Limit Theorems for Associated Whitney Numbers of Dowling Lattices

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Abstract

The Whitney number $W_m(n,k)$ of the rank-n Dowling lattice $Q_n(G)$, based on the group G having order m, is the number of elements in $Q_n(G)$ of co-rank k. The associated numbers $U_m(n,k) = k!\,W_m(n,k)$ and $V_m(n,k) = k!\,m^k W_m(n,k)$ were studied by M. Benoumhani [*Adv. in Appl. Math.* **19** (1997), no. 1, 106–116] where a generating function was derived using algebraic techniques and logconcavity was shown for $\{U_m(n,k)\}$ and for $\{V_m(n,k)\}$. We give a central limit theorem and a local limit theorem on $\mathbb{R}$ for $\{U_m(n,k)\}$ and for $\{V_m(n,k)\}$. In addition, asymptotic formulas for $\max_k U_m(n,k)$, $\max_k V_m(n,k)$ and their modes are given.

Keywords: Central limit theorem; Local limit theorem; Asymptotic formulas

1. Introduction

An array $\{a(n,k) : n \geq 0,\ 0 \leq k \leq D(n)\}$ of nonnegative real numbers, with $P_n = \sum_{k=0}^{D(n)} a(n,k) \neq 0$, satisfies a *central limit theorem* (is *asymptotically normal*) *with mean* μ_n *and variance* σ_n^2 if

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}} \left| \sum_{0\leq k\leq \mu_n + x\sigma_n} \frac{a(n,k)}{P_n} - (2\pi)^{-1/2} \int_{-\infty}^{x} e^{-t^2/2}\, dt \right| = 0 . \quad (1)$$

The array satisfies a *local limit theorem on* $S \subseteq \mathbb{R}$ *with mean* μ_n *and variance* σ_n^2 provided

$$\lim_{n\to\infty} \sup_{x\in S} \left| \frac{\sigma_n a(n, \lfloor \mu_n + x\sigma_n \rfloor)}{P_n} - (2\pi)^{-1/2} e^{-x^2/2} \right| = 0 . \tag{2}$$

Typically, a central limit theorem for a sequence of associated random variables gives one for the array from which a local limit theorem follows under certain conditions. (See the paragraph describing Harper's method before Lemma 1.)

Based on a finite group G having order $m \geq 1$, Dowling [6] constructed a finite geometric lattice $Q_n(G)$ of rank $n \geq 1$ using the partial G-partitions of an n-set. The Whitney number $W_m(n,k) \in \mathbb{N}$ is the number of elements in $Q_n(G)$ having co-rank k (so $W(Q_n(G),k) = W_m(n,n-k)$). Then $W_m(n,0) = W_m(n,n) = 1$ for $n \geq 1$, $W_m(n,k) \neq 0$ for $0 \leq k \leq n$ with $n \geq 1$ and $W_m(n,k) = 0$ for $k > n \geq 1$. Dowling [6] derived the recurrence relation

$$W_m(n,k) = W_m(n-1,k-1) + (1+mk)W_m(n-1,k) \tag{3}$$

for $1 \leq k \leq n$ and $n \geq 2$. Define $W_m(0,0) = 1$ and $W_m(0,k) = 0$ for $k \geq 1$. Then (3) is correct for $k, n \geq 1$ and $W_m(n,k) \neq 0$ for $0 \leq k \leq n$.

The associated numbers $U_m(n,k) = k!\,W_m(n,k)$ and $V_m(n,k) = k!\,m^k W_m(n,k)$ $(k,n \geq 0)$ were studied by Benoumhani [2]. There a combinatorial interpretation, due to Dowling, was given; a generating function was derived using algebraic techniques; and logconcavity was shown for $\{U_m(n,k)\}$ and for $\{V_m(n,k)\}$. In this paper, we derive the generating function analytically for completeness; refine a result of [2] implying logconcavity; and give a central limit theorem and a local limit theorem on $\mathbb{R}$ for $\{U_m(n,k)\}$ and for $\{V_m(n,k)\}$. In addition, we give asymptotic formulas for $\max\{U_m(n,k) : 0 \leq k \leq n\}$, $\max\{V_m(n,k) : 0 \leq k \leq n\}$ and for their modes. We choose to use the method of Harper [9] to establish our limit theorems due to the nature of these arrays.

