph $K_{a,b}$ so that the necessary parity conditions will hold and again the construction can be finished by using Lemma 4 on both A and B.

Subcase 3.2 *Suppose that* $c > a+b-1$. To begin the construction we place K_{a+b} on the vertices $A \cup B$. Now each vertex of A needs its degree increased by $c-a-b+1$ while each vertex of B needs its degree increased

Assume that σ is graphical. Thus there exists a graph G of order $\sum_{k=1}^{n} b_k$ containing b_k vertices of degree b_k for $1 \le k \le n-2$, b_{n-1} vertices of degree b_n, and b_n vertices of degree b_{n-1}. Since the sum of the degrees of the vertices is even, $\sum_{k=1}^{n-2} b_k^2 + 2b_{n-1}b_n$ is even, which implies that $\sum_{k=1}^{n-2} b_k$ is even. Now consider when $j \ne n$. For each k $(1 \le k \le n)$, let B_k denote the set of b_k vertices of degree $\sigma(b_k)$. Let x denote the number of edges that join the vertices of B_{n-2} to the vertices of $V(G) - B_{n-2}$. Since the degree of each vertex of B_k is b_k for $1 \le k \le n-3$ and the degree of each vertex in B_{n-1} is b_n, while the degree of each vertex of B_n is b_{n-1}, we see that at most $\sum_{k=1}^{n-3} b_k^2 + 2b_{n-1}b_n$ edges leave the set $V(G) - B_{n-2}$. Thus $x \le \sum_{k=1}^{n-3} b_k^2 + 2b_{n-1}b_n$. Further, a vertex in B_{n-2} has degree at most $b_{n-2} - 1$ in $\langle B_{n-2} \rangle$. So at least one edge per vertex of B_{n-2} must leave the set B_{n-2}. Hence $x \ge b_{n-2}$. Therefore $b_{n-2} \le \sum_{k=1}^{n-3} b_k^2 + 2b_{n-1}b_n$.

Now assume that $\sum_{k=1}^{n-2} b_k$ is even and when $j \ne n$, suppose further that $b_{n-2} \le 2b_{n-1}b_n + \sum_{k=1}^{n-3} b_k^2$. We now construct a graph G with the desired properties. For each k $(1 \le k \le n)$, let B_k denote a set of b_k vertices. On the sets B_{n-1} and B_n place the complete bipartite graph K_{b_{n-1},b_n} so that each vertex of B_{n-1} has degree b_n and each vertex of B_n has degree b_{n-1}. On each set B_k $(1 \le k \le n-2)$ we place the complete graph K_{b_k}. Now in the graph $\langle \cup_{k=1}^{n-2} B_k \rangle$, each vertex needs one more edge to satisfy its degree requirement. We proceed by joining each vertex of B_1 to one vertex of B_2 (so that no vertex of B_2 gets its degree increased by more than one). Now the vertices of B_1 all have degree b_1 and exactly $b_2 - b_1$ vertices of B_2 still need their degree increased by one. Now join each of these $b_2 - b_1$ vertices of B_2 to one vertex of B_3 (so that no two vertices of B_2 get joined to the same vertex of B_3). So all the vertices of B_2 have degree b_2 and exactly $b_3 - b_2 - b_1$ vertices of B_3 need their degree increased by one. Continue in this manner until all the vertices of B_k have degree b_k for each k $(1 \le k \le n-3)$.

The remainder of the construction depends on the parity of n.

Case 1 *Assume that n is even.* Now we have $\sum_{i=1}^{n/2-1}(b_{2i} - b_{2i-1})$ vertices of B_{n-2} that need their degree increased by one. Let B'_{n-2} denote this subset of B_{n-2}. By assumption, $\sum_{k=1}^{n-2} b_k$ is even so that $\sum_{i=1}^{n/2-1}(b_{2i} - b_{2i-1})$ is even. Since the number of vertices in B'_{n-2} is even, we may form $(1/2)\sum_{i=1}^{n/2-1}(b_{2i} - b_{2i-1})$ pairs of vertices. Then for each pair u,v in B'_{n-2} we remove one edge xy from $\langle (\cup_{k=1}^{n-3} B_k) \cup B_{n-1} \cup B_n \rangle$ and add the two edges ux and vy. This process does not change the degrees of x and y but increases the degree of each u and v by one. We must now ensure that there are enough edges available to do this. We have $(1/2)\sum_{k=1}^{n-3} b_k(b_k - 1)$ edges from the complete graphs placed on the sets $B_1, B_2, \ldots, B_{n-3}$. Counting the edges between consecutive sets $B_1, B_2, \ldots, B_{n-3}$, we see that there are

