Constructing Flow-Equivalent Graphs

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Abstract

Two graphs are said to be flow-equivalent, if they have the same number of nowhere-zero λ-flows, i.e., they have the same flow polynomial. In this paper, we present a few methods of constructing non-isomorphic flow-equivalent graphs.

1 Introduction

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. $F(G, \lambda)$ is a polynomial in λ which gives the number of nowhere-zero λ-flows in G independent of the chosen orientation. Many properties of the flow polynomial, as well as more details on nowhere-zero flows can be found in [3] and [4]. Let M be a multigraph. Let $G(M)$ denote the graph obtained from M by replacing every multiple edge by a simple edge. Two multigraphs M_1 and M_2 are *amallamorphic* if $G(M_1)$ is isomorphic to $G(M_2)$. Two graphs R and S are said to be *flow equivalent* if $F(R, \lambda) = F(S, \lambda)$. For convenience, we sometimes use $\lambda = 1 - \omega$.

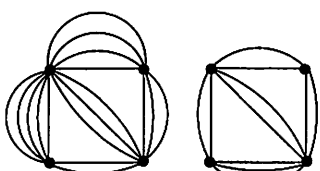

Figure 1: Amallamorphic graphs

In computing the flow polynomial of an amallamorph M of the graph $G(M)$ where the edges of $G(M)$ become sheafs of edges (multiple edges), it is efficient to remove an entire sheaf in one step instead of removing one edge at a time. Read and Whitehead[3] obtain the "**SRF**" or the **Sheaf Removal Formula**:

$$F(M,\omega) = (-1)^m \left[\frac{\omega^m - 1}{1-\omega} F(K,\omega) + F(H,\omega) \right]$$

In SRF, M is a multigraph having a sheaf of m edges, K is the graph obtained from M by contracting the sheaf to a vertex, and H is the graph obtained from M by deleting the sheaf. A letter labelling an edge in this paper indicates the edge multiplicity of that edge. If G has a bridge, then $F(G,\lambda) = 0$. If e is any edge of G, then $F(G,\lambda) = F(G'',\lambda) - F(G',\lambda)$, where G' and G'' are obtained from G by deleting and contracting the edge e, respectively. By a result of Jaeger [1], if G is planar, then $P(G^*,\lambda) = \lambda \cdot F(G,\lambda)$, where G^* is the planar dual of G and $P(G^*,\lambda)$ is the chromatic polynomial of G^*.

2 A Theorem of Equivalence

Given P_2, the path on 2 vertices, let X_{a_k} be the multigraph of sheaf multiplicity a_k whose underlying graph is P_2. The reader can find $F(X_{a_k},\lambda)$ and more in [4]. We now present a result which can not be obtained through planar duality of graphs.

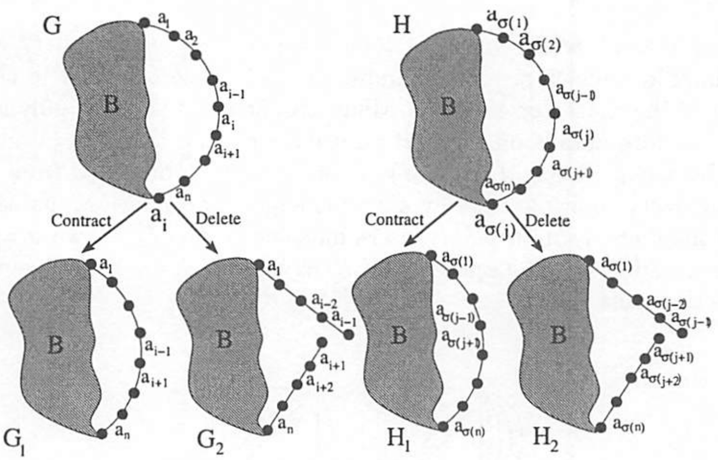

Figure 2

Theorem 2.1 *Let G be the graph in Figure 2, which is made up of a (possibly nonplanar) subgraph B and a path of length n of edge bundles X_{a_i} connecting 2 vertices of B. Then the flow polynomial of G is invariant* *under any permutation of the edge bundles,i.e, if $\sigma \in S_n$ applied to the edge bundles $a_1, a_2, \ldots a_n$ of G results in the graph H, then $F(G,\omega) = F(H,\omega)$.*

