King Graph Ramsey Numbers

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ABSTRACT. A king graph KG_n has n^2 vertices corresponding to the n^2 squares of an $n \times n$-chessboard. From one square (vertex) there are edges to all squares (vertices) being attacked by a king. For given graphs G and H the Ramsey number $r(G, H)$ is the smallest n such that any 2-coloring of the edges of KG_n contains G in the first or H in the second color. Results on existence and nonexistence of $r(G, H)$ and some exact values are presented.

1. Introduction

The king graph KG_n has the squares of an $n \times n$ chessboard as its vertices and an edge between two squares if a king on one square attacks the other square, that is, between two squares having at least one point in common. In other words, a square on n^2 lattice points of the unit square lattice are the vertices of KG_n and the sides and the diagonals of the unit squares are the edges of KG_n (Figure 1).

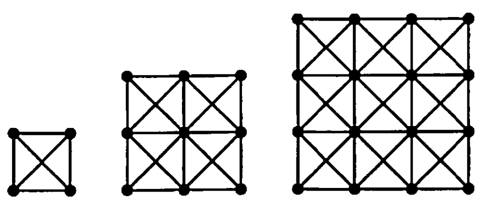


Figure 1. KG_n for $n = 2, 3, 4$.

The king graph Ramsey number $r(G, H)$ is defined as the minimum n such that every 2-coloring of the edges of KG_n contains the given graph G of the first color (green) or the given graph H of the second color (red). The existence of $r(G, H)$ is possible only for graphs G and H being subgraphs of some KG_n. We restrict ourselves to connected graphs G and H. The path P_2 with two vertices may be excluded since $r(P_2, H)$ is the smallest order of KG_n containing H.

We have $r(G, H) = r(H, G)$ (symmetry) and $r(G', H') \leq r(G, H)$ if $G' \subseteq G$ and $H' \subseteq H$ (monotonicity) in general.

Instead of the sequence of the complete graphs K_n used as host graphs in the case of the classical Ramsey numbers [10], here the sequence of king graphs is used for $r(G,H)$. Other sequences of host graphs for corresponding variations of the classical Ramsey numbers are discussed in [1–9].

2. Existence

For king graph Ramsey numbers $r(G,H)$ the existence is not guaranteed for all pairs of graphs G and H as it is the case for the classical Ramsey numbers. We will use $r(G,H) = \infty$ if $r(G,H)$ does not exist.

Theorem 1. If G is non-bipartite and H is not a path then $r(G,H) = \infty$.

Proof. The 2-coloring in Figure 2 can be used for every KG_n since the green subgraph (thin edges) is bipartite and the red subgraph consists of distinct paths. □

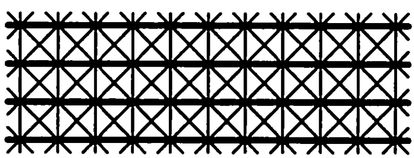


Figure 2. Bipartite graphs and paths in a 2-coloring.

Theorem 2. For cycle graphs C_s, C_t we have $r(C_s, C_t) = \infty$.

Proof. If s or t is odd then Theorem 1 can be used. For s and t even, $s \leq t$, we distinguish the cases $s = 4$ for $t = 4, 6, 8, 10$, and $t \geq 12$, $s = 6$ for $t = 6$ and $t \geq 8$, and $s, t \geq 8$. In Figures 3 to 10 we present 2-colorings which can be used for KG_n. The colorings neither contain a green C_s nor

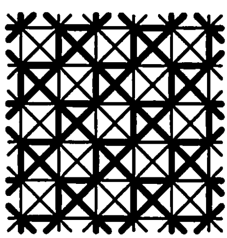


Figure 3. $r(C_4, C_4) = \infty$.

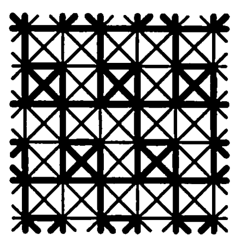


Figure 4. $r(C_4, C_6) = \infty$.