$\sum_{i=1}^{n/2-2} b_{2i-1}$ edges between consecutive sets $B_1, B_2, \ldots, B_{n-4}$. Also there are $\sum_{i=1}^{n/2-2}(b_{2i} - b_{2i-1})$ edges between the sets B_{n-3} and B_{n-4} for a total of $\sum_{i=1}^{n/2-2} b_{2i}$ edges between consecutive sets $B_1, B_2, \ldots, B_{n-3}$. Finally, we have $b_{n-1}b_n$ edges from our complete bipartite graph of the sets B_{n-1} and B_n. To have enough edges to increase each pair of vertices in B'_{n-2} we must have the following inequality hold:

$$(1/2) \sum_{i=1}^{n/2-1} (b_{2i} - b_{2i-1}) \leq (1/2) \sum_{k=1}^{n-3} b_k(b_k - 1) + \sum_{i=1}^{n/2-2} b_{2i} + b_{n-1}b_n.$$

By the hypothesis, we have $b_{n-2} \leq \sum_{i=1}^{n-3} b_k^2 + 2b_{n-1}b_n$.
Thus

$$b_{n-2} \leq \sum_{k=1}^{n-3} b_k^2 + \sum_{k=1}^{n-3} b_k - \sum_{k=1}^{n-3} b_k + 2b_{n-1}b_n$$

or

$$b_{n-2} \leq \sum_{k=1}^{n-3} b_k^2 + (\sum_{i=1}^{n/2-2} b_{2i} + \sum_{i=1}^{n/2-1} b_{2i-1}) - \sum_{k=1}^{n-3} b_k + 2b_{n-1}b_n$$

so that by adding $\sum_{i=1}^{n/2-2} b_{2i} - \sum_{i=1}^{n/2-1} b_{2i-1}$ to both sides, we obtain

$$b_{n-2} + \sum_{i=1}^{n/2-2} b_{2i} - \sum_{i=1}^{n/2-1} b_{2i-1} \leq \sum_{k=1}^{n-3} b_k^2 - \sum_{k=1}^{n-3} b_k + 2 \sum_{i=1}^{n/2-2} b_{2i} + 2b_{n-1}b_n.$$

Therefore

$$\sum_{i=1}^{n/2-1} (b_{2i} - b_{2i-1}) \leq \sum_{k=1}^{n-3} b_k(b_k - 1) + 2 \sum_{i=1}^{n/2-2} b_{2i} + 2b_{n-1}b_n$$

and multiplying through by 1/2, we obtain the desired inequality.

Case 2 *Assume that n is odd.* In this case there are $b_{n-2}+\sum_{i=1}^{(n-3)/2}(b_{2i-1} - b_{2i})$ vertices of B_{n-2} that need their degree increased by one. Let B'_{n-2} denote these vertices. As in Case 1, it is clear that B'_{n-2} contains an even number of vertices. Thus we proceed as before and delete one edge xy of $\langle(\cup_{k=1}^{n-3}) \cup B_{n-1} \cup B_n\rangle$ for each pair u, v of vertices in B'_{n-2} and add the two edges xu and yv, thereby increasing the degrees of u and v each by one and leaving the degrees of all other vertices unchanged. Again, we must count the available edges and make sure there are enough to increase the degree of each vertex in B'_{n-2}. We have $(1/2)\sum_{k=1}^{n-3} b_k(b_k - 1)$ edges from the complete graphs placed on the sets $B_1, B_2, \ldots, B_{n-3}$. There are also $\sum_{i=1}^{(n-3)/2} b_{2i-1}$ edges between consecutive sets $B_1, B_2, \ldots, B_{n-3}$ and $b_{n-1}b_n$ edges from the complete bipartite graph placed on the sets B_{n-1} and B_n. Thus we must have the following inequality hold:

$$(1/2)(b_{n-2}+\sum_{i=1}^{(n-3)/2}(b_{2i-1}-b_{2i})) \leq (1/2)\sum_{k=1}^{n-3} b_k(b_k-1)+\sum_{i=1}^{(n-3)/2} b_{2i-1}+b_{n-1}b_n.$$

Using similar algebraic steps as in Case 1, we see that this inequality is equivalent to $b_{n-2} \leq 2b_{n-1}b_n + \sum_{k=1}^{n-3} b_k^2$, which is provided by the initial hypothesis and thus concludes the construction. □

In Section 1, we stated the Erdös-Gallai and Havel-Hakimi criteria for a given sequence to be graphical and in the proofs of Theorems 6 and 7 we have utilized construction methods to prove that certain permutations are graphical. We now introduce a criterion of Fulkerson, Hoffman, and McAndrew [2] (see also [6]) that determines if a given sequence is graphical.

