Constrained Ramsey Numbers of Matchings

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Abstract

The rainbow Ramsey number $RR(G_1, G_2)$ or constrained Ramsey number $f(G_1, G_2)$ of two graphs G_1 and G_2 is defined to be the minimum integer N such that any edge-coloring of the complete graph K_N with any number of colors must contain either a subgraph isomorphic to G_1 with every edge the same color or a subgraph isomorphic to G_2 with every edge a different color. This number exists if and only if G_1 is a star *or* G_2 is acyclic. In this paper, we present the conjecture that the constrained Ramsey number of nK_2 and mK_2 is $m(n-1)+2$, along with a proof in the case $m \leq \frac{3}{2}(n-1)$.

keywords: rainbow Ramsey, constrained Ramsey, generalized Ramsey

1 Introduction

The rainbow Ramsey number $RR(G_1, G_2)$ or constrained Ramsey number $f(G_1, G_2)$ of two graphs G_1 and G_2 is defined to be the minimum integer N such that any edge-coloring of the complete graph K_N on N vertices using any number of colors must contain either a subgraph isomorphic to G_1 with every edge the same color or a subgraph isomorphic to G_2 with every edge a different color. Both terms have been used in the literature. For instance, see [11] and [12] for rainbow Ramsey number and [15] for constrained Ramsey number. Since *constrained Ramsey number* seems to be the more commonly used terminology, we will use *constrained* in this paper. Both of these terms are generalizations of a parameter $RM(G)$ defined by Bialostocki and Voxman [1], so we adopt their more descriptive notation $RM(G_1, G_2)$ instead of $f(G_1, G_2)$ for the constrained Ramsey number of G_1 and G_2.

The following existence theorem appears in both [15] and [11], and follows quickly from an earlier result by Erdös and Rado (see [13, p. 129]).

Theorem 1. *The constrained Ramsey number $RM(G_1, G_2)$ exists if and only if G_1 is a star or G_2 is a forest.*

For simplicity, we will say that a graph is *monochromatic* if all of its edges are colored the same color, and we will say that a graph is *rainbow* if all of its edges are colored different colors. Thus, the constrained Ramsey number $RM(G_1, G_2)$ is the minimum N so that any edge-coloring of K_N constains either a monochromatic G_1 or a rainbow G_2. We call a 1-regular graph a *matching*. Notice that any 1-regular graph consists of n disjoint copies of the complete graph on 2 vertices, for some integer n. Such a graph is commonly denoted by nK_2.

The constrained Ramsey number is the natural off-diagonal generalization of a parameter defined by Bialostocki and Voxman[1]. They defined $RM(G)$ for a graph G to be the minimum integer N such that any coloring of the edges of the complete graph K_N with any number of colors must contain either a monochromatic or a rainbow copy of G. This number exists if and only if G is an acyclic graph. One of the major results in [1] is the following.

Theorem 2 (Bialostocki, Voxman). *For every positive integer n, the number*

$$RM(nK_2) = n(n-1) + 2.$$

In this paper, we will consider the natural generalization of this result for the constrained Ramsey number, that is, we consider $RM(nK_2, mK_2)$.

1.1 Constrained Ramsey Numbers and Matchings

In [8], Cockayne and Lorimer presented a formula for the generalized Ramsey number for matchings:

Theorem 3 (Cockayne,Lorimer). *For any positive integers $c, n_1, n_2, \ldots, n_c$, where $n_1 \geq n_i$ for $2 \leq i \leq c$, the generalized Ramsey number*

$$r(n_1K_2, n_2K_2, \ldots n_cK_2) = n_1 + 1 + \sum_{i=1}^{c}(n_i - 1). \quad (1)$$

In particular, if $n_1 = n_2 = \ldots = n_c$, we have

Corollary 1. *If n is any positive integer, then*

$$r(nK_2, nK_2, \ldots, nK_2) = (c+1)(n-1) + 2.$$

We also have the following corollary.

Corollary 2. *For any positive integers n and m,*

$$RM(nK_2, mK_2) \geq m(n-1) + 2.$$

Proof. A graph colored with c or fewer colors cannot possibly contain a rainbow copy of $(c+1)K_2$. If the graph is colored with $c+1$ or more colors, then such a subgraph is possible. Thus, taking $m = c+1$,

$$RM(nK_2, mK_2) \geq r(nK_2, nK_2, \dots nK_2) = m(n-1) + 2.$$

□

We may also easily see the inequality $RM(nK_2, mK_2) \geq m(n-1)+2$ directly. Color the graph $K_{m(n-1)+1}$ as follows. Color all of the edges of a subgraph isomorphic to K_{2n-1} with color 1. Choose $n-1$ additional vertices and color all of the edges among these vertices and between these vertices and those already colored with color 2. For each color $i = 3, 4, \dots m-1$, choose $n-1$ additional vertices and color the edges among those vertices and between those vertices and the part of the graph already colored with color i. The resulting graph has $2n-1+(m-2)(n-1) = m(n-1)+1$ vertices and contains no set of n independent edges in the same color. Since only $m-1$ colors appear, it also cannot contain a set of m independent edges in different colors.

