# A Note on 3-Equitable Labelings Of Multiple Shells

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**Abstract**: Let $G$ be a graph with vertex set $V$ and edge set $E$. A vertex labelling $f : V \longrightarrow \{0,1,2\}$ induces an edge labelling $\overline{f} : E \longrightarrow \{0,1,2\}$ defined by $\overline{f}(uv) = \mid f(u) - f(v) \mid$. Let $v_f(0), v_f(1), v_f(2)$ denote the number of vertices $v$ with $f(v) = 0, f(v) = 1$ and $f(v) = 2$ respectively. Let $e_f(0), e_f(1), e_f(2)$ be similarly defined. A graph is said to be **3-equitable** if there exists a vertex labeling $f$ such that $\mid v_f(i) - v_f(j) \mid \leq 1$ and $\mid e_f(i) - e_f(j) \mid \leq 1$ for $0 \leq i, j \leq 2$. In this paper, we show that every multiple shell $MS\{n_1^{t_1}, \cdots, n_r^{t_r}\}$ is 3-equitable for all positive integers $n_1, \cdots, n_r, t_1, \cdots, t_r$.

## INTRODUCTION

Throughout this paper, all graphs are finite, simple and undirected. Let $V(G)$ and $E(G)$ denote the vertex set and the edge set of a graph $G$. A mapping $f : V(G) \longrightarrow \{0,1,2\}$ is called a ternary vertex labeling of $G$ and $f(v)$ is called the label of the vertex $v$ under $f$. For an edge $e = u\ v$, the induced edge labeling $f : E(G) \longrightarrow \{0,1,2\}$ is given by $\overline{f}(e) = \mid f(u) - f(v) \mid$. Let $v_f(0), v_f(1), v_f(2)$ be the number of vertices of $G$ having labels 0, 1 and 2 respectively under $f$ and let $e_f(0), e_f(1), e_f(2)$ be the number of edges having labels 0, 1 and 2 respectively under $f$.

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**Definition:** A ternary vertex labeling of a graph $G$ is called a **3-equitable labeling** if $| v_f(i) - v_f(j) | \leq 1$ and $| e_f(i) - e_f(j) | \leq 1, 0 \leq i, j \leq 2$. A graph $G$ is called **3-equitable** if it admits a 3-equitable labeling.

$k$-equitable graphs were first introduced by Cahit.[3] By cordial graphs he means 2-equitable graphs. In the same paper he proved that the cycle $C_n$ and the wheel $W_n$ are 3-equitable iff $n$ is not $\equiv 3 (\bmod 6)$. In [2] Bapat and Limaye proved that following families are 3-equitable: (1) Helm graph $H_n, (n \geq 4)$, (2) A Flower graph $FL_n$, (3) One point union $H_n^{(k)}$ of $k$-copies of $H_n, k \geq 1$, (4) One point union $K_4^{(k)}$ of $k$ copies of $K_4$, (5) A $K_4$-snake of $n$ blocks, each equal to $K_4$, (6) A $C_t$- snake of $n$ blocks $t = 4, 6$ and $t = 5$ with $n$ not congruent to 3 modulo 6.

**Definition:** A shell $S_n$ of width $n$ is a graph obtained by taking $n - 3$ concurrent chords in a cycle $C_n$ on $n$ vertices. The vertex at which all the chords are concurrent is called the **apex**. The two vertices on $C_n$ adjacent to the apex have degree 2, the apex has degree $n - 1$ and all the other vertices have degree 3 each.

The shell $S_n$, also called a fan $f_{n-1}$, was proved to be cordial for all $n \geq 4$.[3]

**Definition:** A multiple shell $MS\{n_1^{t_1}, \cdots, n_r^{t_r}\}$ is a graph formed by $t_i$ shells of width $n_i$ each, $1 \leq i \leq r$, which have a common apex.

Thus a multiple shell is a one point union of many shells. If there are in all $k$ shells with a common apex, then it is called a $k$-tuple shell. If $k = 2$ (respectively $k = 3$) we call such a shell a **double** (respectively **triple** ) Shell. A multiple shell is said to be balanced if it is of the form $MS\{p^t\}$ or of the form $MS\{p^t, (p+1)^s\}$. The balanced multiple shell $MS\{n^t\}$ was proved to be cordial by Sze-Chin Shee and Yong Song Ho,[5] where as any multiple shell $MS\{n_i^{t_1}, \cdots, n_r^{t_r}\}$ was proved to be cordial by Andar, Boxwala and Limaye.[1]

In this paper, we show that all multiple shells are 3-equitable.

## 3-EQUITABLE LABELINGS OF SHELLS

We begin by listing first, some ternary labelings of the shell $S_n, n \geq 3$, which will be used to construct a cordial labeling of multiple shells. Let $V(S_n) = \{u, v_1, \cdots, v_{n-1}\}, E(S_n) = \{u\, v_i \mid 1 \leq i \leq n-1\} \bigcup \{v_i\, v_{i+1} \mid 1 \leq i \leq n-2\}$. Here $u$ is the apex. Clearly $| V(S_n) |= n$ and $| E(S_n) |= 2n-3$. In what follows the apex vertex $u$ will always be labeled 0. For any labeling $f$, let $v_f(0,1,2) = (v_f(0), v_f(1), v_f(2))$ and $e_f(0,1,2) = (e_f(0), e_f(1), e_f(2))$.