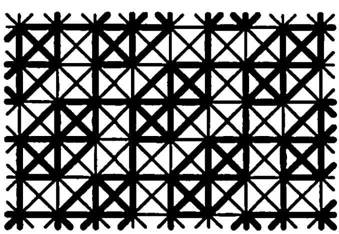


Figure 5. $r(C_4, C_8) = \infty$.

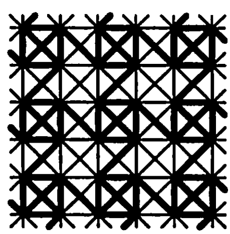


Figure 6. $r(C_6, C_6) = \infty$.

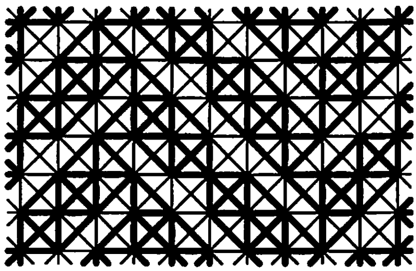


Figure 7. $r(C_4, C_{10}) = \infty$.

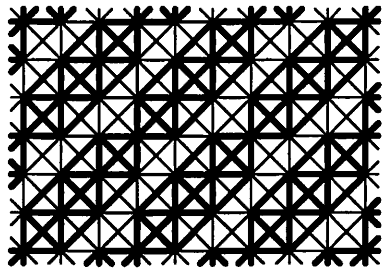


Figure 8. $r(C_4, C_{2i}) = \infty$ $(i \geq 6)$.

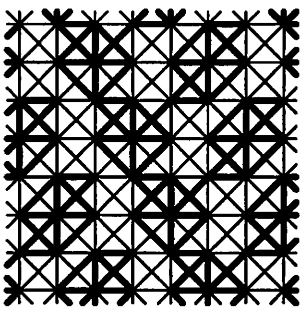


Figure 9. $r(C_6, C_{2i}) = \infty$ $(i \geq 4)$.

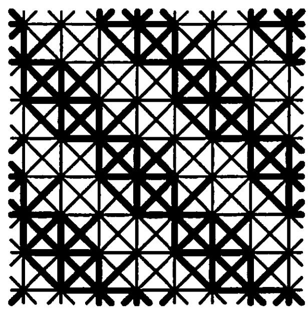


Figure 10. $r(C_{2i}, C_{2j}) = \infty$ $(i, j \geq 4)$.

a red C_t since all cycles surrounding edges having the same color only are too small and all cycles surrounding at least one edge of the other color are too large. □

Corollary 1. The existence of $r(G, H)$ is possible only in the following three distinct cases where

(1) one of G and H is non-bipartite and the other one is a path,

(2) one of G and H is bipartite but not a tree and the other one is a tree,

(3) G and H both are trees.

Proof. If G is non-bipartite then by symmetry and Theorem 1 it is necessary for the existence of $r(G, H)$ that H is a path. This corresponds to case (1) and it remains that G and H both are bipartite. If G is bipartite but not a tree (case (2)) then G contains a cycle and H is a tree by monotonicity and Theorem 2. It remains case (3) where G and H are trees. □

We now consider some special cases of $r(G, H)$.

Theorem 3. For stars $K_{1,s}$ and $K_{1,t}$, we have

$$r(K_{1,s}, K_{1,t}) = \begin{cases} 3 & \text{if } s + t \leq 9,\ s \geq 3,\ t \geq 2, \\ \infty & \text{if } s + t \geq 10. \end{cases}$$

Proof. Red diagonals and green sides in KG_2 prove $r(K_{1,s}, K_{1,t}) \geq 3$. By the pigeon-hole principle the central vertex of KG_3 is incident to s green or t red edges if $s + t \leq 9$ so that $r(K_{1,s}, K_{1,t}) \leq 3$.