Theorem 8 *(Fulkerson, Hoffman, McAndrew) Let $s : d_1, d_2, \ldots, d_n$ be a sequence with $d_1 \geq d_2 \geq \cdots \geq d_n \geq 1$, where $n \geq 2$. Then s is graphical if and only if for each $k = 1, 2, \ldots, n$, and m with $k + m \leq n$,*

$$\sum_{i=1}^{k} d_i \leq k(n-m-1) + \sum_{i=n-m+1}^{n} d_i.$$

Using this theorem, we now prove part 1 of Conjecture 5.

Theorem 9 *Let $S = \{a_1, a_2, \ldots, a_n\}$ be a set of integers such that $1 \leq a_1 < a_2 < \cdots < a_n$ and let π be a permutation of S such that $\pi(a_n) = a_n$. Then π is graphical if and only if $\sum_{i=1}^{n} a_i\pi(a_i)$ is even and $a_n \leq \sum_{i=1}^{n-1} a_i\pi(a_i)$.*

Proof Let $s = a_1+a_2+\cdots+a_n$ and let $\sigma = \pi^{-1}$. For a given r ($1 \leq r \leq n$), we define $d_i = a_r$ if and only if $\sum_{j=r+1}^{n} \sigma(a_j) + 1 \leq i \leq \sum_{j=r}^{n} \sigma(a_j)$. We use Theorem 8 to show that the sequence $d_1, d_2, \ldots, d_s$ is graphical. Notice that this sequence is the sequence

$$a_n, \ldots, a_n, a_{n-1}, \ldots, a_{n-1}, \ldots, a_2, \ldots, a_2, a_1, \ldots, a_1,$$

where the first a_n terms are a_n, the next $\sigma(a_{n-1})$ terms are a_{n-1}, the next $\sigma(a_{n-2})$ terms are a_{n-2}, ..., and the last $\sigma(a_1)$ terms are a_1. Thus we must show that for each $k = 1, 2, \ldots, s$ and m with $0 \le m \le s - k$, the following inequality holds:

$$\sum_{i=1}^{k} d_i \le k(s-m-1) + \sum_{i=s-m+1}^{s} d_i. \tag{1}$$

We now divide the proof into three cases.

Case 1 *Suppose that* $k = a_n$. By the hypothesis, we have $a_n \le \sum_{i=1}^{n-1} a_i \pi(a_i)$ so that

$$a_n^2 \le a_n^2 - a_n + \sum_{i=1}^{n-1} a_i \pi(a_i).$$

Since $k = a_n$, we have $\sum_{i=1}^{a_n} d_i = a_n^2$. If $m = s - a_m$, then the right hand side of (1) is

$$\begin{aligned} a_n(s-(s-a_n)-1) + \sum_{i=s-(s-a_n)+1}^{s} d_i &= a_n^2 - a_n + \sum_{i=1}^{n-1} a_i \sigma(a_i) \\ &= a_n^2 - a_n + \sum_{i=1}^{n-1} a_i \pi(a_i) \end{aligned}$$

and we see that inequality (1) holds for $k = a_n$ and $m = s - a_n$. Now let $0 \le m < s - a_n$. Then $s - a_n - m - 1 \ge 0$ and $\sum_{i=s-m+1}^{s} d_i \ge 0$ so that

$$\begin{aligned} \sum_{i=1}^{a_n} d_i = a_n^2 &\le a_n(a_n + (s - a_n - m - 1)) + \sum_{i=s-m+1}^{s} d_i \\ &= a_n(s-m-1) + \sum_{i=s-m+1}^{s} d_i \end{aligned}$$

and thus inequality (1) holds for $k = a_n$ and for every m $(0 \le m \le s - a_n - 1)$.