In the case when $m = n$, Theorem 2 shows that this inequality is in fact an equality[1].

We suspect that this result can be generalized as follows:

Conjecture 1. *For every pair of positive integers n and m, where $n \geq 3$ and $m \geq 2$,*

$$RM(nK_2, mK_2) = m(n-1) + 2.$$

First, we handle the trivial special cases $n = 1$, $n = 2$, and $m = 1$ not included in the conjecture. Any graph with at least one edge must contain both a monochromatic and a rainbow K_2, so $RM(K_2, mK_2) = RM(nK_2, K_2) = 2$. If a graph contains at least n independent edges, then either two of the edges are different colors or all of them are the same color. Thus, $RM(nK_2, 2K_2) = 2n$. Similarly, if a graph contains at least m independent edges, then it must contain either a rainbow mK_2 or a monochromatic $2K_2$. However, a graph with fewer than $2m$ vertices could be colored with every edge a different color to avoid these two graphs. Therefore, $RM(2K_2, mK_2) = 2m$.

Bialostocki and Voxman's proof can be adapted to show Conjecture 1 in the case $m < n$.

Theorem 4. *For any two positive integers n and m, where $2 \leq m < n$,*

$$RM(nK_2, mK_2) = m(n-1) + 2.$$

Proof. We will proceed by induction on m. The formula holds when $m = 2$, as discussed above. For some $m \geq 3$, suppose the edges of $K_{m(n-1)+2}$ are colored with any number of colors. If fewer than m colors are used, then we may apply Corollary 1 with $c = m - 1$ to see that some monochromatic copy of nK_2 must appear. Thus, we may assume without loss of generality that at least m colors are used.

Choose one edge of each of m different colors that appear in such a way that the number of independent edges in this set is maximal. Let H represent these edges and let $V(H)$ represent the vertices incident with these edges. If $|V(H)| = 2m$, then we have a rainbow copy of mK_2 and we are done. Assume that $|V(H)| \leq 2m - 1$.

Let $M = V(K_{m(n-1)+2}) - V(H)$. If there is any color which appears in the graph induced by M and not in H, then the number of independent edges in H is not maximal, which contradicts our choice of H. If *every* color which appears in H also appears in M, then we may choose some color in H which does not appear on an independent edge and replace that edge with an edge of the same color in M to produce a set of representatives of the colors with more independent edges than H. Again, this contradicts our choice of H. Thus, the colors appearing in M must be a proper subset of the set of colors appearing in H.

Since $m < n$, the set M contains at least

$$\begin{aligned} |M| &\geq (n-1)m + 2 - (2m-1) \\ &= nm - 3m + 3 \\ &\geq nm - 2m - n + 1 + 3 \\ &= (n-2)(m-1) + 2 \end{aligned}$$

vertices. Therefore, by the inductive hypothesis, the subgraph generated by M contains either a monochromatic copy of $(n-1)K_2$ or a rainbow copy of $(m-1)K_2$. Since H contains one edge of each color appearing in M and at least one edge of a color not appearing in M, we may add an edge from H to the subgraph in M to produce either a monochromatic nK_2 or a rainbow mK_2. □

Next we will show that the same formula holds for $m = n + 1$. Two of the smaller values must be shown separately.

Theorem 5. *The constrained Ramsey number* $RM(3K_2, 4K_2) = 10$.

Proof. By Corollary 2, we know that $RM(3K_2, 4K_2) \geq 10$. Suppose the edges of K_{10} are colored with any number of colors. Consider any set of 5 independent edges, say ab, cd, ef, gh and ij. If 4 or more colors appear, or if some color appears at least 3 times, we are done. Without loss of generality, we may assume that the edges ab, cd, ef, gh and ij are colored with colors $1, 1, 2, 2$, and 3, respectively.

Notice that if color 3 is used on any of the edges ac, bd, ad, bc, then it cannot be used on any of the edges eg, fh, eh, fg without creating a monochromatic $3K_2$ in color 3. Thus, we may assume that this color appears on at most one of these sets of four edges. Assume without loss of generality that color 3 does not appear on the edges ac, bd, ad, bc. Notice that color 2 cannot appear on these edges either without creating a monochromatic $3K_2$.

