Abelian Group Labels on Hamiltonian Cycles

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Abstract

We consider labeling edges of graphs with elements from abelian groups. Particular attention is given to graphs where the labels on any two hamiltonian cycles sum to the same value. We find several characterizations for such labelings for cubes, complete graphs and complete bipartite graphs. This extends work of [1, 8, 9, 10]. We also consider the computational complexity of testing if a labeled graph has this property and show it is NP-complete even when restricted to integer labelings of 3-connected, cubic, planar graphs with face girth at least five.

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Introduction

Consider the weighted graph in Figure 1. Note the sum of the weights on any hamiltonian cycle is 16. In [1, 8, 9, 10] studies are made of weighted complete graphs having the property that the sum of the weights on any hamiltonian cycle is constant, independent of the choice of hamiltonian cycle. We note that given this information about a weighted graph, solving the Traveling Salesperson Problem is particularly easy. Further, [8] shows that testing if a complete graph of order n has this property can be done in $O(n^2)$ time, where the weights are all integers. In this document we extend some of the work in [1, 8, 9, 10] to abelian groups and consider families other than complete graphs.

For undefined terms and concepts the reader is referred to [4,7]. By *graph* we shall mean a finite simple graph. Further, G will represent a graph of order n. By *group* we shall mean an abelian group. We shall let (A,+) represent an arbitrary group. Given $x \in A$ let 2x represent x+x. We shall refer to 2x as *twice* the value of x. We shall say y is *even* if it is twice the value of some group element. A *vertex labeling* of G is a mapping from V(G) into A. An edge labeling is defined similarly. When clarity allows, we will refer to an edge labeling simply as a labeling. An edge labeling γ is *induced* by a vertex labeling α if $\gamma(uv)=\alpha(u)+\alpha(v)$, for each edge uv. We will say a labeling is *induced* if it is induced by some vertex labeling. Given a labeled graph G, the *weight* of G, denoted $\omega(G)$, is the sum of the edge labels. This term will often be applied to edges in a subgraph. Given a labeled graph G, if we speak of a subgraph H we will mean that H is labeled with the labeling on E(G) restricted to E(H). We wish to study labelings which have the property that given any pair of hamiltonian cycles, the cycles have the same weight.

Let us say a labeling is *hamilton-stable* if every pair of hamiltonian cycles has the same weight. A *circuit* is a closed walk. Suppose the vertices of a circuit appear sequentially as $v_1, v_2, \dots, v_k$. We shall represent the circuit as $(v_1, v_2, \dots, v_k)$. At times it will be important to similarly represent a circuit by sequentially listing its edges e.g. $(e_1, e_2, \dots, e_k)$. In a graph labeled by γ, an even cycle $(e_1, e_2, \dots, e_k)$ is *balanced* if $\gamma(e_1)-\gamma(e_2) + \dots -\gamma(e_k) = 0$. A labeling γ is *C4-balanced* if each 4-cycle is balanced. Equivalently, for each (e_1, e_2, e_3, e_4) we have $\gamma(e_1)+\gamma(e_3) = \gamma(e_2)+\gamma(e_4)$. As we shall see, in many cases labelings are hamilton-stable if and only if they are C_4-balanced. Let us say the labeling of a circuit $(e_1, e_2, \dots, e_k)$ is *even* if $\gamma(e_1,)+\gamma(e_2,) + \dots + \gamma(e_k)$ is even. Equivalently for odd circuits, the labeling is even if $\gamma(e_1) - \gamma(e_2) + \gamma(e_3) - \gamma(e_4) + \dots + \gamma(e_k)$ is even. If the labels on any pair of k-factors have the same weight, we will say the labeling is *k-stable*. Thus, if a labeling is 2-stable then it is hamilton-stable.

Observation 1. Given r and s where $1 \le s \le r$, a labeling of an r-regular graph is s-stable if and only if it is (r-s)-stable.

Proof. Suppose γ is an s-stable labeling of an r-regular graph G. Suppose, H is an (r-s)-regular subgraph. Let F be the edge complement of H. Note that F is s-regular. Since each edge of G is in H or F but not both we note w(F)+w(H)=w(G). Since each pair of s-factors has the same weight we note
each pair of (r-s)-factors must have the same weight. ☐

We shall see that for several classes of graphs, stronger statements are available.

Clearly, if a labeling is induced, then the sum of the labels on a hamiltonian cycle will equal twice the value of any vertex labeling that induce the edge labeling. Hence, if a labeling is induced then it is hamilton-stable. Figure 1 shows the converse does not hold. The labels on each hamiltonian cycle sum with regular addition to 16. We restate a result of [6] that shows the labeling in Figure 1 is not induced.

Theorem 2. An edge labeling γ of G is induced if and only if both of the following hold:

- Every odd circuit has an even labeling.

And

- Every even cycle is balanced.

We note that the first condition is not satisfied in Figure 1. Also, it follows from this theorem that every induced labeling is

C_4-balanced.

Cubes

Let Q_n denote the n-dimensional cube.

Theorem 3. A labeling of Q_n is induced if and only if it is C_4-balanced.

Proof. By Theorem 2, if Q_n is induced then it is C_4-balanced. So suppose γ is an edge labeling of Q_n that is C_4-balanced. We show by induction on n that γ is induced by some vertex labeling. If n is zero or one the result is clear. So suppose $n \geq 2$ and the result holds true for all labelings of cubes with dimension less than n.

