The actions of fractional automorphisms

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Abstract

A fractional automorphism of a graph is a doubly stochastic matrix which commutes with the adjacency matrix of the graph. If we apply an ordinary automorphism to a set of vertices with a particular property, such as being independent or dominating, the resulting set retains that property. We examine the circumstances under which fractional automorphisms preserve the fractional properties of functions on the vertex set.

1 Introduction

In [6] a fractional isomorphism between two graphs with adjacency matrices A, B is defined to be a doubly stochastic matrix S with the property that $AS = SB$. This definition is found by generalising the view of ordinary graph isomorphisms as permutation matrices. It is natural to consider the case when $A = B$; any doubly-stochastic matrix S such that $SA = AS$ can be considered a *fractional automorphism.* It is understood that a matrix has the property of being a fractional automorphism (or isomorphism) subject to a certain ordering of the vertices, imposed by the ordering used in the adjacency matrix.

Fractional automorphisms have been studied, though not under that name, by Tinhofer in [4, 5] and Godsil in [2]. It is obvious that the set of all fractional automorphisms of a graph with adjacency matrix A, which we shall denote by $S(A)$, contains the convex hull of the set of automorphisms taken as permutation matrices; a graph is called *compact* if these two sets

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are in fact equal. While several classes of graphs are known to be compact, as yet no good characterisation of compact graphs has been found.

It is shown in [2] that every fractional automorphism of a graph G determines a nontrivial equitable partition of the vertex set. A partition π is *equitable* if for any cell $C \in \pi$, the induced subgraph $V[C]$ is regular and, for any pair of distinct cells $C_1, C_2 \in \pi$ the induced bipartite graph between C_1 and C_2 is biregular.

One view of fractionalising graph parameters is that one takes an integer linear program and considers the linear relaxation. If G is a graph with adjacency matrix A and incidence matrix B, for example, then a *fractional dominating vector* is a $[0,1]$-vector which satisfies the matrix inequality $(A+I)x \geq 1$; note that if we restrict the values of the components of x to 0 and 1, then x is the characteristic vector of a dominating set of G. Similarly, a *fractional independent vector* is a vector x satisfying $B^T x \leq 1$, and a *fractional covering vector* is a vector x which satisfies $B^T x \geq 1$. A good introduction to fractional graph theory is [7].

In [3], the authors asked the question: suppose that x is a fractional dominating, independent, or covering vector, and let S be a fractional automorphism. Is Sx necessarily dominating, independent, or covering, respectively? It was shown that, if G is a compact graph, then all fractional automorphisms preserve the properties of being fractionally dominating, independent, or covering. (The authors used the term "automorphism-closed" to mean "compact".) An extension of this result, also proved, is that if G is regular and $S(A)$ is precisely the convex hull of the automorphisms together with the constant stochastic matrix, then all fractional automorphisms of G preserve the properties under discussion.

2 Fractional domination is preserved

It turns out that the question of when a fractional automorphism's action on a fractional dominating vector results in a fractional dominating vector is easily answered, and the answer is "always".

Theorem 1 *Let S be a fractional automorphism of a graph G with adjacency matrix A, and let x be a fractional dominating vector. Then Sx is also fractional dominating.*

Proof. By the definition of fractional domination, we know that $(A+I)x \geq 1$. So:

$$\begin{aligned}(A+I)(Sx) &= ((A+I)S)x \\ &= (AS+IS)x \\ &= (SA+SI)x \\ &= S((A+I)x) \\ &\geq S \cdot 1 \\ &= 1 \text{ since } S \text{ is doubly-stochastic}\end{aligned}$$

□

Note that a similar result holds for fractional closed neighbourhood packing, which is the LP dual to fractional domination.

3 Fractional independence and another look at automorphisms

Our result on fractional domination hinges on the fact that the adjacency matrix A is used to define both the fractional dominating vector and a fractional automorphism. We have no such coincidence for fractional independent or covering vectors, however. Note that a vector x is fractional independent if and only if $1 - x$ is fractional covering, where 1 here stands for the vector of all 1's, as it does above, so in considering the action of doubly stochastic matrices on fractional independent or covering vectors, it suffices to consider only one or the other.

This underscores a deeper concern about our definition of fractional automorphisms, namely the actions of such an operator on the edge set. We know that an ordinary automorphism can be defined as a permutation of the vertex set, and that this induces a permutation on the edge set. There does not, however, seem to be an obvious analogue of this induced permutation for fractional automorphisms.

We are thereby led to consider an alternative way of characterising automorphisms. Rather than being a single permutation P of the vertices which preserves adjacency, we can instead view an automorphism as a pair (P, Q) of permutations, on the vertices and edges respectively, which jointly preserve incidence: that is, for some vertex-edge incidence matrix B, $PB = BQ$.