The edges of KG_n can be partitioned into 8 classes. For each of the four directions we use 2 classes such that adjacent edges belong to different classes. Then the edges of $s - 1$ classes are colored in green and the edges of the $8 - (s - 1) \leq t - 1$ remaining classes in red proving $r(K_{1,s}, K_{1,t}) = \infty$ for $s + t \geq 10$. □

Theorem 4. The values of $r(K_{1,s}, H)$, $s = 6, 7$, and 8 are as in Table 1.

	P_3	P_4	$K_{1,3}$	otherwise
$K_{1,6}$	3	5	3	∞
$K_{1,7}$	3	∞	∞	∞
$K_{1,8}$	∞	∞	∞	∞

Table 1. $r(K_{1,s}, H)$ for $s = 6, 7$, and 8.

Proof. Since $P_3 = K_{1,2}$, the first and third column of Table 1 follow from Theorem 3. In the second column we obtain $r(K_{1,6}, P_4) \geq 5$ from the 2-coloring of KG_4 with one red edge from each border vertex to the geometrically nearest inner vertex and green edges otherwise.

In a KG_5 without a green $K_{1,6}$ and without a red P_4 the central vertex has to be incident to three red edges. The edges incident to the other vertices of these three red edges complete a red P_4 or a green $K_{1,6}$. This contradiction proves $r(K_{1,6}, P_4) \leq 5$.

A 2-coloring of the infinite king graph where every vertex is a vertex of a red triangle and all other edges are green proves $r(K_{1,7}, P_4) = \infty$ and thus $r(K_{1,8}, P_4) = \infty$ by monotonicity.

For the fourth column we color the infinite king graph such that every vertex belongs to a red K_4 and all other edges are green. This proves $r(K_{1,6}, H) = \infty$ if $H \not\subseteq K_4$. For non-bipartite graphs H, Theorem 1 implies $r(K_{1,6}, H) = \infty$. It remains $H = C_4$. However, the 2-coloring in Figure 3 proves $r(K_{1,6}, C_4) = \infty$. The remaining entries in Table 1 follow by monotonicity. □

The existence of $r(K_{1,5}, H)$ may be possible for more graphs H than in Table 1, however, the number of vertices of H must be at most 12.

Theorem 5. If H is not a subtree of the graph in Figure 11 then

$$r(K_{1,5}, H) = \infty.$$

Proof. The 2-coloring in Figure 12 does not contain a green $K_{1,5}$ and the components of the red subgraph are as in Figure 11. To see that H cannot contain cycles, Theorem 1 can be used for odd cycles. The 2-coloring in Figure 3 proves $r(K_{1,5}, C_4) = \infty$. For $t \geq 3$ we obtain $r(K_{1,5}, C_{2t}) = \infty$

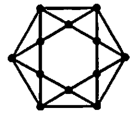


Figure 11. Red component of the 2-coloring in Figure 12.

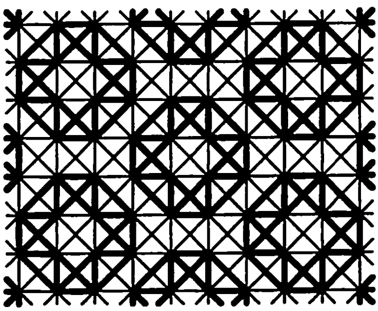


Figure 12. $r(K_{1,5}, H) = \infty$ for H not being a subgraph of the graph in Figure 11.

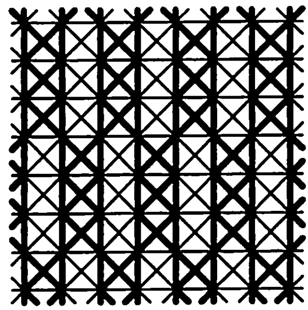


Figure 13. $r(K_{1,5}, C_{2t}) = \infty$ for $t = 4$.

from 2-colorings as in Figure 13 for $t = 4$ where the red subgraph consists of parallel lines and diagonals in blocks of $t - 1$ consecutive squares. □

Theorem 6. If $s \leq 4$ then $r(K_{1,s}, H)$ exists for infinitely many graphs H.