We may conceive of Q_n as two copies of Q_{n-1} with a 1-factor joining corresponding vertices. Let us denote these two copies of Q_{n-1} as H and K. If we restrict γ to H, the resulting labeling is C_4-balanced. So we apply induction and let α be a vertex labeling on H that induces the restricted labeling. Let us extend α in the following fashion. Given a vertex u in K, let v be the corresponding vertex in H. Define $\alpha(u)$ to be $\gamma(uv)-\alpha(v)$. It is clear that α induces the labels on all edges between H and K. So, suppose xy is an edge of K. It remains to show that $\gamma(xy)=\alpha(x)+\alpha(y)$. Let x' and y' be those vertices in H that correspond respectively to x and y. By hypothesis, $\gamma(xx')+\gamma(yy')=\gamma(xy)+\gamma(x'y')$. Thus,

$$\gamma(xy)=\gamma(xx')+\gamma(yy')-\gamma(x'y')=[\alpha(x')+\alpha(x)]+[\alpha(y')+\alpha(y)]-[\alpha(x')+\alpha(y')]=\alpha(x)+\alpha(y). \qquad \square$$

Consider the cube Q_3 in Figure 2 labeled with elements from Z_2. Each hamiltonian cycle has weight 1. By Theorem 2, since this labeling is not C_4-balanced it is not induced. Thus, a cube may be hamilton-stable yet fail to be C_4-balanced. Our next result shows that for larger dimensions, this is not the case.

Theorem 4. For $n\geq 4$, a labeling of Q_n is C_4-balanced if and only if it is hamilton-stable.

Proof. If the labeling is C_4-balanced then by Theorem 3 it is induced and is hence hamilton-stable. So, suppose $\gamma$is a hamilton-stable labeling of Q_n. Suppose (u,v,s,t) are vertices in a 4-cycle. We may mark the vertices of Q_n with n-bit binary words so that two vertices are adjacent if and only if they differ in exactly one corresponding bit. Without loss of generality, suppose u,v,s,t have only zeros with the following exceptions: v contains a 1 in its first bit, the first two bits of s are 1 and the second bit of t is 1. Now, let x,y and z be those vertices that contain only zeros with the following exceptions: x contains a 1 in its third bit, y contains 1 in its first and third bit and z contains 1 in its first three bits.

We claim there is a t-v path that contains all vertices of Q_n except u,x,y,z,s. We prove this by induction on n. In the base case where n=4 consult Figure 3. The upper and lower Q_3 represent the two hyperplanes where the fourth bit is 0 and 1 respectively. Take the shaded edges and add the edges pp' and vv'. This is a desired t-v path for Q_4. So, suppose n≥5 and the claim is true for (n-1)-cubes. As in the preceding proof we may assume Q_n is formed by taking H and K, two copies of Q_{n-1} and adding a 1-factor between corresponding vertices. Without loss of generality we may assume the vertices p,s,t,u,v,x,y,z all lie within H. By induction, there is a t-v path in H, say P, that contains all the vertices of H except u,x,y,z,s. Let a and b be vertices that are adjacent in P. Let a' and b' be their counterparts in K. Note, there is a hamiltonian cycle in K that contains the edge a'b'. Let us remove this edge from the hamiltonian cycle. Remove also the edge ab from P. Now add to this pair of paths the edges aa' and bb'. This creates a t-v path in Q_n that contains all the vertices except u,x,y,z,s. Hence, the claim is established.

Consider the path delivered by the preceding claim. If we add the edges uv and st, together with the u-s path u,x,y,z,s, then we create a hamiltonian cycle. We could create a second hamiltonian cycle if we instead added the edges ut and vs, along with the same u-s path. Given that these two cycles differ by a pair of edges, together with the fact that the labeling is hamilton-stable, we may conclude the labels on uv and st and the labels on ut and vs sum to the same value. □

Thus, while being hamilton-stable is a global property, it is equivalent to a local property.

Theorem 5. A labeling of Q_3 with elements from the real numbers is hamilton-stable if and only if it is C_4-balanced.

Proof. The proof in one direction follows from Theorem 3. So let us denote the edges of Q_3 as in Figure 4. And suppose edge i is labeled with x_i, an element of R. Further, suppose this labeling is hamilton-stable but not C_4-balanced. Suppose the labels on each hamiltonian cycle sum to λ. Suppose, without loss of generality that $x_1+x_3 \neq x_2+x_4$. Equivalently, $x_1-x_2+x_3-x_4=\beta\neq 0$. By rescaling, if necessary, we can assume λ is either 0 or 1. This provides us with a system of seven linear equations:

$$x_1+x_3+x_4+x_5+x_7+x_8+x_{10}+x_{11}=\lambda$$
$$x_1+x_2+x_4+x_5+x_6+x_8+x_{11}+x_{12}=\lambda$$
$$x_1+x_2+x_3+x_5+x_6+x_7+x_9+x_{12}=\lambda$$
$$x_2+x_3+x_4+x_6+x_7+x_8+x_9+x_{10}=\lambda$$
$$x_1+x_3+x_6+x_8+x_9+x_{10}+x_{11}+x_{12}=\lambda$$
$$x_2+x_4+x_5+x_7+x_9+x_{10}+x_{11}+x_{12}=\lambda$$
$$x_1-x_2+x_3-x_4=\beta$$

Whether λ is 0 or 1, the first six equations force β to be 0.