We fractionalise this as follows. An *edge-based fractional automorphism* of a graph with incidence matrix B is a pair (S, T) of doubly-stochastic matrices such that $SB = BT$. We can think of S as defining the action on the vertices, and T the action on the edges.

A pair (S, T) is an edge-based fractional automorphism with respect to orderings of the vertices and edges of the graph. When we are speaking of sets of edge-based fractional automorphisms, it will be understood that these orderings are fixed. Note that, with respect to any such fixed orderings, the set of edge-based fractional automorphisms of G is a convex set in the Cartesian product of the $n \times n$ with the $m \times m$ real matrices, where n is the number of vertices of G and m is the number of edges.

Theorem 2 *Let G be a graph with incidence matrix B, and let the pair (S^T, T) is an edge-based fractional automorphism of G. Then if x is a fractional independent vector, then so is Sx.*

Proof. The definition of fractional independence assures us that $B^T x \leq 1$. Thus:

$$\begin{aligned} B^T(Sx) &= (B^T S)x \\ &= (T^T B^T)x \\ &= T^T(B^T)x \\ &\leq T^T \cdot 1 \\ &= 1 \end{aligned}$$

□

Corollary 1 *Let S be a fractional automorphism of a graph G. Then if x is a fractional independent vector and T a matrix such that (S^T, T) is an edge-based fractional automorphism, then Sx is a fractional independent vector.*

This partial result begs a further question: let S be a fractional automorphism. Under what circumstances does there exist a doubly-stochastic matrix T such that the pair (S, T) forms an edge-based fractional automorphism? The converse question is also interesting: let (S, T) be an edge-based fractional automorphism. Is it necessarily the case that S by itself is a fractional automorphism? We have achieved partial answers to both of these questions.

Godsil ([2]) showed that each equitable partition of a graph determines a unique idempotent fractional automorphism of the graph. Suppose that the cells of an equitable partition are $C_1, ..., C_k$, with cardinalities $c_1, ..., c_k$. The idempotent fractional automorphism associated with the partition is the $n \times n$ matrix with the reciprocal of c_i in each entry with both row and column index associated with vertices in $C_i, i = 1, \ldots, k$, and zeroes at any position not in a $C_i \times C_i$ submatrix. (Although the fixed ordering of the vertices with respect to which we have fractional automorphisms may not be such, it is helpful to imagine that the ordering of the vertices has those of C_1 first, and then those of C_2, etc. Then the matrix described is a block diagonal matrix with constant entries in each diagonal block: the only entries possible that would make the resulting matrix doubly stochastic.) Let us denote by $\mathcal{F}_G$ the set of all such idempotent fractional automorphisms of the graph G.

Proposition 1 *Let S be a fractional automorphism of G. If S is in the convex hull of the union of $\mathcal{F}_G$ and the automorphisms of G, then there exists a doubly-stochastic $m \times m$ matrix T such that the pair (S,T) is an edge-based fractional automorphism.*

Proof. By the convexity of the set of edge-based fractional automorphism pairs, remarked on above, it suffices to verify the claim for each member of $\mathcal{F}_G$ and for each automorphism of G. For automorphisms of G, this verification has already been noted.

Suppose that $S \in \mathcal{F}_G$; let $C_1, \ldots, C_k$ be the cells of the underlying equitable partition, with cardinalities $c_1, \ldots, c_k$. Without loss of generality, since we are dealing with only one fractional automorphism, let us suppose that the fixed ordering of the vertices is such that S is a block diagonal matrix.

We shall partition the edges of G into sets $E_{i,j}, i \leq j$, with cardinalities $e_{i,j}$, consisting of the edges with one end in C_i and the other in C_j. Let the edges then be ordered so that the edges in $E_{1,1}$ come first, then $E_{1,2}, \ldots, E_{1,k}$, then $E_{2,2}, \ldots, E_{2,k}$, etc. This is without loss of generality; if there is already some ordering of the edges fixed, then we can do everything that follows with reference to columns indexed by the edges in the $E_{i,j}$, in the resulting vertex-edge incidence matrix, but it will be much easier for the reader to follow if we have the edges supposedly ordered as described.

Let B be the vertex-edge incidence matrix of G with respect to the orderings supposed of the vertices and edges of G, and let T be the $m \times m$ block-diagonal matrix with blocks indexed by $E_{i,j} \times E_{i,j}$ down its main diagonal, with each such block having constant entry equal to the reciprocal of $e_{i,j}$. Clearly T is doubly stochastic; it remains to be shown that $SB = BT$.