Proof. To see the existence of $r(K_{1,s}, P_t)$ we start at a central vertex of KG_n without a green $K_{1,s}$. Since $s \leq 4$ there exists a sequence of red edges each going either upwards or horizontally to the right. This guarantees a red P_t if n is sufficiently large. □

The existence of $r(G, H)$ for graphs with vertices of degrees at most 4 is restricted furthermore by the following two theorems.

Theorem 7. $r(K_4, H) = \infty$.

Proof. The 2-coloring in Figure 14 neither contains a green K_4 nor a connected red $H \neq P_2$. □

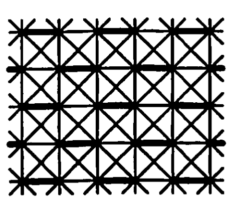


Figure 14. $r(K_4, H) = \infty$.

Theorem 8.
$$r(K_{2,3},H)=\begin{cases}3 & \text{if } H=P_3,\\ 5 & \text{if } H=P_4,\\ \infty & \text{otherwise.}\end{cases}$$

Proof. It may be checked that $K_{2,3}$ can occur in KG_n as in one of the two possibilities of Figure 15 only. In both cases diagonal edges in both direc-

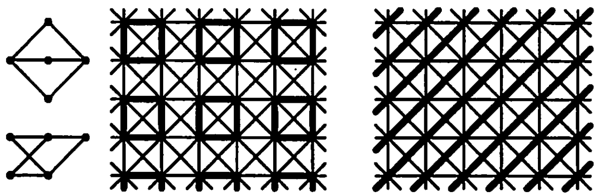


Figure 15. $r(K_{2,3},H)=\infty$ for $H \not\subseteq P_4$.

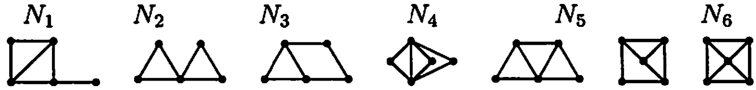
tions and two consecutive edges of squares in one direction are used. Thus there is no green $K_{2,3}$ in both 2-colorings of Figure 15. One 2-coloring contains red cycles C_4 only and the other one red paths only so that $r(K_{2,3},H)$ can exist for $H=P_3$ and $H=P_4$ only. The proofs of $r(K_{2,3},P_3)=3$ and $r(K_{2,3},P_4)\geq 5$ are straightforward and $r(K_{2,3},P_4)\leq 5$ has been checked by computer. □

3. Small graphs

For the three cases of Corollary 1, in Table 2 to 4 we gather exact values of $r(G,H)$ for all graphs with up to 5 or 6 vertices. We may remark that the entries ∞ being underlined follow by monotonicity. For all the values not being covered by the preceding theorems appropriate 2-colorings for the lower bounds are easy to find and the proofs for the upper bounds are straightforward or found by computer.

	C_3	N_1	N_2	C_5	N_3	N_4	N_5	N_6	N_7	N_8	N_9	N_{10}	N_{11}	N_{12}	N_{13}
P_3	3	3	3	3	3	3	3	3	3	3	3	3	3	4	∞
P_4	4	4	4	4	4	4	4	4	4	5	4	8	6	7	$\underline{\infty}$
P_5	5	5	5	5	5	5	5	5	5	∞	5	$\underline{\infty}$	$\underline{\infty}$	$\underline{\infty}$	$\underline{\infty}$
P_6	6	6													

N_1 N_2 N_3 N_4 N_5 N_6



N_7 N_8 N_9 N_{10} N_{11} N_{12} N_{13}

Table 2. Paths versus non-bipartite graphs.