Hence, the system is inconsistent. □

Given that the linear system described in the proof of Theorem 5 has no solution with real numbers, and the integers and rational numbers are subsets of the reals, Theorem 5 implies corresponding statements for Z and Q.

From Theorems 3 and 4 we note that for $n \geq 4$, a labeling of the n-cube is hamilton-stable if and only if it is induced. In this case the cube must be k-stable, for all k.

Theorem 6. If a labeling of Q_n is 1-stable then it is hamilton-stable.

Proof. The result is trivial for n=0,1,2 so suppose $n \geq 3$. Suppose γ is a 1-stable labeling of Q_n. Let (e_1, e_2, e_3, e_4) be a cycle in Q_n. Remove from Q_n all vertices incident with the edges e_1, e_2, e_3, e_4 and let F be the set of edges in a 1-factor of this subgraph. Note, the labels on $\{e_1, e_3\} \cup F$ and the labels on $\{e_2, e_4\} \cup F$ sum to the same values. Thus, $\gamma(e_1)+\gamma(e_3) = \gamma(e_2) + \gamma(e_4)$. As the labeling is C_4-balanced we can apply Theorem 3 and conclude it is induced and hence our desired result. □

Theorem 7. For $n \geq 4$ and k fixed between 1 and n-1, a labeling of Q_n is hamilton-stable if and only if it is k-stable.

Proof. If a labeling of Q_n is hamilton-stable it is induced. Hence, the labels on any k-factor sum to the same value. So assume γ is k-stable where k is fixed and $1 \le k \le n-1$. Let (e_1, e_2, e_3, e_4) be a cycle in Q_n. Remove from Q_n all vertices in this 4-cycle and let T be a 2-factor in what remains. Let U be a 1-factor in T. Let R be $Q_n - (\{e_1, e_2, e_3, e_4\} \cup E(T))$. We note R is bipartite and (n-2)-regular. Such graphs are Type 1 (see [3], page 93). So, edge color R with n-2 colors. Take k-1 of these color classes and add to it the edges of U as well as e_1 and e_3. This forms a k-factor. We could form a different k-factor by replacing e_1 and e_3 with e_2 and e_4. Since γ is k-stable we conclude $\gamma(e_1)+$
$\gamma(e_3)=$
$\gamma(e_2)+\gamma(e_4)$. Thus, Q_n is C_4-balanced, hence hamilton-stable.

□

Bipartite Graphs

Theorem 8. A labeling of $K_{m,n}$ is induced if and only if it is C_4-balanced.

Proof. The necessity is immediate from Theorem 2. Suppose γ is a C_4-balanced labeling on $K_{m,n}$. From Theorem 2 we need only show every even cycle is balanced. We proceed by induction on k, the cycle size. Consider the even cycle $(v_1, v_2, \ldots, v_k)$. If k=4 then by hypothesis the cycle is balanced. So suppose $k \geq 6$ and every even cycle of length less than k is balanced. Hence, $\gamma(v_1v_2)-\gamma(v_2v_3)+\gamma(v_3v_4)-\gamma(v_4v_1)=0$. Also, $\gamma(v_1v_4)-\gamma(v_4v_5)+\gamma(v_5v_6)\ldots-\gamma(v_kv_1)=0$. Adding the two expressions shows $(v_1, v_2, \ldots, v_k)$ is balanced.

□

Let us say graphs G and H are *degree similar* if V(G)=V(H) and for each vertex v in these graphs, $\deg_G(v)=\deg_H(v)$. Now, suppose a graph contains edges uv and wx but not vw nor ux. If we replace the first two edges with the last two, we produce graphs that are degree similar. Let us call this operation of replacement a *switching*. Degree similar graphs are related in the following way.

Remark 9. If two graphs are degree similar, one can be transformed into the other by a sequence of switchings.

A proof is available in [2] (page 151). We consider an alternate proof which is similar to the proof of the Havel-Hakimi Theorem (see {3], page 12).

Proof. First, we note that if G can be formed from H by a sequence of switchings, then H can be formed from G. We proceed to prove the remark by induction on n, the order of the graph. Suppose the remark is true for all graphs of order less than n. Let G and H be graphs of order n with degree sequence $d_1, d_2, \ldots, d_n$ where $d_1 \leq d_2 \leq \ldots \leq d_n$ and having the same vertex set. Label the vertices as $v_1, v_2, \ldots, v_n$ so that $\deg(v_k)=d_k$. Let u be v_n. If u has the same neighborhood in G as H then G-u and H-u have the same degree sequence. We can apply the induction hypothesis and transform G-u into H-u by a sequence of switchings. Adding u and incident edges, we can use the same switchings to transform G into H. So suppose u does not have the same neighborhood in both graphs.

We will transform G and H with switchings into graphs G' and H' where u has the same neighborhood in G' and H'. Then, a sequence of switchings moves G to G', we follow this by a sequence moving G' to H' and then a sequence moving H' to H.