Case 2 *Suppose that* $k > a_n$. Then d_k must fall somewhere in the sequence $d_{a_n+1}, \ldots, d_s$. Thus there exists an integer r $(1 \le r \le n-1)$ such that $d_k = a_r$ and

$$1 + \sum_{i=r+1}^{n} \sigma(a_i) \le k \le \sum_{i=r}^{n} \sigma(a_i).$$

We define the integer j $(1 \leq j \leq \sigma(a_r))$ so that $k = \sum_{i=r+1}^{n} \sigma(a_i) + j$. Now we see that

$$\sum_{i=1}^{k} d_i = a_n^2 + a_{n-1}\sigma(a_{n-1}) + \cdots + a_{r+1}\sigma(a_{r+1}) + ja_r = \sum_{i=r+1}^{n} a_i\sigma(a_i) + ja_n.$$

Let m be an integer with $0 \leq m \leq s - k$, or equivalently, we can say $k \leq s - m \leq s$. Since $k > a_n$ we see that $k \geq a_n + 1$ so that $a_n + 1 \leq s - m$, or $a_n \leq s - m - 1$. Observe that the right hand side of (1) is

$$\begin{aligned} k(s - m - 1) + \sum_{i=s-m+1}^{s} d_i \geq (\sum_{i=r+1}^{n} \sigma(a_i) + j)a_n &= \sum_{i=r+1}^{n} a_n\sigma(a_i) + ja_n \\ &\geq \sum_{i=r+1}^{n} a_i\sigma(a_i) + ja_r, \end{aligned}$$

and thus inequality (1) holds for $k > a_n$ and for every m $(0 \leq m \leq s - k)$.
Case 3 *Suppose that* $k < a_n$. Observe that $\sum_{i=1}^{k} d_i = ka_n$. First, let m be a nonnegative integer such that $m \leq s - a_n - 1 = a_1 + a_2 + \cdots + a_{n-1} - 1$. Notice that $\sum_{i=s-m+1}^{s} d_i \geq 0$ and by assumption $a_n \leq s - m - 1$ so that $ka_n \leq k(s - m - 1) + \sum_{i=s-m+1}^{s} d_i$. Thus inequality (1) holds for every m with $0 \leq m \leq a_1 + a_2 + \ldots a_{n-1} - 1$. Next let $a_1 + a_2 + \cdots + a_{n-1} \leq m \leq s - k$. We define the integer j $(0 \leq j \leq a_n - k)$ such that $m = a_1 + a_2 + \cdots + a_{n-1} + j$. Notice that since $k < a_n$ we have $kj \leq a_nj$ and by assumption $a_n \leq \sum_{i=1}^{n} a_i\pi(a_i)$ so that $k \leq \sum_{i=1}^{n} a_i\pi(a_i)$. Using these facts, we observe that $kj + k \leq a_nj + \sum_{i=1}^{n} a_i\pi(a_i)$ and by adding $ka_n - (kj + k)$ to both sides of this inequality, we have

$$ka_n \leq k(a_n - j - 1) + a_nj + \sum_{i=1}^{n-1} a_i\pi(a_i).$$

Noticing that $a_n - j - 1 = s - m - 1$ and $a_nj + \sum_{i=1}^{n-1} a_i\pi(a_i) = \sum_{i=s-m+1}^{s} d_i$, we see that inequality (1) holds for $k < a_n$ and for every m $(a_1 + a_2 + \cdots + a_{n-1} \leq m \leq s - k)$ completing the proof. □

We conclude with a revision of part 2 of Conjecture 5.

Conjecture 10 *Let* $S = \{a_1, a_2, \ldots, a_n\}$ *be a set of integers with* $1 \leq a_1 < a_2 < \cdots < a_n$ $(n \geq 2)$ *and let* π *be a permutation on* S *with* $\pi(a_n) \neq a_n$ *and* $\sum_{i=1}^{n} a_i\pi(a_i)$ *even. Let* m *be the smallest integer from* $\{1, 2, \ldots, n\}$ *such that*

$$a_m\pi(a_m) = max\{a_i\pi(a_i)|1 \leq i \leq n\}.$$

Let $M = \{i|\pi(a_i) > a_i\}$ *and* $k = |M|$. *Let* $i_1 = m$ *and for each* j *with* $2 \leq j \leq k$ *let* i_j *be defined so that*

$$a_{i_1}\pi(a_{i_1}) \geq a_{i_2}\pi(a_{i_2}) \geq \cdots \geq a_{i_k}\pi(a_{i_k}).$$

For each r *(*$1 \leq r \leq k$*), let* $S_r = \sum_{j=1}^{r} a_{i_j}\pi(a_{i_j})$.
Then

1. *if* $a_m < \pi(a_m)$, *then* π *is graphical if and only if for each* r *(*$1 \leq r \leq k$*),*

$$S_r \leq \sum_{\pi(a_i)\leq S_r} a_i\pi(a_i) + \sum_{\pi(a_i)>S_r} a_i S_r + S_r(S_r - 1)$$

and

2. *if* $a_m \geq \pi(a_m)$, *then* π *is graphical.*

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