We write $f(n) = o(g(n))$ if $\lim_{n\to\infty} f(n)/g(n) = 0$; $f(n) = O(g(n))$ provided $|f(n)| \leq C|g(n)|$ for $n \geq N$; and $f(n) \sim g(n)$ if $\lim_{n\to\infty} f(n)/g(n) = 1$. All asymptotics are with $n \to \infty$. The expectation of a random variable X is denoted $\mathbb{E}(X)$ and its variance $\mathrm{Var}(X)$. We refer the reader to Durrett [7] for probability. The nonnegative integers are denoted by $\mathbb{N}$: the real numbers by $\mathbb{R}$ and the complex numbers by $\mathbb{C}$. The greatest integer at most $x \in \mathbb{R}$ is denoted $\lfloor x \rfloor$. Throughout the paper, m is a fixed positive integer and is usually omitted from the notation.

2. Results

For $k, n \geq 0$, let $U_m(n,k) = k!\,W_m(n,k) \in \mathbb{N}$ and $V_m(n,k) = k!\,m^k W_m(n,k) \in \mathbb{N}$. Then $U_m(n,0) = 1$ and $U_m(n,n) = n!$ for $n \geq 0$. $U_m(n,k) \neq 0$ for $0 \leq k \leq n$ and $U_m(n,k) = 0$ for $k > n \geq 0$. In addition, (3) implies

$$U_m(n,k) = kU_m(n-1,k-1) + (1+mk)U_m(n-1,k) \tag{4}$$

for $k, n \geq 1$. For completeness, we first derive the generating functions for $\{U_m(n,k)\}$ and for $\{V_m(n,k)\}$ analytically.

Let

$$h(x,y) = \sum_{n\geq 0}\sum_{k\geq 0} U_m(n,k)\frac{x^n}{n!}y^k = e^x + \sum_{n\geq 1}\sum_{k\geq 1} U_m(n,k)\frac{x^n}{n!}y^k. \tag{5}$$

Then (4) implies

$$h_x(x,y) = y^2 h_y(x,y) + yh(x,y) + h(x,y) + myh_y(x,y).$$

Hence, $z = h(x,y)$ satisfies the linear first order PDE $z_x - (y^2+my)z_y = (1+y)z$ with initial condition $z(0,y) = 1$. The characteristic curves are given by $dy/dx = -(y^2+my)$, which has solution $\left(y/(y+m)\right)^{1/m} = ce^{-x}$. Along such a curve, $dz/dy = -(1+y)z/(y^2+my)$, which has solution $z = k(c)y^{-1/m}(y+m)^{-(m-1)/m} = k\left(y^{1/m}(y+m)^{-1/m}e^x\right)y^{-1/m}(y+m)^{-(m-1)/m}$ where k is an "arbitrary function". Applying the initial condition, we obtain $k\left(y^{1/m}(y+m)^{-1/m}\right) = y^{1/m}(y+m)^{(m-1)/m}$ or, equivalently, $k(s) = sm/(1-s^m)$. Then

$$\begin{aligned} h(x,y) &= \left(\frac{y}{y+m}\right)^{\frac{1}{m}} e^x \left(\frac{m}{1-\frac{ye^{mx}}{y+m}}\right) y^{-\frac{1}{m}}(y+m)^{-\frac{m-1}{m}} \\ &= \frac{e^x}{1-\frac{y}{m}\left(e^{mx}-1\right)} \qquad (x,y \in \mathbb{R};\ x^2+y^2 \leq R_m) \end{aligned} \tag{6}$$

as in [2; Theorem 3]. Obviously, the analogous generating function for $\{V_m(n,k)\}$ is $h(x,my)$.

Due to the nature of our arrays (see Lemma 1 and Lemma 4), we choose to use the method of Harper [9] to establish our limit theorems. Harper's method can be formalized as follows: Let $\{P_n(w)\}_{n\geq 0}$ be a sequence of nonzero polynomials with nonnegative real coefficients, where $P_n(w) = \sum_{k=0}^{D(n)} a(n,k)w^k$ is of degree $D(n)$ and all $D(n)$ roots of $P_n(w)$ are real and nonpositive. Then the array of coefficients $\{a(n,k) : n \geq 0,\ 0 \leq k \leq D(n)\}$ satisfies a central limit theorem with

$$\mu_n = \frac{P_n'(1)}{P_n(1)} \quad \text{and} \quad \sigma_n^2 = \frac{P_n'(1)}{P_n(1)} + \frac{P_n''(1)}{P_n(1)} - \left(\frac{P_n'(1)}{P_n(1)}\right)^2 \tag{7}$$