Case 1. **One of the edges ac, bd, ad, bc is some new color.** Suppose without loss of generality that ac is a new color, color 4. Since ac, bd, ef, and ij are independent edges, edge bd must be one of the colors $2, 3$ or 4, or else we have a rainbow $4K_2$.

We may assume that bd is color 4. If the edge ce is any color except 2 or 3, then we have a rainbow $4K_2$, using either ab or bd along with ce, gh, and ij. Similarly, we may assume that df is colored either 2 or 3. If ce and df are the same color, then together with either gh or ij they form a monochromatic $3K_2$. Thus, without loss of generality, ce is color 3 and df is color 2.

By the same argument, one of the edges ag and bh is color 2 and the other is color 3. However, we now have $3K_2$ in color 3.

Case 2. **The edges ac, bd, ad, bc are all color 1.** If any edge from the set of vertices a, b, c, d to the set e, f, g, h is a new color, then we have a rainbow $4K_2$.

Consider the edges ae, cg, bf, and dh, colored in the three colors $1, 2, 3$. If color 1 appears twice, then we have $3K_2$ in color 1. Similarly, if color 3 appears twice, we have a monochromatic $3K_2$. If color 2 appears twice incident with ef or twice incident with gh, then we have $3K_2$ in color 2. We may assume that color 2 appears twice, once incident with the edge ef and once incident with gh. Without loss of generality, edges ae and cg are color 2, edge bf is color 1 and edge dh is color 3.

Consider edge ai. If this edge is in some new color, then ai, cg, bf and dh form a rainbow $4K_2$. If it is color 1, then it forms a monochromatic $3K_2$ along with bf and cd. If it is color 2, then it forms a monochromatic $3K_2$ along with ef and gh. Thus, we may assume without loss of generality that edge ai is color 3. Similarly, we may assume that edge cj is color 3. But then edges ai, cj and dh form a monochromatic $3K_2$. □

Theorem 6. *The constrained Ramsey number $RM(4K_2, 5K_2) = 17$.*

Proof. The lower bound follows from Corollary 2.

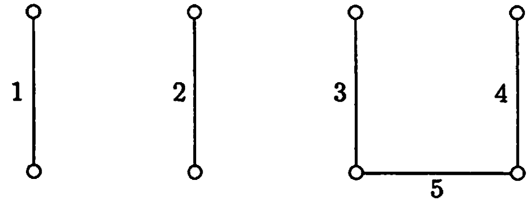

Figure 1: Possible Location for Edge of Color 5 in Theorem 6

Suppose that the edges of K_{17} are colored with any number of colors. If 4 or fewer colors are used, then by Corollary 1, there is a monochromatic subgraph isomorphic to $4K_2$. Thus, we may assume that at least 5 colors are used.

Since $RM(4K_2, 4K_2) = 14 \leq 17$, we may also assume without loss of generality that there is a rainbow subgraph isomorphic to $4K_2$; we will label the colors 1, 2, 3, and 4. Some color 5 must appear somewhere in the graph. If color 5 appears on an edge independent from the edges of the $4K_2$, we are done.

Suppose an edge of color 5 appears incident with two of the edges of the $4K_2$, as shown in Figure 1. Since $RM(3K_2, 3K_2) = 8 \leq 9$, there must be either a monochromatic or a rainbow $3K_2$ on the remaining 9 vertices. If there is a monochromatic $3K_2$ in some new color, then we have a rainbow $5K_2$ in colors 1, 2, 3, 4, and this new color. If there is a monochromatic $3K_2$ in one of the colors 1, 2, 3, 4, or 5, then we may add the appropriate edge to obtain a monochromatic $4K_2$. Thus, we may assume wlog that there is a rainbow $3K_2$, necessarily using three of the four colors 1, 2, 3, and 4. In particular, there is an edge in color 3 or an edge in color 4, so, up to interchanging colors, we may assume that we have a subgraph as shown in Figure 2.

Let $N = V(K_{17}) - \{a, b, c, d, e, f, g, h, i\}$. If N contains an edge in any color other than 1, 2, and 3, then we have a rainbow $5K_2$. Since $|N| = 8 = RM(3K_2, 3K_2)$, there must be either a monochromatic $3K_2$ in color 1, 2, or 3 or a rainbow $3K_2$ on colors 1, 2, and 3 on N. If N contains a monochromatic $3K_2$, then we have a monochromatic $4K_2$ in the original graph. Thus, we may assume that N contains three independent edges in colors 1, 2, and 3, respectively. The remaining independent edge in N must be color 1, 2, or 3, say wlog color 1. Without loss of generality, we have the graph shown in Figure 3.