Set $d=d_n$. If u is adjacent to $v_{n-d}, v_{n-d+1}, \ldots, v_{n-1}$ then set G' equal to G. Otherwise, u is adjacent to some v_k where $k \leq n-d-1$ and not adjacent to some v_j where $n-d \leq j \leq n-1$. Since $\deg(v_k) \leq \deg(v_j)$ and v_k and u are adjacent but v_j and u are nonadjacent, there is some v_m where v_m is adjacent to v_j but not to v_k. Let us now perform the switching where uv_k and v_mv_j is removed and replaced with v_kv_m and uv_j. If u is not adjacent to $v_{n-d}, v_{n-d+1}, \ldots, v_{n-1}$ then repeat this process until it is adjacent to exactly these vertices. Call the resulting graph G'.

Perform a similar argument to produce a sequence of switchings that transform H into H', a graph with the same degree sequence where u is adjacent to $v_{n-d}, v_{n-d+1}, \ldots, v_{n-1}$. □

We will not make immediate use of this previous remark.

Corollary 10. If a labeling of $K_{m,n}$ is C_4-balanced then any two degree similar subgraphs have the same weight.

Hence, if a labeling of $K_{n,n}$ is C_4-balanced it must be hamilton-stable. Consider a labeling of the 4-cycle that is not C_4-balanced. This shows a labeling of $K_{2,2}$ may be hamilton-stable yet not C_4-balanced.

Theorem 11. If $n \geq 3$, a labeling of $K_{n,n}$ is C_4-balanced if and only if it is hamilton-stable.

Proof. Suppose γ is a hamilton-stable labeling of $K_{n,n}$. Suppose the partite sets of $K_{n,n}$ are labeled $\{v_1, v_3, \ldots, v_{2n-1}\}$ and $\{v_2, v_4, \ldots, v_{2n}\}$. Without loss of generality we will show (v_1, v_2, v_3, v_4) is balanced and conclude our desired result. Consider the paths (v_2, v_5, v_4) and $(v_1, v_6, v_7, v_8, v_9 \ldots, v_{2n}, v_3)$. We can now proceed as we did in the proof of Theorem 4 to reach our desired result. □

Corollary 12. If $n \geq 3$, a labeling of $K_{n,n}$ is induced if and only if it is hamilton-stable.

By a slight modification of the proof of Theorem 8 we can show the following.

Theorem 13. If $n \geq 3$ and k is fixed, even and $4 \leq k \leq 2n-2$ then a labeling of $K_{n,n}$ is C_4-balanced if and only if every k-cycle is balanced.

We also have an analogue for $K_{n,n}$ of Theorem 7.

Theorem 14. For $n \geq 3$ and k fixed where $1 \leq k \leq n-1$ then a labeling of $K_{n,n}$ is hamilton-stable if and only if it is k-stable.

Proof. For n=3 the result is trivial. So suppose $n \geq 4$. If the labeling is hamilton-stable then it is induced and hence k-stable. If the labeling is
1-stable then any hamiltonian cycle can have its edge set decomposed into two 1-factors. Hence, the weight on any hamiltonian cycle is twice the value of any 1-factor. If $k \geq 2$ and the labeling is k-stable, then proceed as in the
proof of Theorem 7 to show the graph is C_4-balanced and hence induced. □

Complete Graphs

Let us say a group is *2-torsion free* if it contains no elements of order two. For example, all groups of odd order are 2-torsion free, as are the reals and integers. In a 2-torsion free group, x=y if and only if $2x=2y$. For an edge labeling γ, denote by $2\gamma$the edge labeling that maps e to $2\gamma(e)$. If γ labels G with elements of a 2-torsion free group, then γ is hamilton-stable if and only if 2γ is hamilton-stable. The following extends a result of [8] from the integers to all 2-torsion free groups.

Theorem 15. A labeling γ of K_n with elements from a 2-torsion free group is C_4-balanced if and only if 2γ is induced.

Proof. Suppose 2γ is induced by elements from a 2-torsion free group. Let us say α induces 2γ. Let (v_1,v_2,v_3,v_4) be a 4-cycle in K_n. Note,

$$2\gamma(v_1v_2)+2\gamma(v_3v_4)=[\alpha(v_1)+\alpha(v_2)]+[\alpha(v_3)+\alpha(v_4)]=$$
$$[\alpha(v_2)+\alpha(v_3)]+[\alpha(v_4)+\alpha(v_1)]=2\gamma(v_2v_3)+2\gamma(v_4v_1).$$

Hence, $\gamma(v_1v_2)+\gamma(v_3v_4)=\gamma(v_2v_3)+\gamma(v_4v_1)$. Thus, γ is C_4-balanced.

So suppose a 2-torsion free labeling γ of K_n is C_4-balanced. We will see that both conditions of Theorem 2 are satisfied for 2γ. The condition for odd cycles is trivially satisfied.

We show by induction on k that every even cycle $(e_1,e_2, \dots, e_k)$ is balanced. By hypothesis, the statement holds true for $k=4$. So suppose $k\geq 6$ and this statement is true for each even cycle of size less than k. Remove from this cycle the edges e_1, e_2 and e_k. This creates a path with k-3 edges. Let e be the edge incident with the end-points of this path. We apply the induction hypothesis and note $-\gamma(e) + \gamma(e_3) - \gamma(e_4) \ldots + \gamma(e_{k-1})$ is zero. By hypothesis $\gamma(e_1) - \gamma(e_2) + \gamma(e) - \gamma(e_k)$ is zero. Adding the two expressions and doubling each term shows each even cycle, labeled with 2γ, is balanced.