By direct calculation the $C_i \times E_{i,i}$ block of SB has constant entry $\frac{2}{c_i}$; for $i < j$ the $C_i \times E_{i,j}$ block of SB has constant entry $\frac{1}{c_1}$; for $i < j$ the $C_i \times E_{i,j}$ block of SB has constant entry $\frac{1}{c_j}$; and all other entries of SB are zero.

Let $a_{i,j}$ be the number of neighbors in C_j that each vertex in C_i has; this is well-defined by the equitability of the underlying partition. By direct calculation, the $C_i \times E_{i,i}$ block of BT has constant entry $\frac{a_{i,i}}{e_{i,i}}$. Since $a_{i,i} = c_i - 1$ and $e_{i,i} = \binom{c_i}{2}$, it is clear that $\frac{a_{i,i}}{e_{i,i}} = \frac{2}{c_i}$. A similar counting argument will show that the off-diagonal elements of BT match up with those of SB. □

Proposition 2 *Let (S, T) be an edge-based fractional automorphism of G. If S and T are symmetric, and G is regular, then S is a fractional automorphism of G.*

Proof. Recall that, if G is k-regular with adjacency and incidence matrices A and B, respectively, then $BB^T = A + kI$. Consider, then:

$$\begin{aligned} S(A + kI) &= S(BB^T) \\ &= BTB^T \\ &= BB^T S, \text{ by symmetry} \\ &= (A + kI)S \end{aligned}$$

Since $(kI)S = S(kI) = kS$, this implies that $SA = AS$. □

4 Fractional automorphisms of general incidence structures

One added advantage of edge-based fractional automorphisms is that they give us a way to fractionalise the idea of an automorphism on an incidence structure such as a finite geometry or a combinatorial design. Such structures have no useful notion of "adjacency" as graphs do, and so the more usual formulation of a fractional graph automorphism will not carry over.

Let us be concrete: an incidence structure consists of an ordered pair (P, B), where the set B (the "blocks") is composed of subsets of P (the "points"). Every incidence structure has an incidence matrix M, with rows indexed by P and columns by B, where

$$M_{i,j} = \begin{cases} 1 & \text{if point } i \text{ is contained in block } j, \\ 0 & \text{otherwise} \end{cases}$$

Hence, we can define a fractional automorphism of an incidence structure to be a pait (S, T) of doubly-stochastic matrices such that $SM = MT$. We can also discuss fractional isomorphisms: Two incidence structures with matrices M, N are fractionally isomorphic when there exist a pair of doubly-stochastic matrices (S, T) such that $SM = NT$.

One well-studied class of incidence structures is that of BIBDs: a balanced incomplete block design with parameters (v, k, λ) is an incidence structure with the following properties: $|P| = v$; if $z \in B, |z| = k$; and if $p, q \in P$, then there are precisely λ elements of B containing both p and q. This extreme regularity allows us to determine precisely which BIBDs are fractionally isomorphic.

Theorem 3 *Two BIBDs are fractionally isomorphic if and only if they have the same parameters.*

Proof. It is common to denote by b the number of blocks $|B|$ in a BIBD, and by r the number of blocks which contain a given point. We note first that, for two designs to be fractionally isomorphic they must share both v and b; otherwise their incidence matrices will be of different sizes. (b and r are determined if one has fixed v, k, and λ; the relationships between these parameters are that $r(k-1) = \lambda(v-1)$ and $bk = vr$. A proof of these equations can be found in Chapter 1 of [1].)

Suppose that M and N are the incidence matrices of designs with identical values for v and b, and with k-values k_M and k_N, and r-values r_M and r_N. Let S and T be doubly-stochastic matrices such that $SM = NT$.

It is easy to see that the row- and column-sums of the matrix M are r_M and k_M, respectively; a similar remark holds for the matrix N. Thus, the matrix SM will also have column-sums equal to k_M, by the doubly-stochastic nature of S. Similarly, the matrix NT will have row-sums equal to r_N. The sums of the entries of SM and NT are thus bk_M and vr_N, respectively. But since the two matrices are equal, we have that $bk_M = vr_N$. However, we know that $bk_M = vr_M$; and hence $r_M = r_N$ and $k_M = k_N$. Since we can express λ in terms of v, k, r, the parameters of the two designs must therefore be equal.

So let M and N be the incidence matrices of designs with identical parameters. Let $S = \frac{1}{v}J_v$ and $T = \frac{1}{b}J_b$, where J_n indicates the $n \times n$ matrix with all entries 1; it is easy to check that $SM = NT$. □

Perhaps more interesting is the case of pairwise-balanced designs (PBDs), a generalisation of BIBDs where there is no fixed block size. One might conjecture that two PBDs must share the same profile of block sizes - that is, have the same multiset of block sizes – for them to be fractionally isomorphic; it is unclear whether this would be sufficient, however.

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