Let $M = V(K_{17}) - \{a, b, c\}$. Since $|M| = 14 = RM(4K_2, 4K_2)$, we may assume wlog that M contains a rainbow $4K_2$. If this $4K_2$ does not contain

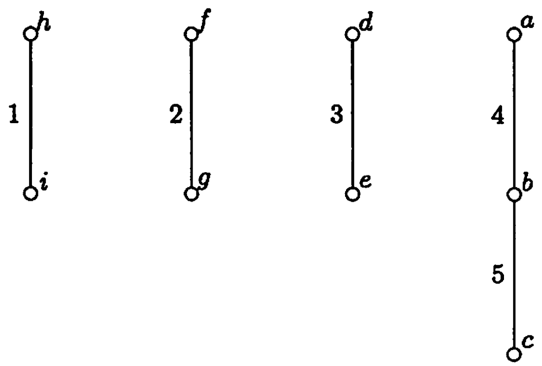

Figure 2: Other Possible Location for Edge of Color 5 in Theorem 6

an edge of color 4 and an edge of color 5, then we may add edge *bc* or edge *ab* to obtain a rainbow $5K_2$. Thus, we may assume that an edge of color 4 and an edge of color 5 appear in M.

If the color 4 edge appears anywhere in M besides the edges ng, nf, og, of, pd, pe, qe, and/or qd, then we have a rainbow $5K_2$. Without loss of generality, we may assume that edge ng is color 4.

Consider edge op. If op is color 1, then we have a $4K_2$ in color 1. If op is color 2, 4, or 5, or some new color, then we have a rainbow $5K_2$. Thus, op must be color 3. Similarly, oq, oe, od, fp, fq, fe, and fd must all be color 3.

Consider edge qd. If qd is color 1, we have a monochromatic $4K_2$ in color 1; if qd is color 2, 4, or 5, or some new color, then we have a rainbow $5K_2$. Thus, qd and, similarly, edges qe, pe, and pd must all be color 3.

Now, if any edge on the vertices h, i, j, k, l, and m is color 3, we have a $4K_2$ in color 3. If any one of these edges is color 2, 4, or 5 or some new color, then we have a rainbow $5K_2$. Thus, we may assume that vertices h, i, j, k, l, and m induce a complete graph in color 1.

Finally, consider the six edges hd, ie, jf, ko, lp, and mq. If two or more of these edges are color 1 or if two or more are color 3, then we have a monochromatic $4K_2$. If any one of these edges is color 2, 4, or 5, or a new color, then we have a rainbow $5K_2$. There are no other possibilities; we must have either a monochromatic $4K_2$ or a rainbow $5K_2$. □

The proof for $n \geq 5$ and $m = n + 1$ actually shows a slightly more general case. First, we will need a few technical lemmas.

Lemma 1. *Assume that* $RM(nK_2, (m-1)K_2) = (m-1)(n-1)+2$. *Suppose* $K_{m(n-1)+2}$ *is edge-colored with any number of colors. Then ei-*

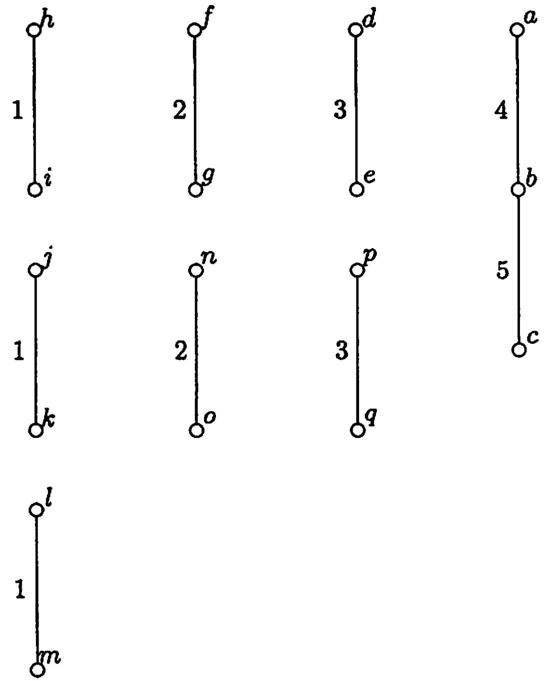

Figure 3: Subgraph Which Must Exist, WLOG, in Theorem 6

ther $K_{m(n-1)+2}$ contains a monochromatic nK_2 or a rainbow mK_2, or any set of independent edges in a given color can be extended to a set of $\lceil \frac{n}{2} \rceil$ independent edges in that color.