□

We note the proof in one direction did not require the group to be 2-torsion free. That is, if a labeling γ of K_n is C_4-balanced then 2γ is induced.

By Remark 9, any labeling of K_n that is C_4-balanced must be 2-stable.

Our next two results are extensions of theorems found in [8]. It applies to all abelian groups.

Theorem 16. A labeling of K_n is hamilton-stable if and only if it is C_4-balanced.

Proof. We will consider only the case where $n \geq 5$.

Suppose γ is a C_4-balanced labeling of K_n. By the preceding observation it is
2-stable and hence hamilton-stable.

So, suppose the labeling γ of K_n is hamilton-stable. Let u,v,x,y be vertices in K_n. Let P be a u-v path that contains all vertices of the graph except x and y. Note, if we take the edges of P and add to it ux, xy, and yv then we form a hamiltonian cycle. If we remove ux and yv and replace them with uy and xv then we form a second hamiltonian cycle which differs from the first by only two edges. Hence, $\gamma(ux)+\gamma(yv) = \gamma(uy)+\gamma(xv)$. Accordingly, γ is C_4-balanced.

□

Corollary 17. A labeling of K_n is hamilton-stable if and only if for some even k where $4 \le k \le n$, the labeling is C_k-balanced.

Using techniques similar to those above one can prove this by showing that a graph is C_4-balanced if and only if for some even k where $4 \le k \le n$, the labeling is C_k-balanced.

Theorem 18. Given n and r where rn is even and $1 \le r \le n-2$, a labeling of K_n is hamilton-stable if and only if it is r-stable.

Our proof is similar to one found in [8].

Proof. Remark 9 taken with Theorem 16 yield the necessity in this proof.

So suppose γ is an r-stable labeling of K_n where $1 \le r \le n-2$ and rn is even. We wish to show γ is C_4-balanced and thus, by Theorem

16, is hamilton-stable. In the argument that follows, u,v,w,y shall denote distinct vertices in K_n and also in an r-regular graph H that we will arrange to be an r-factor in K_n. Find two nonadjacent vertices in H and call them u and v.

If N(u)=N(v) then N(u) cannot induce a complete graph, for otherwise some vertex would be adjacent to at least r+1 vertices. Let w and y be nonadjacent vertices in N(u). Let us rename, if necessary, the other vertices of H so that $V(H)=V(K_n)$. Hence, we have an r-factor in K_n and if we remove the edges uw and vy and replace them with uv and wy we create another r-factor which has, by hypothesis, the same weight. Thus $\gamma(uw)+\gamma(vy) = \gamma(wy)+\gamma(uv)$. Hence, γ is C_4-balanced.

If $N(u)\neq N(v)$ then let w be an element of N(u) which is not adjacent to v. Further, let y be an element of N(v) which is not adjacent to u. Removing uw and vy and replacing them with wv and uy produces another r-regular graph. Again, rename the remaining vertices, if necessary to produce an r-factor of K_n. If we remove the edges wu and vy and replace them with wu and vy, we form a second r-factor with the same weight. Thus, $\gamma(uw)+\gamma(vy) = \gamma(wu)+\gamma(yv)$. And again, γ is C_4-balanced. □

The complete graph in Figure 1 is labeled with the integers under normal addition. This is hamilton-stable. Yet, as observed following the statement of Theorem 2, it is not induced. However, if we double the value on each edge, we produce an induced labeling.

Theorem 16, applied to 2-torsion free groups, along with Theorem 15 yields the following.

Corollary 19. A labeling γ of K_n from a 2-torsion free group is hamilton-stable if and only if $2\gamma$is induced.

Suppose a labeling γ of a graph from the group of real numbers has the property that 2γ is induced. Then γ is also induced. Hence, the following result which was discovered and partially discovered in a variety of forms [1, 9, 10].

Corollary 20. The complete graph labeled with real numbers is hamilton-stable if and only if it is induced.

Theorem 16 along with Remark 9 gives the following.

Corollary 21. A labelingof K_n is hamilton-stable if and only if for every G and H, subgraphs of K_n which are degree similar, $\omega(G)=\omega(H)$.

Computational Complexity

In this section, we will assume that group operations can be performed in a fixed unit of time. From Theorems 4, 11 and 16 we see that for cubes, complete graphs and balanced complete bipartite graphs, hamilton-stable and C_4-balanced are equivalent notions, provided the graphs are large enough. Hence, we can test labeled graphs of order n in these families in $O(n^4)$ time to see if they are hamilton-stable. From [6] (Proof of Theorem 4) we know that a labeled graph can be checked in $O(n^2)$ time to see if it is induced. Hence, in the case of cubes and complete bipartite graphs, we can test in $O(n^2)$ time for hamilton-stability. This holds true for complete graphs when labeled with elements from a 2-torsion free group.

Suppose G is a graph with a hamiltonian cycle which contains the edge uv. If we replace uv with the subgraph shown in Figure 5 let us call the resulting graph G_{uv}. Note, G_{uv} has at least two hamiltonian cycles. Now, suppose G has e edges and we label them using each of the integers $2^1, 2^2, 2^3, \ldots, 2^e$. Using normal addition, if G has two hamiltonian cycles, their weights will be different. Hence, with this labeling, G will be hamilton-stable if and only if it contains at most one hamiltonian cycle. If we select a designated vertex, say v, and look at each vertex u adjacent with v, we can form G_{uv} in polynomial time. Thus, if there is a polynomial time algorithm that accepts all labeled graphs and determines if they are hamilton-stable, then we can determine in polynomial time if a graph is hamiltonian.