	P_3	P_4	$K_{1,3}$	P_5	T_1	$K_{1,4}$	P_6	T_2	T_3	T_4	T_5	$K_{1,5}$
C_4	2	4	3	4	4	5	4	4	4	5	5	∞
B_1	3	4	3	4	4	5	4	4	4	5	5	$\underline{\infty}$
B_2	3	5	∞	∞	$\underline{\infty}$	$\underline{\infty}$	$\underline{\infty}$	$\underline{\infty}$	$\underline{\infty}$	$\underline{\infty}$	$\underline{\infty}$	$\underline{\infty}$
B_3	3	4	5	5	5	6						
B_4	3	4	4	4	4	5						
B_5	3	4	5	6								

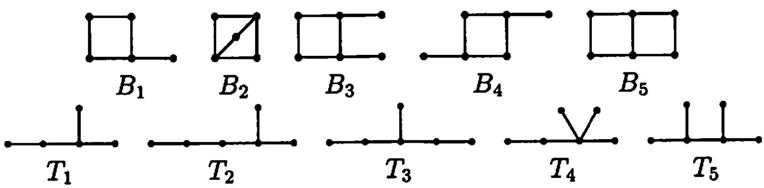


Table 3. Bipartite graphs not being trees versus trees.

	P_3	P_4	$K_{1,3}$	P_5	T_1	$K_{1,4}$	P_6	T_2	T_3	T_4	T_5	$K_{1,5}$
P_3	2	2	3	3	3	3	3	3	3	3	3	3
P_4		3	3	3	3	3	3	3	3	3	3	4
$K_{1,3}$			3	3	3	3	3	3	3	3	3	3
P_5				3	3	4	3	3	3	4	3	5
T_1					3	3	3	3	3	3	3	4
$K_{1,4}$						3	5	4	4	3	5	3
P_6							3	3	3	4	4	5
T_2								3	3	4	4	5
T_3									3	4	4	5
T_4										3	5	4
T_5											4	5
$K_{1,5}$												∞

Table 4. Trees versus trees.

4. Cycles and paths

For paths versus paths the existence is guaranteed.

Theorem 9. $r(P_s, P_t) < \infty$.

Proof. Since the graphs of the triangle gameboards are subgraphs of the king graphs, the existence of $r(P_s, P_t)$ follows from Theorem 3 in [5]. □

Theorem 10. For $s \leq 5$ the numbers $r(C_s, P_t)$ do exist.

Proof. If the red subgraph for a sufficiently large KG_n is connected then a red P_t exists. Otherwise, there are red components separated by curves intersecting green edges only. If there is a branching of these curves, that is, the four vertices of a unit square belong to three different red components, then this square contains a green C_3 and a green C_4. Moreover, the green edges of the neighboring squares force a green C_5. If there is no branching of the separating curves then at least one red component connects vertices of opposite sides of KG_n and thus contains a red P_t. □

Theorem 11. The number $r(C_s, P_t)$ does not exist for

(1) $s = 6$, $t \geq 25$, (2) $s = 7$, $t \geq 9$, (3) $s = 8$, $t \geq 13$,

(4) $s = 4i$, $4i + 1$, $4i + 2$, $4i + 3$, $t \geq 4i + 1$, $i \geq 2$, $s \neq 8$.

Proof. Consider 2-colorings as in Figure 16 for $i = 2$ and 3 with red components having $4i$ vertices. Since there are only green cycles C_3, C_4,

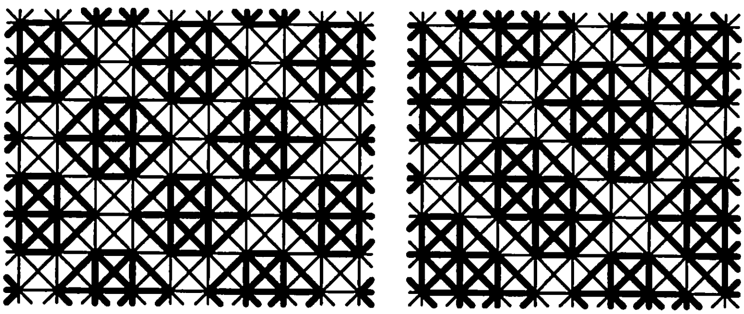


Figure 16. $r(C_s, P_t) = \infty$ for $i = 2$ and 3 in cases (2) and (4) of Theorem 11.