Proof. Suppose there is a set of k independent edges in the same color, say color 1. Let M be the set of $2k$ vertices incident with these edges. If

$$\begin{aligned} 2k &\leq m(n-1)+2-RM(nK_2,(m-1)K_2) \\ &= m(n-1)+2-[(m-1)(n-1)+2] \\ &= n-1, \end{aligned}$$

then we may assume that there is either a monochromatic nK_2 or a rainbow $(m-1)K_2$ on the remaining vertices. If the rainbow $(m-1)K_2$ does not contain color 1, then we may add an edge in color 1 to produce a rainbow mK_2. Otherwise, the rainbow $(m-1)K_2$ contains an edge in color 1 independent from the edges in M. We may add the vertices incident with this edge to M and repeat the argument. Continuing in this fashion, we can extend the set M until $|M| = 2k$, where $2k > n-1$, that is, until $k > (n-1)/2$. □

We will primarily use this lemma in the following form.

Corollary 3. *Assume* $RM(nK_2, (m-1)K_2) = (m-1)(n-1)+2$ *and* $n \geq 5$. *If* $K_{m(n-1)+2}$ *is edge-colored with any number of colors, then either the graph contains a monochromatic* nK_2 *or a rainbow* mK_2, *or any edge or pair of independent edges in a single color can be extended to a set of three independent edges in that color.*

Lemma 2. *Assume that* $RM(nK_2, pK_2) = p(n-1)+2$ *for every positive integer* $p < m$. *Suppose* $K_{m(n-1)+2}$ *is edge-colored with any number of colors and suppose the resulting graph does not contain either a monochromatic* nK_2 *or a rainbow* mK_2. *If* M *is a set of vertices and* S *is a set of* c *colors,* $c \geq 1$, *such that*

(1) there is a set of c *independent edges on the vertices of* M *containing an edge in each color of* S and

(2) $|M| \leq c(n-1)$,

then there is an edge in $K_{m(n-1)+2}$ *independent of* M *colored with one of the colors of* S.

Proof. Let M be such a set. Since

$$\begin{aligned} |M| &\leq c(n-1) \\ &= (m(n-1)+2) - ((m-c)(n-1)+2) \\ &= (m(n-1)+2) - RM(nK_2, (m-c)K_2), \end{aligned}$$

the remainder of the graph must contain either a monochromatic nK_2 or a rainbow $(m-c)K_2$. If none of the colors of S appear in the rainbow $(m-c)K_2$, then it can be extended to a rainbow mK_2. Thus, we may assume that there is a rainbow $(m-c)K_2$ independent from M containing an edge in one of the colors of S. □

We are now ready to prove the main result. Notice that for $n \geq 5$, we have $n+1 \leq \frac{3}{2}(n-1)$.

Theorem 7. *For* $n \geq 5$ *and* $2 \leq m \leq \frac{3}{2}(n-1)$, *the constrained Ramsey number*

$$RM(nK_2, mK_2) = m(n-1)+2$$

Proof. Notice that $RM(nK_2, mK_2) \geq m(n-1)+2$ by Corollary 2, so we only need show $RM(nK_2, mK_2) \leq m(n-1)+2$. We proceed by strong induction on m, using Theorems 2 and 4 as the base. Thus, we assume that the formula holds for $RM(nK_2, pK_2)$ for all $p < m$ and that $m > n \geq 5$. Suppose $K_{m(n-1)+2}$ is edge-colored with any number of colors. Since $m(n-1)+2 \geq (m-1)(n-1)+2 = RM(nK_2, (m-1)K_2)$, we may assume without loss of generality that there is a rainbow $(m-1)K_2$, say in colors $\{1, 2, \ldots m-1\}$. Now, since $m \leq 2(n-1)$, it follows that there are at least $m(n-1)+2-2(m-1) \geq (m-2)(n-2)+2 = RM((n-1)K_2, (m-2)K_2)$ vertices remaining. If a monochromatic $(n-1)K_2$ appears in a new color, then we may add an edge in this new color to the rainbow $(m-1)K_2$ to produce a rainbow mK_2. If a monochromatic $(n-1)K_2$ appears in one of the colors $1, 2, \ldots m-1$, then this subgraph along with the appropriate edge from the rainbow $(m-1)K_2$ yields a monochromatic nK_2.

Thus, we may assume without loss of generality that a rainbow $(m-2)K_2$ appears, independent from the $(m-1)K_2$. If any new color appears on this $(m-2)K_2$, then we have a rainbow mK_2. Thus, without loss of generality, we may assume that the $(m-2)K_2$ is colored with colors $1, 2, \ldots m-2$.

Since $m \leq (3/2)(n-1)$, there are at least $m(n-1)+2-2(m-1)-2(m-2) \geq (m-3)(n-3)+2 = RM((n-2)K_2, (m-3)K_2)$ vertices remaining. If there is a monochromatic $(n-2)K_2$ on these vertices in one of the colors $1, 2, \ldots m-2$, then we have a monochromatic nK_2. If, on the other hand, there is a monochromatic $(n-2)K_2$ or a rainbow $(m-3)K_2$ containing some new color, then we have a rainbow mK_2. Thus, we may assume, without loss of generality, that we have one of the following three cases.