This leads us to define the following problem.

HAMILTON STABLE

Instance: A labeled graph G.
Question: Is G not hamilton-stable?

Theorem 22. With the preceding notation, the Hamilton Stable problem is NP-complete.

Proof. As noted, the problem of determining if a graph is hamiltonian reduces in polynomial time to the Hamilton Stable problem. Now, given two hamiltonian cycles, we can check their weights in polynomial time. Hence, we can demonstrate in polynomial time if the answer to Hamilton Stable is yes.

□

We know (see [5], page 199) that showing a graph is hamiltonian remains NP-complete for bipartite graphs. But if G is bipartite, then G_{uv} is as well. Hence, the Hamilton Stable problem remains NP-complete when restricted to bipartite graphs labeled with integers.

It is also known [5] that determining if a graph is hamiltonian remains NP-complete for planar, cubic, 3-connected graphs where no face is bounded by fewer than five edges. Consider now the graph in Figure 6. Note, there are at least two xy paths that contain all vertices except z. The same can be said about xz and yz paths. Further, if $v \notin \{x, y, z\}$ is a vertex in this graph, then there exist vx,vy and vz paths that are pairwise disjoint, except for v. So suppose G contains v, a vertex of degree three, that is adjacent to x,y and z. Let us replace v with the configuration in Figure 6 and call the resulting graph G'. Note, G is hamiltonian if and only if G' contains at least two hamiltonian cycles. Further, if G is planar, cubic and 3-connected where no face is bounded by fewer than five edges, then so also is G'. Hence we have the following

Theorem 23. The Hamilton Stable problem remains NP-complete when restricted to integer labeled graphs that are planar, cubic, 3-connected and have face girth at least five.

Open Questions

Given a labeled complete graph, can we check in $O(n^2)$ if the graph is hamilton-stable, even if the group contains elements of order two?

This leaves open similar questions for other classes of graphs such as grids in the plane as well as on other surfaces, such as the projective plane, cylinder and torus. More generally, for which families of graphs and which groups will hamilton-stable be equivalent to C_4-balanced or induced? Is there a simple way to characterize such families? Is there a simple way to characterize families of graphs that are hamilton stable if and only if they are induced? This extends a question posed in [10]. We can show that if H and K are degree similar subgraphs of $K_{m,n}$, then H can be transformed into K with a sequence of switchings using only edges in $K_{m,n}$. For which other families does this hold?

In [8] hamilton-stable complete graphs with distinct integer weights are studied. That is, no distinct edges are given the same integer. Can this be extended to other groups? In particular, if a complete graph can be labeled with distinct real numbers so that the labeling is hamilton-stable, can it be similarly labeled with the integers?

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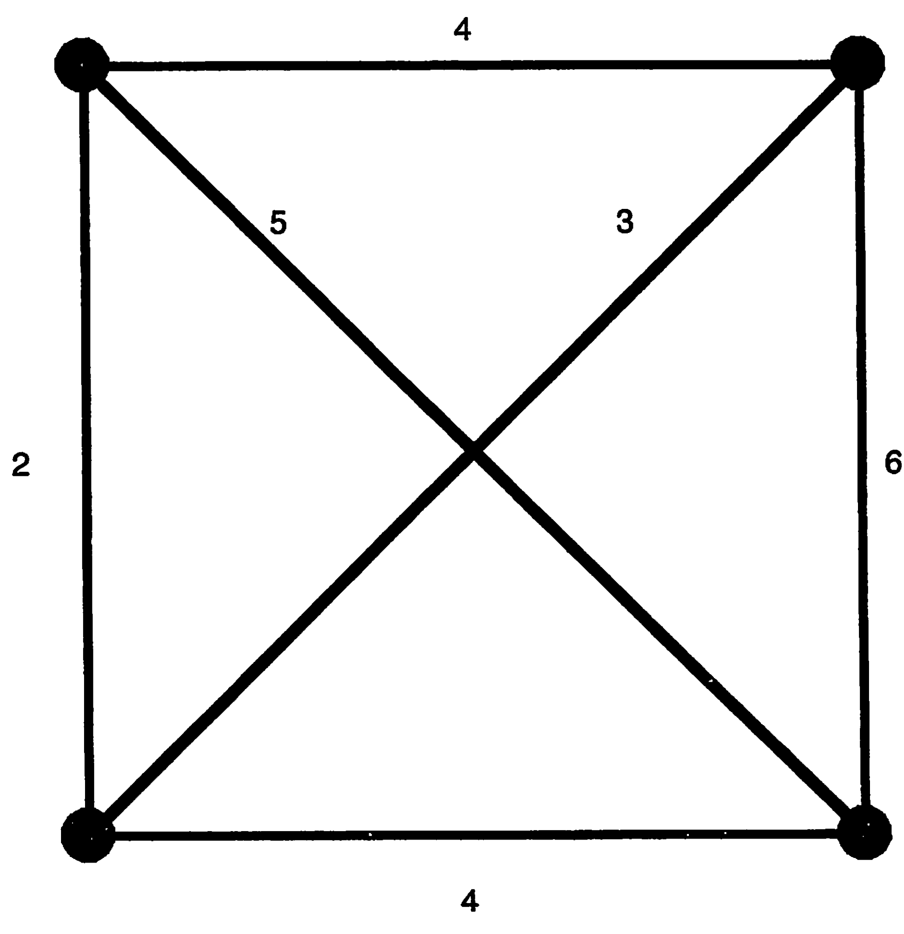