if $\sigma_n \to \infty$ as $n \to \infty$ (the derivatives are with respect to w). A proof of Harper's method can be found in [1; proof of Theorem 2] and a proof of an extension of Harper's method can be found in [4; Theorem 1] (see [11] for a survey of Pólya frequency sequences). Both results use the natural probabilistic interpretation $P(X_n = k) = a(n,k)/P_n(1)$, $0 \leq k \leq D(n)$, of the array $\{a(n,k)\}$. Then (referring to (7)), $E(X_n) = \mu_n$ and $\mathrm{Var}(X_n) = \sigma_n^2$. The proofs of both results use the central limit theorem in an essential way to show $(X_n - \mu_n)/\sigma_n \xrightarrow{d} N(0,1)$ which is equivalent to (1) (see [7; p. 70]). If, in addition, the nonzero coefficients of each $P_n(w)$ are consecutive, then the array of coefficients satisfies a local limit theorem on $\mathbb{R}$ with the above μ_n and σ_n^2 provided $\sigma_n \to \infty$ as $n \to \infty$. A (slightly incorrect) proof can be found in [1; Theorem 2]; a correction can be found in [3; Theorem II]; and a proof can found in [8; Theorem 7.1.4] when all $a(n,k)$ are integers.

For $n \geq 0$, let

$$P_n(w) = P_{n,m}(w) = \sum_{k=0}^{n} U_m(n,k)w^k \tag{8}$$

so that $P_0(w) = 1$, $P_1(w) = 1 + w$ and $P_2(w) = 1 + (m+2)w + 2w^2$. Note that all coefficients of $P_n(w)$ are positive integers. For $n \geq 1$, (4) implies

$$P_n(w) = (1 + w)P_{n-1}(w) + (mw + w^2)P_{n-1}'(w). \tag{9}$$

Benoumhani [2] showed that all n roots of $P_n(w)$ are real and negative for $n \geq 1$. Our first result refines that of [2] and uses a different approach.

Lemma 1. Fix $m \geq 2$. Then all n roots of $P_n(w) = P_{n,m}(w)$ are distinct and in the interval $(-m, 0)$ for $n \geq 0$. Consequently, the sequence $U_m(n,0), \ldots, U_m(n,n)$ is strictly logconcave for $n \geq 0$.

Proof. Our result is correct for $n = 0, 1$. Suppose $P_{n-1}(w)$ has roots $-m < r_{n-1} < \cdots < r_1 < 0$ where $n \geq 2$. Then $P'_{n-1}(r_{n-1}), \ldots, P'_{n-1}(r_1)$ alternate sign with $P'_{n-1}(r_1) > 0$. From (9), $P_n(r_k) = r_k(r_k + m)P'_{n-1}(r_k)$, so $P_n(r_{n-1}), \ldots, P_n(r_1)$ alternate sign with $P_n(r_1) < 0$ and $P_n(0) > 0$. By the Intermediate Value Theorem, $P_n(w)$ has a root in each interval $(r_{n-1}, r_{n-2}), \cdots, (r_2, r_1), (r_1, 0)$. From (9), $P_n(-m) = (-m+1)P_{n-1}(-m)$. For odd n, $P_n(-m) < 0$ and $P_n(r_{n-1}) > 0$, while for even n, $P_n(-m) > 0$ and $P_n(r_{n-1}) < 0$. In either case, $P_n(w)$ has a root in $(-m, r_{n-1})$. Our result follows by induction on n. ■

Remark. A slight modification of the proof shows that all n roots of $P_{n,1}(w)$ are distinct and in the interval $[-1, 0)$ for $n \geq 0$. Consequently, the sequence $U_1(n, 0), \ldots, U_1(n, n)$ is strictly logconcave for $n \geq 0$.

In view of (7), we now find asymptotic expansions for $P_n(1)$, $P'_n(1)$ and $P''_n(1)$.