Case 1 **There is a monochromatic $(n-2)K_2$ in color $m-1$.** Label the vertices as shown in Figure 4, so that edges u_iv_i and w_ix_i are color i for $1 \leq i \leq m-2$.

From corollary 1, if only $m-1$ colors were used to color the edges of $K_{m(n-1)+2}$, then there must be a monochromatic nK_2. Thus, we may assume that there is some new color, say color m, appearing on these vertices. According to corollary 3, we may also assume that this color appears on at least 3 independent edges. If any edge in color m is not an edge u_iw_i, u_ix_i, v_iw_i or v_ix_i for some i, $1 \leq i \leq m-2$, then we have a rainbow mK_2. At most 2 of the 3 independent edges in color m can appear incident with u_i, v_i, w_i and x_i for any given i. Thus, we may assume without loss of generality that edges v_1w_1 and v_2w_2 are color m.

We will proceed by induction. Let

$$M_{\leq i} = \{u_j, v_j, w_j, x_j | 1 \leq j \leq i\}$$

Then the graph induced by $M_{\leq 2}$ contains a pair of independent edges in *any* two of the three colors $1, 2$, and m, that is, it contains two independent edges in colors 1 and 2, two independent edges in colors 1 and m, and two independent edges in colors 2 and m.

Suppose, for any i, $1 \leq i \leq m-2$, that the graph induced by $M_{\leq i}$ contains a set of i independent edges in *any* i of the colors $1, 2, \ldots i$, and m. Since $|M_{\leq i}| = 4i$, we may apply lemma 2 with $c = i$ and $S = \{1, 2, \ldots i\}$. Since $n \geq 5$, we have $4i \leq c(n-1)$. Thus, there must be some edge independent from $M_{\leq i}$ in one of the colors $1, 2, \ldots i$. If this edge is not u_jw_j, u_jx_j, v_jw_j or v_jx_j for some j, where $i < j \leq m-2$, then we have a rainbow mK_2 using this edge in, say, color k, a matching on $M_{\leq i}$ in the colors $\{1, 2, \ldots i, m\} - \{k\}$, and a matching in the remainder of the graph in colors $i+1, i+2, \ldots m-1$. Thus, we may assume without loss of generality that the new edge in color k, $1 \leq k \leq i$, is the edge $v_{i+1}w_{i+1}$. Let C be any subset of $i+1$ colors from the set $\{1, 2, \ldots i+1, m\}$. If C contains color $i+1$, then the graph induced by $M_{\leq i+1}$ contains a set of independent edges in the colors of C, since $M_{\leq i}$ contains a set of independent edges in colors $C - \{i+1\}$. If C does not contain color $i+1$, then $C = \{1, 2, \ldots i, m\}$. Since the graph induced by $M_{\leq i}$ contains a set of independent edges in colors $\{1, 2, \ldots i, m\} - \{k\}$, the graph induced by $M_{\leq i+1}$ contains a set of independent edges in the colors of C.

Continuing inductively, we may assume that $M_{\leq m-2}$ contains a set of $m-2$ independent edges in *any* $m-2$ of the colors $\{1, 2, \ldots m-2, m\}$. If we apply lemma 2 with $c = m-2$ and $S = \{1, 2, \ldots m-2\}$, then we may assume that there is an edge independent from $M_{\leq m-2}$ in one of the colors $1, 2, \ldots m-2$. Then this edge, say in color k, an independent edge in color $m-1$, and a set of independent edges in $M_{\leq m-2}$ in colors $\{1, 2, \ldots m-2, m\} - \{k\}$ form a rainbow mK_2.

***Case 2* There is a rainbow $(m-3)K_2$ not containing color $m-1$.**
Without loss of generality, we may assume that there is a subgraph as shown in Figure 5. As in case 1, we may assume that some new color, say m, appears on at least three independent edges. If any edge in this new color is not adjacent to either the edge in color $m-1$ shown in Figure 5 or *both* of the edges of color $m-2$, then we have a rainbow mK_2. Since at most two independent edges can be adjacent to the edge in color $m-1$, we may assume that at least one edge of color m appears adjacent to both edges of color $m-2$.