Figure 1

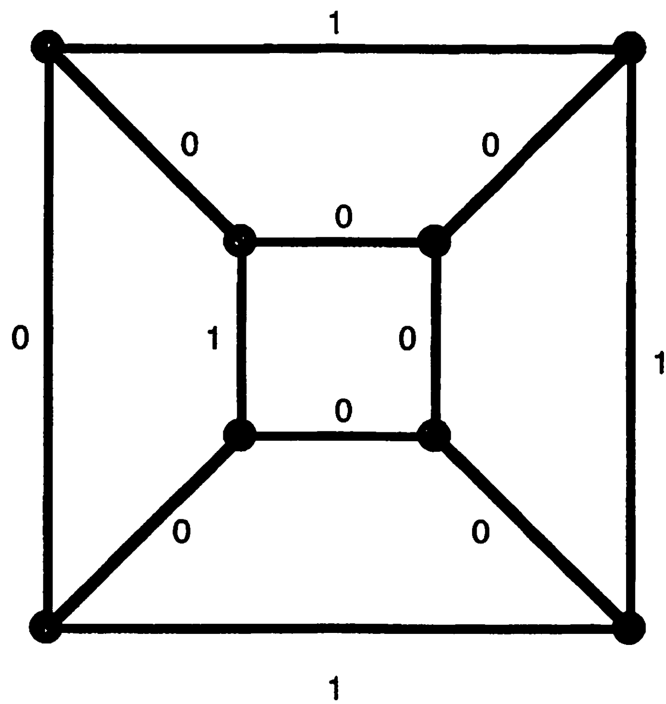

Figure 2

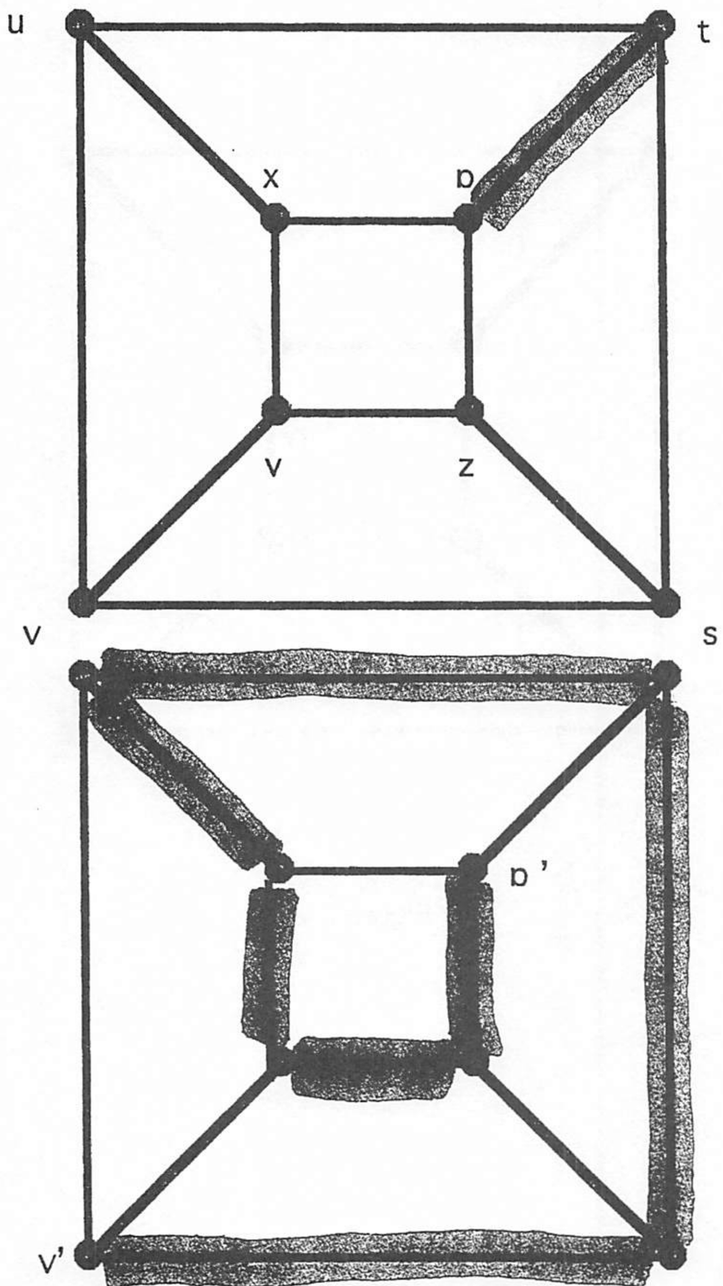

Figure 3