Proof: Let us proceed by induction on n, the length of the path. For $n = 1$ the result holds trivially. Suppose that the result is true for n-1 and consider the graphs G and H of Figure 2. Under the permutation $\sigma \in S_n$, given any $1 \le i \le n$, we can find a unique j such that $\sigma(j) = i$. Apply SRF to the edge bundle a_i of G and the the edge bundle $a_{\sigma(j)}$ of H. This will yield the following equations:

$$F(G,\omega) = (-1)^{a_i}\left[\frac{\omega^{a_i}-1}{1-\omega}F(G_1,\omega) + F(G_2,\omega)\right]$$
$$F(H,\omega) = (-1)^{a_{\sigma(j)}}\left[\frac{\omega^{a_{\sigma(j)}}-1}{1-\omega}F(H_1,\omega) + F(H_2,\omega)\right]$$

Since $\sigma(j) = i$ and by inductive hypothesis, $F(G_1,\omega) = F(H_1,\omega)$. We also observe that

$$F(G_2,\omega) = \prod_{k=1}^{i-1} F(X_{a_k},\omega) \cdot \prod_{k=i+1}^{n} F(X_{a_k},\omega) \cdot F(B,\omega)$$
$$F(H_2,\omega) = \prod_{k=1}^{j-1} F(X_{a_{\sigma(k)}},\omega) \cdot \prod_{k=j+1}^{n} F(X_{a_{\sigma(k)}},\omega) \cdot F(B,\omega)$$

This shows $F(G_2,\omega) = F(H_2,\omega)$ and therefore $F(G,\omega) = F(H,\omega)$. ■

3 Invariance Under Transposition

The planar duals of the graphs G and H of the Figure 3 are what Xu, Liu, and Peng[6] called n-bridge graphs, which are graphs consisting of s paths joining two vertices. Combined with the result of Jaeger[1], we obtain the following corollary:

Corollary 3.1 *Let C_n be the underlying simple graph of the graph G whose edge multiplicities are $\vec{a} = (a_1, a_2, \ldots, a_n)$. Pick any $\sigma \in S_n$ and apply σ to the edge bundles of G and call the new graph G_σ, whose edge multiplicities now are $\sigma(\vec{a}) = (a_{\sigma(1)}, a_{\sigma(2)}, \ldots, a_{\sigma(n)})$. Then the flow polynomial of G is permutation invariant, i.e.,*

$$F(G,\omega) = F(G_\sigma,\omega).$$

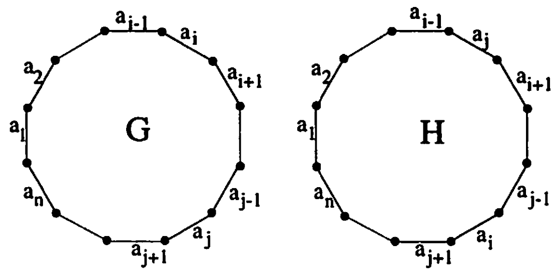

Figure 3: H is G with a transposition applied to 2 bundles of G

The planar duals of the graphs G and H of the Figure 5 are what Read[2] called broken wheels, depicted in Figure 4. While proving the strong logarithmic concavity of the wheels W_k, Read[2] proved the invariance of the chromatic polynomial of W_n under any permutation of the outer rims. In the language of colorings, Read [2] also found the chromatic polynomial of W_n which we state below. Here $N = \sum_{i=1}^{k} a_i$.

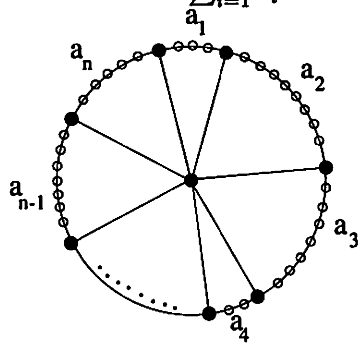

Figure 4: The Wheel W_n with n spokes

$$Q_n(\lambda) = \frac{1}{\lambda}\left[(\lambda-1)^{n+1} + (-1)^n\right]$$

$$P(W_n, \lambda) = \lambda \prod_{i=1}^{k} Q_{a_i}(\lambda) + (-1)^N \lambda(\lambda - 2)$$

Combined with the result of Jaeger[1], we obtain the following corollary:

Corollary 3.2 *Let W_n, the wheel on n+1 vertices, be the underlying simple graph of the graph G, where the rim edges of G have multiplicity 1 and the spokes of G have edge multiplicities $\vec{a} = (a_1, a_2, \ldots, a_n)$. Pick any $\sigma \in S_n$ and apply σ to the spokes of G and call the new graph G_σ whose edge multiplicities now are $\sigma(\vec{a}) = (a_{\sigma(1)}, a_{\sigma(2)}, \ldots, a_{\sigma(n)})$. Then the flow* *polynomial of G is permutation invariant, i.e.,*

$$F(G, \lambda) = F(G_\sigma, \lambda).$$

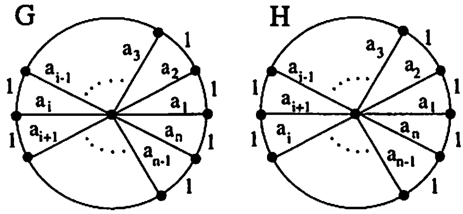

Figure 5: *G* and *H* with a transposition applied to 2 spokes of *G*

4 A Method of Construction

Proposition 4.1 *Given positive integers $a_1, a_2, \ldots, a_n$, a family of non isomorphic flow equivalent graphs can be constructed.*

Proof: Start with 2^{n-1} copies of X_{a_1}. For the first 2^{n-2}, we add two single edges and an edge bundle of multiplicity a_2, while for the other 2^{n-2}, we add a single edge and an edge bundle of multiplicity a_2, as shown in the left side of Figure 6. We now repeat the same process: For the first and third 2^{n-3}, we add two single edges and an edge bundle of multiplicity a_2, while for the second and fourth 2^{n-3}, we add a single edge and an edge bundle of multiplicity a_2, as shown in the right side of Figure 6.

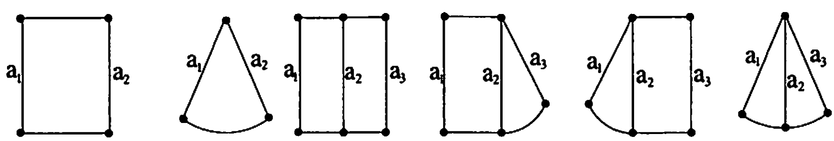

Figure 6

If we continue in this manner, we will arrive at a family of 2^{n-1} many graphs which were built on a starting subgraph and "bricks" of triangular or square shapes. Since the deletion os any edge bundle of multiplicity one will result in a graph which possesses a bridge, then The graphs obtained in each step of this method of construction have the same flow polynomial. ■

Figure 7 depicts a generalization of the above procedure.

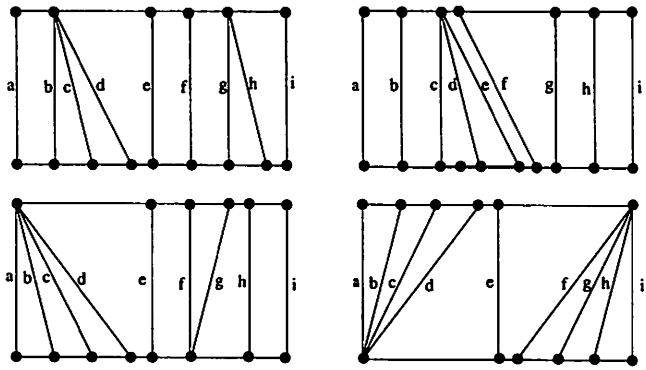

Figure 7

Corollary 4.2 *Given the graph shown in Figure 8, any one of the following edge bundle exchanges will result in a flow-equivalent amallamorph:*

1. Exchange a_1 with $a_i + 1$ and a_i with $a_1 - 1$ for $1 < i < k$
2. Exchange a_k with $a_i + 1$ and a_i with $a_k - 1$ for $1 < i < k$
3. Exchange a_j with $a_i + 1$ and a_i with $a_j - 1$ for $1 < i, j < k$ ■

Proof: The reader can verify the results by appling the SRF to the appropriate edge bundles of M.

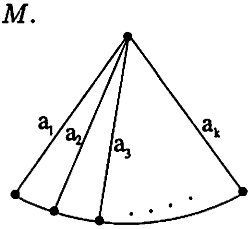

Figure 8

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