Let M be the set of vertices incident with the edges of colors $m-2$ and $m-1$ shown in the figure. We may apply lemma 2 with $c = 2$ and $S = \{m-2, m-1\}$. Since $6 \leq 2(n-1)$ for $n \geq 5$, we may assume that there is an edge in color $m-1$ or color $m-2$ independent from M. If an edge in color $m-2$ appears, then we have a rainbow mK_2; we may assume that an edge in color $m-1$ appears. Let M' be the set of vertices in M along with the two endpoints of this new edge of color $m-1$. Apply lemma 2 to M' with $c = 2$ and $S = \{m-2, m-1\}$, since $8 \leq 2(n-1)$ for $n \geq 5$. Thus, there must be another edge in color $m-1$ independent from M'.

Now, from corollary 3, we may also assume that there is an edge in

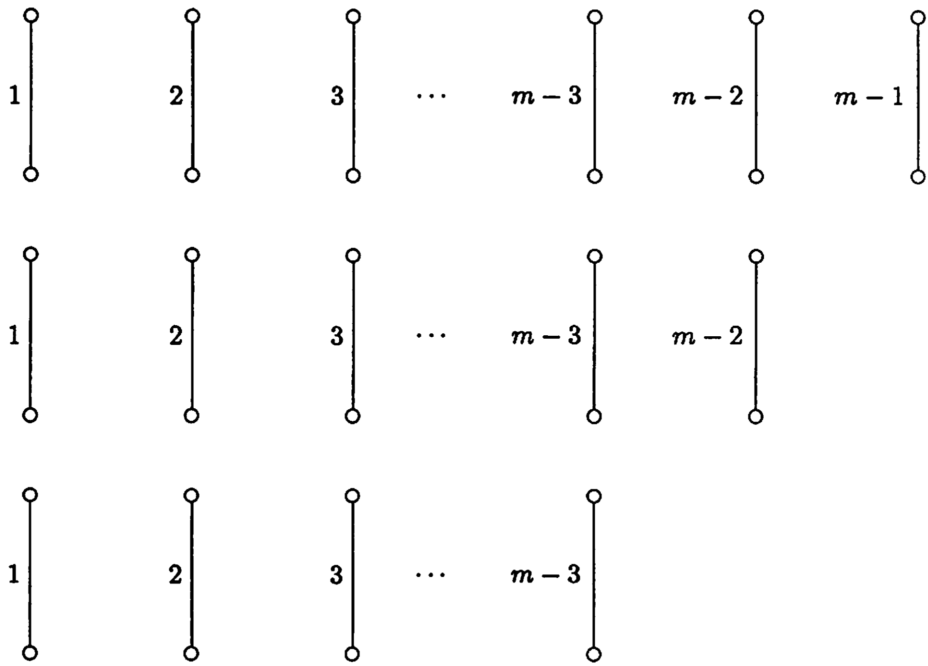

Figure 5: Case 2 of Theorem 7

color $m-2$ independent from the two edges in that color shown in Figure 5. If this edge is not adjacent to the edge in color $m-1$, then we have a rainbow mK_2. So we may assume that there is an edge in color $m-2$ adjacent to the edge of color $m-1$. Since there are two independent edges in $V(K_N)-M$ in color $m-1$, there is an edge in color $m-1$ independent from this new edge in color $m-2$. Consider these two edges in colors $m-1$ and $m-2$, respectively, and the edge of color m. If there is still a set of $m-3$ independent edges in colors $1, 2, \ldots m-3$ on the remainder of the graph, then we have a rainbow mK_2.

Since we are using three vertices of $V(K_N)-M$, it is possible that these three vertices are incident with three different edges in the same color, say color $m-3$. Let L be the set of vertices in M along with the 6 vertices adjacent to the edges in color $m-3$. We may apply lemma 2 to L with $S = \{m-3, m-2, m-1\}$. Since $12 \leq 3(n-1)$ for $n \geq 5$, there must be some edge independent from L in one of these three colors. Observe that with this edge and the edges in L, we can obtain an independent set of edges in colors $m-3, m-2, m-1$ and m. There must be an independent set of edges in colors $1, 2, \ldots m-4$ on the vertices remaining, so we have a rainbow mK_2.