From (5), (6), (8),

$$h(z, w) = \sum_{n=0}^{\infty} P_n(w) \frac{z^n}{n!} = \frac{e^z}{1 - \frac{w}{m}\left(e^{mz} - 1\right)}$$

which is analytic in $\mathbb{C} \times \mathbb{C}$ (see [10] for terminology) except for simple poles where $w = m/(e^{mz} - 1)$. Let $\alpha = \ln(m+1)/m$ and

$$h_1(z) = h(\alpha z, 1) = \sum_{n=0}^{\infty} P_n(1) \frac{(\alpha z)^n}{n!} = \frac{-me^{\alpha z}}{(m+1)\left(\frac{e^{\alpha m z}}{m+1} - 1\right)}.$$

$$h_2(z) = h_w(\alpha z, 1) = \sum_{n=0}^{\infty} P'_n(1) \frac{(\alpha z)^n}{n!} = \frac{m\left(e^{\alpha(m+1)z} - e^{\alpha z}\right)}{(m+1)^2\left(\frac{e^{\alpha m z}}{m+1} - 1\right)^2}.$$

$$h_3(z) = h_{ww}(\alpha z, 1) = \sum_{n=0}^{\infty} P''_n(1) \frac{(\alpha z)^n}{n!}$$

$$= \frac{-2m\left(e^{\alpha(2m+1)z} - 2e^{\alpha(m+1)z} + e^{\alpha z}\right)}{(m+1)^3\left(\frac{e^{\alpha m z}}{m+1} - 1\right)^3}.$$

Then $h_k(z)$ is analytic on $|z| < 1 + \delta = (\alpha^2 + 4\pi^2)^{1/2}$ except for a pole of order k at $z = 1$ $(1 \le k \le 3)$. Hence,

$$g_k(z) = (z-1)^k h_k(z) = \sum_{n=0}^{\infty} b_{k,n}(z-1)^n$$

is analytic on $|z| < 1 + \delta$ $(1 \le k \le 3)$. We have $(g_k'(1) = b_{k,1})$.

$$g_1(z) = \frac{e^\alpha + \alpha e^\alpha (z-1) + \cdots}{-\alpha(m+1) - \frac{\alpha^2}{2} m(m+1)(z-1) + \cdots}$$

so $b_{1,0} = -e^\alpha/\alpha(m+1)$.

$$g_2(z) = \frac{e^\alpha m^2 + \alpha e^\alpha m^2(m+2)(z-1) + \cdots}{\alpha^2 m^2(m+1)^2 + \alpha^3 m^3(m+1)^2(z-1) + \cdots}$$

so $b_{2,0} = e^\alpha/\alpha^2(m+1)^2$, $b_{2,1} = 2e^\alpha/\alpha(m+1)^2$ and

$$g_3(z) = \frac{-2e^\alpha m^3 - 2\alpha e^\alpha m^3(2m+3)(z-1) - \cdots}{\alpha^3 m^3(m+1)^3 + \frac{3}{2}\alpha^4 m^4(m+1)^3(z-1) + \cdots}$$

so $b_{3,0} = -2e^\alpha/\alpha^3(m+1)^3$ and $b_{3,1} = -e^\alpha(m+6)/\alpha^2(m+1)^3$.

By Darboux's Theorem (see [12: Theorem 8.4]),

$$\begin{aligned} P_n(1) &= n!\,\alpha^{-n}\left\{-b_{1,0} + o(n^{-1})\right\} \\ P_n'(1) &= n!\,\alpha^{-n}\left\{b_{2,0}n + (b_{2,0} - b_{2,1}) + o(n^{-1})\right\} \\ P_n''(1) &= n!\,\alpha^{-n}\left\{-\frac{b_{3,0}}{2}n^2 + \left(b_{3,1} - \frac{3b_{3,0}}{2}\right)n + O(1)\right\} \end{aligned} \tag{10}$$

as $n \to \infty$. Hence,

$$\begin{aligned} \frac{P_n'(1)}{P_n(1)} &= -\frac{b_{2,0}}{b_{1,0}}n - \frac{b_{2,0} - b_{2,1}}{b_{1,0}} + o(1). \\ \left(\frac{P_n'(1)}{P_n(1)}\right)^2 &= \frac{b_{2,0}^2}{b_{1,0}^2}n^2 + \frac{2b_{2,0}^2 - 2b_{2,0}b_{2,1}}{b_{1,0}^2}n + o(n). \\ \frac{P_n''(1)}{P_n(1)} &= \frac{b_{3,0}}{2b_{1,0}}n^2 + \frac{3b_{3,0} - 2b_{3,1}}{2b_{1,0}}n + o(n) \end{aligned} \tag{11}$$

as $n \to \infty$. Consequently, (7), (11) give

$$\begin{aligned}\mu_n &= -\frac{b_{2,0}}{b_{1,0}} n - \frac{b_{2,0} - b_{2,1}}{b_{1,0}} + o(1) \\ &= \frac{m}{(m+1)\ln(m+1)}(n+1) - \frac{2}{m+1} + o(1)\end{aligned} \tag{12}$$

and

$$\begin{aligned}\sigma_n^2 &= -\frac{2b_{1,0}b_{2,0} + 2b_{1,0}b_{3,1} - 3b_{1,0}b_{3,0} + 4b_{2,0}^2 - 4b_{2,0}b_{2,1}}{2b_{1,0}^2} n + o(n) \\ &= \frac{m(m - \ln(m+1))}{(m+1)^2 \ln^2(m+1)} n + o(n) \to \infty\end{aligned} \tag{13}$$

as $n \to \infty$, since $b_{1,0}b_{3,0} - 2b_{2,0}^2 = 0$.