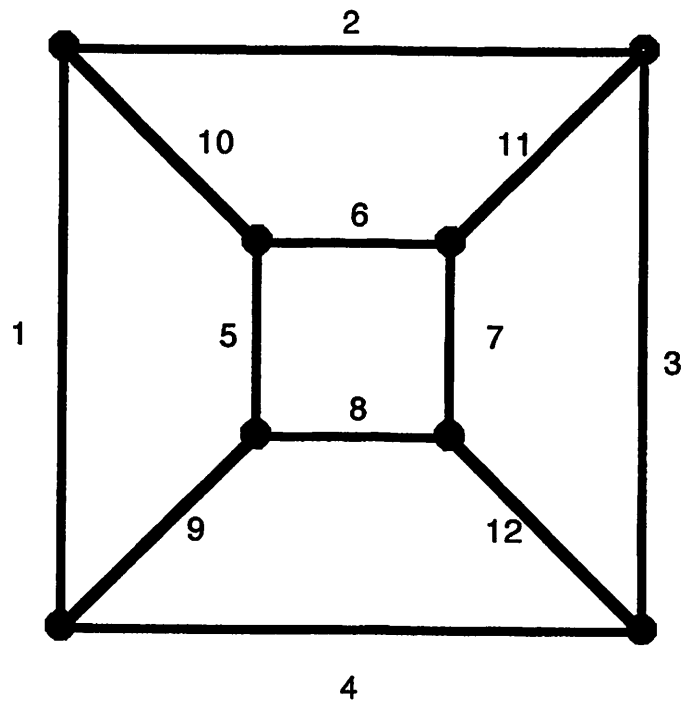

Figure 4

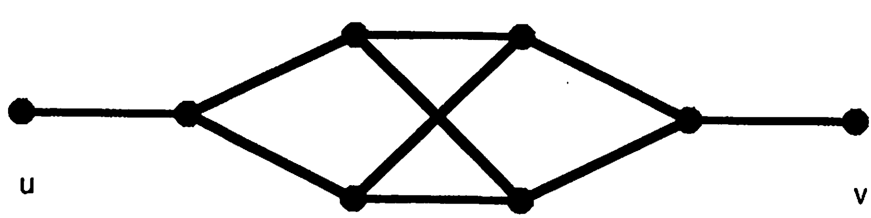

Figure 5

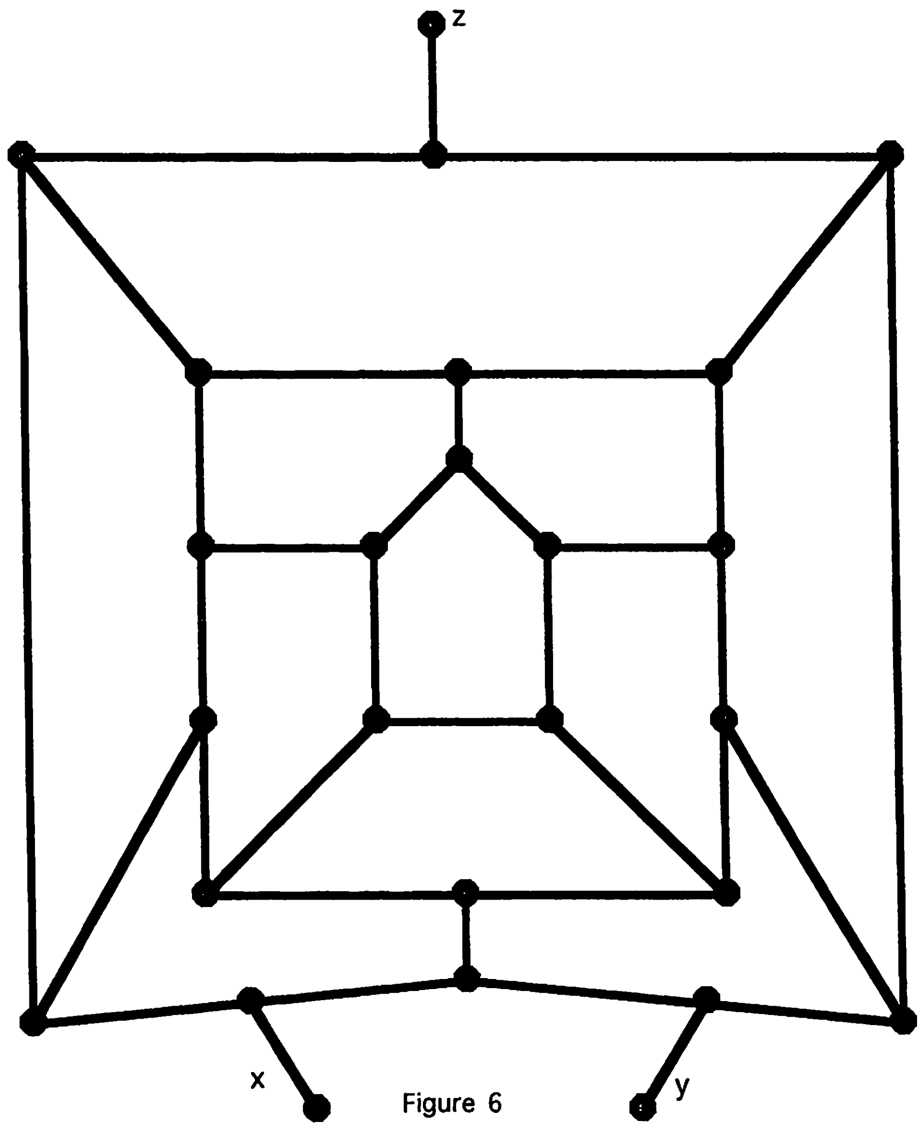

Figure 6