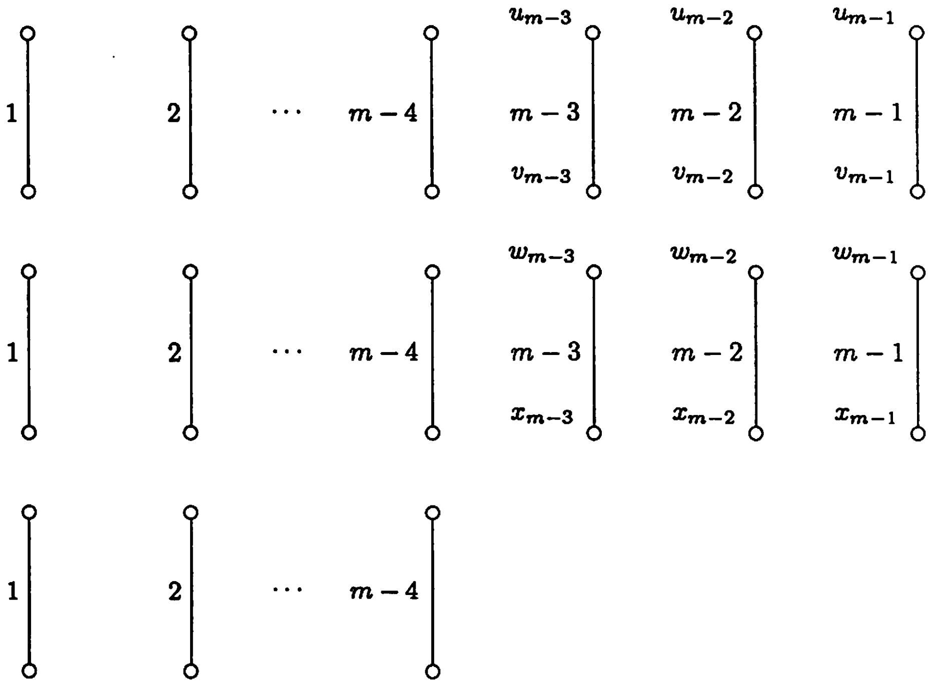

Figure 6: Case 3 of Theorem 7

Case 3 **There is a rainbow $(m-3)K_2$ containing color $m-1$.** We may assume that we have the graph shown in Figure 6, with edges u_iv_i and w_ix_i in color i, for $i = m-3, m-2, m-1$.

As in the previous two cases, we may assume that there is some new color, say color m, appearing on at least three independent edges. If any edge in color m is not one of the edges u_iw_i, u_ix_i, v_iw_i or v_ix_i for $i = m-3, m-2$, or $m-1$, then we have a rainbow mK_2. Since at most two independent edges can be chosen from $\{u_iw_i, u_ix_i, v_iw_i, v_ix_i\}$ for each i, we may assume without loss of generality that edges $v_{m-2}w_{m-2}$ and $v_{m-1}w_{m-1}$ are color m.

Let $M = \{u_{m-2}, v_{m-2}, w_{m-2}, x_{m-2}, u_{m-1}, v_{m-1}, w_{m-1}, x_{m-1}\}$. If we apply lemma 2 to M with $c = 2$ and $S = \{m-2, m-1\}$, we have some edge in color $m-2$ or $m-1$ independent from M. If this edge is not one of the edges $u_{m-3}w_{m-3}$, $u_{m-3}x_{m-3}$, $v_{m-3}w_{m-3}$ or $v_{m-3}x_{m-3}$, then we have a rainbow mK_2. Assume wolog that edge $v_{m-3}w_{m-3}$ is color $m-2$ or $m-1$. Let $M' = \{u_i, v_i, w_i, x_i | i = m-3, m-2, m-1\}$, and let $S = \{m-3, m-2, m-1\}$. According to lemma 2, there is some edge in one of the colors $m-3, m-2, m-1$ independent from M'. Thus, there is a rainbow mK_2. □

We have seen that the formula

$$RM(nK_2, mK_2) = m(n-1) + 2$$

from Conjecture 1 holds for $m \leq \frac{3}{2}(n-1)$. In general, for $n \geq 2$, we have

$$m(n-1) + 2 \leq RM(nK_2, mK_2) \leq 2(n-1)m$$

The lower bound was discussed previously. Notice that the upper bound holds for $n = 2$ and for $m = 1$ provided $n \geq 2$. For any $n \geq 3$ and $m \geq 2$, suppose $RM(nK_2, (m-1)K_2) \leq 2(n-1)(m-1)$ and $RM((n-1)K_2, mK_2) \leq 2(n-2)m$. Consider any edge-coloring of $K_{2(n-1)m}$. If the resulting graph does not contain a rainbow mK_2, then without loss of generality it must contain a monochromatic $(n-1)K_2$. If we remove these $2(n-1)$ vertices, there are $2(n-1)(m-1)$ vertices remaining. Thus, there is either a monochromatic nK_2 or a rainbow $(m-1)K_2$ on the remaining vertices. Without loss of generality, then, we have a monochromatic $(n-1)K_2$, say in color c, and a disjoint rainbow $(m-1)K_2$. Either the rainbow $(m-1)K_2$ contains an edge in color c or it does not. If it contains an edge in color c, then this edge along with the monochromatic $(n-1)K_2$ form a monochromatic nK_2. Otherwise, an edge in color c from the $(n-1)K_2$ may be added to the rainbow $(m-1)K_2$ to produce a rainbow mK_2.

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