C_5, C_6, C_8, and C_s for $s \geq 4i + 4$, cases (2) and (4) are proved.

The coloring in Figure 17 does not contain a green C_8 and the red components have 12 vertices only. This proves case (3).

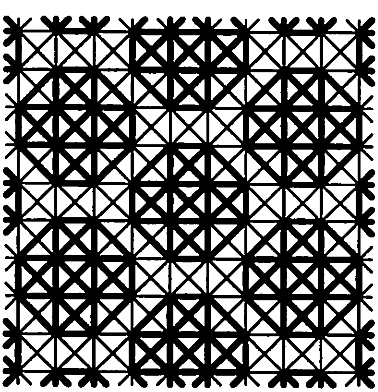


Figure 17. $r(C_8, P_t) = \infty$ for $t \geq 13$.

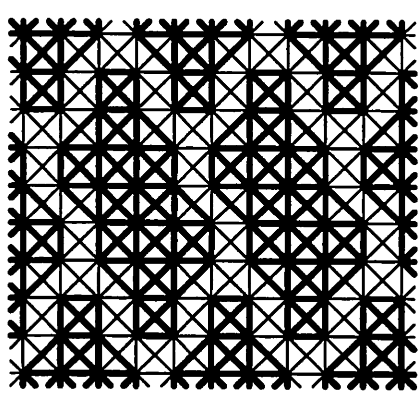


Figure 18. $r(C_6, P_t) = \infty$ for $t \geq 25$.

Case (1) follows from the 2-coloring in Figure 18. There is no green C_6 and the red components have 24 vertices only. □

Thus for fixed $s \geq 6$ the king graph Ramsey numbers $r(C_s, H)$ exist for finitely many graphs H only.

References

[1] L. W. Beineke and A. J. Schwenk: On a bipartite form of the Ramsey problem. Congr. Numer. **15** (1976), 17–22.

[2] A. Benczúr, H. Harborth, and L. K. Jørgensen: Cube graph Ramsey numbers. Abh. Braunschweig. Wiss. Ges. **47** (1996), 151–157.

[3] J.-P. Bode, H. Harborth, and S. Krause: Chessboard Ramsey Numbers. In: A. Bezdek (ed.): Discrete Geometry. In Honor of W. Kuperbergs's 60th Birthday. Marcel Dekker, 2003, 79–84.

[4] J.-P. Bode and H. Harborth: Gameboard Ramsey numbers. (submitted)

[5] J.-P. Bode and H. Harborth: Triangle and hexagon gameboard Ramsey numbers. Congr. Numer. (to appear)

[6] D. Day, W. Goddard, M. A. Henning, and H. C. Swart: Multipartite Ramsey numbers. Ars Combin. **58** (2001), 23–31.

[7] F. Harary, H. Harborth, and I. Mengersen: Generalized Ramsey theory of graphs. XII. Bipartite Ramsey sets. Glasgow Math. J. **22** (1981), 31–41.

[8] H. Harborth and I. Mengersen: Ramsey numbers in octahedron graphs. Discrete Math. **231** (2001), 241–246.

[9] J. H. Hattingh and M. A. Henning: Bipartite Ramsey theory. Utilitas Math. **53** (1998), 255–258.

[10] S. Radziszowski: Small Ramsey numbers. Electronic Journal of Combinatorics (Dynamic surveys), `www.combinatorics.org`