A consequence of our calculations is the following limit theorems for $\{U_m(n,k)\}$.

Theorem 2. Fix $m \geq 1$. The array $\{U_m(n,k) : n \geq 0,\ 0 \leq k \leq n\}$ satisfies a central limit theorem and a local limit theorem on $\mathbb{R}$ with the above μ_n, σ_n^2 (given in (7), (12), (13)) and with $P_n = P_n(1)$ (given in (10)). ■

A result of Darroch [5] (see also [11; pp. 284–285]) implies that $U_m(n) = \max\{U_m(n,k) : 0 \leq k \leq n\}$ occurs for some k in the interval $[\mu_n - 1, \mu_n + 1]$ given in (7), (12). Hence (10), (13), the local limit theorem in Theorem 2 and logconcavity give the following.

Corollary 3. Fix $m \geq 1$. Then $U_m(n)$ occurs for some k in the interval $[\mu_n - 1, \mu_n + 1]$ and

$$U_m(n) \sim U_m(n, \lfloor \mu_n \rfloor) \sim m^{1/2}(m+1)^{1/m}\left[2\pi n(m - \ln(m+1))\right]^{-1/2} n!\, \alpha^{-n}$$

as $n \to \infty$. ■

The array $\{V_m(n,k) : n \geq 0,\ 0 \leq k \leq n\}$ is handled similarly, hence, we merely state the results. First, (3) implies

$$V_m(n,k) = mkV_m(n-1, k-1) + (1 + mk)V_m(n-1, k) \tag{14}$$

for $k, n \geq 1$. For $n \geq 0$, let

$$Q_n(w) = Q_{n,m}(w) = \sum_{k=0}^{n} V_m(n,k) w^k$$

so that $Q_0(w) = 1$, $Q_1(w) = 1 + mw$ and $Q_2(w) = 1 + (m^2 + 2m)w + 2m^2 w^2$. Note that all coefficients of $Q_n(w)$ are positive integers. Then, (14) implies

$$Q_n(w) = (1 + mw)Q_{n-1}(w) + (mw + mw^2)Q'_{n-1}(w)$$

for $n \geq 1$. An argument nearly identical to Lemma 1 gives the following result which refines that of [2].

Lemma 4. Fix $m \geq 2$. Then all n roots of $Q_n(w) = Q_{n,m}(w)$ are distinct and in the interval $(-1, 0)$ for $n \geq 0$. Consequently, the sequence $V_m(n,0), \ldots, V_m(n,n)$ is strictly logconcave for $n \geq 0$. ■

Remark. All n roots of $Q_{n,1}(w)$ are distinct and in the interval $[-1, 0)$ for $n \geq 0$. Consequently, the sequence $V_1(n,0), \ldots, V_1(n,n)$ is strictly logconcave for $n \geq 0$.

Calculations nearly identical to Theorem 2 give the following limit theorems for $\{V_m(n,k)\}$.

Theorem 5. Fix $m \geq 1$. The array $\{V_m(n,k) : n \geq 0,\ 0 \leq k \leq n\}$ satisfies a central limit theorem and a local limit theorem on $\mathbb{R}$ with

$$\mu_n = \frac{n}{2\ln 2} + O(1) \quad , \quad \sigma_n^2 = \frac{1 - \ln 2}{4 \ln^2 2} n + o(n) \quad \text{and } P_n = Q_n(1). \qquad ■$$

Let $V_m(n) = \max\{V_m(n,k) : 0 \leq k \leq n\}$ and $\beta = (\ln 2)/m$.

Corollary 6. Fix $m \geq 1$. Then $V_m(n)$ occurs for some k in the interval $[\mu_n - 1, \mu_n + 1]$ and

$$V_m(n) \sim V_m(n, \lfloor \mu_n \rfloor) \sim 2^{1/m} [2\pi n \ln e/2]^{-1/2} n! \, \beta^{-n} \text{ as } n \to \infty. \qquad ■$$

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