**Case 1:** $n \equiv 1$ **mod** 3. Let $n = 1 + 3s, s \in \mathbb{N}$.

**Type A:** $f(v_{6\alpha+i}) = 1, 0, 2, 2, 0, 1$ for $i = 1, 2, 3, 4, 5, 0$ respectively. Note that $v_f(0,1,2) = (s+1, s, s), e_f(0,1,2) = (2s-1, 2s, 2s)$ and $f$ is a 3-equitable labeling.

**Type B:** $f(v_1) = 0, f(v_2) = 1 = f(v_3)$ and $f(v_{3+6\alpha+i}) = 1, 0, 2, 2, 0, 1$

for $i = 1, 2, 3, 4, 5, 0$ respectively. Note that $v_f(0,1,2) = (s+1, s+1, s-1), e_f(0,1,2) = (2s, 2s+1, 2s-2)$. Here $f$ is not a 3-equitable labeling.

**Type C:** $f(v_1) = 0, f(v_2) = 2 = f(v_3)$ and $f(v_{3+6\alpha+i}) = 1, 0, 2, 2, 0, 1$ for $i = 1, 2, 3, 4, 5, 0$ respectively. Note that $v_f(0,1,2) = (s+1, s-1, s+1), e_f(0,1,2) = (2s, 2s-2, 2s+1)$. Again this is not a 3-equitable labeling.

**Case 2:** $n \equiv 2 \bmod 3$. Let $n = 3s+2$.

**Type A:** $f(v_1) = 1, f(v_2) = 0, f(v_3) = 2 = f(v_4)$ and $f(v_{4+6\alpha+i}) = 2, 0, 1, 1, 0, 2$ for $i = 1, 2, 3, 4, 5, 0$ respectively. Note that $v_f(0,1,2) = (s+1, s, s+1), e_f(0,1,2) = (2s, 2s, 2s+1)$ and $f$ is a 3-equitable labeling.

**Type B:** $f(v_1) = 1, f(v_2) = 0 = f(v_3), f(v_4) = 2$ and $f(v_{4+6\alpha+i}) = 2, 0, 1, 1, 0, 2$ for $i = 1, 2, 3, 4, 5, 0$ respectively. Note that $v_f(0,1,2) = (s+2, s, s), e_f(0,1,2) = (2s+1, 2s, 2s)$. Here $f$ is a 3-equitable labeling for vertices only.

**Type C:** $f(v_1) = 1 = f(v_2), f(v_3) = 0, f(v_4) = 2$ and $f(v_{4+6\alpha+i}) = 2, 0, 1, 1, 0, 2$ for $i = 1, 2, 3, 4, 5, 0$ respectively. Note that $v_f(0,1,2) = (s+1, s+1, s), e_f(0,1,2) = (2s, 2s+1, 2s)$. This is 3-equitable labeling.

**Type D:** $f(v_1) = 0 = f(v_2), f(v_3) = 2 = f(v_4)$ and $f(v_{4+6\alpha+i}) = 2, 0, 1, 1, 0, 2$ for $i = 1, 2, 3, 4, 5, 0$ respectively. Note that $v_f(0,1,2) = (s+2, s-1, s+1), e_f(0,1,2) = (2s+2, 2s-2, 2s+1)$. This is not a 3-equitable labeling.

**Type E:** $f(v_1) = 1 = f(v_2) = f(v_3), f(v_4) = 2$ and $f(v_{3+6\alpha+i}) = 2, 0, 1, 1, 0, 2$

for $i = 1, 2, 3, 4, 5, 0$ respectively. Note that $v_f(0,1,2) = (s, s+2, s), e_f(0,1,2) = (2s, 2s+2, 2s-1)$. This is not a 3-equitable labeling.

**Case 3:** $n \equiv 0$ **mod** 3. Let $n = 3s, s \geq 2$.

**Type A:** $f(v_1) = 1 = f(v_2), f(v_3)0, f(v_4) = 2 = f(v_5)$ and $f(v_{5+6\alpha+i}) = 2, 0, 1, 1, 0, 2$ for $i = 1, 2, 3, 4, 5, 0$ respectively. Note that $v_f(0,1,2) = (s, s, s), e_f(0,1,2) = (2s-1, 2s-1, 2s-1)$ and $f$ is a 3-equitable labeling.

**Type B:** $f(v_1) = 0, f(v_2) = 1, f(v_3) = 0, f(v_4) = 2 = f(v_5)$ and $f(v_{5+6\alpha+i}) = 2, 0, 1, 1, 0, 2$ for $i = 1, 2, 3, 4, 5, 0$ respectively. Note that $v_f(0,1,2) = (s+1, s-1, s), e_f(0,1,2) = (2s-1, 2s-1, 2s-1)$. Here $f$ is a 3-equitable labeling only for the edges.

**Type C:** $f(v_1) = 0, f(v_2) = 2, f(v_3) = 0, f(v_4) = 1 = f(v_5)$ and $f(v_{4+6\alpha+i}) = 1, 0, 2, 2, 0, 1$ for $i = 1, 2, 3, 4, 5, 0$ respectively. Note that $v_f(0,1,2) = (s+1, s, s-1), e_f(0,1,2) = (2s-1, 2s-1, 2s-1)$. This is 3-equitable labeling only for the edges.

**Remark:** In each of the three cases, the labeling A is a 3- equitable labeling, that is, every shell $S_n$ on $n$ vertices is 3 -equitable.

## MULTIPLE SHELLS

A multiple shell $MS\{n_1^{t_1}, \cdots, n_r^{t_r}\}$ is called **homogeneous** (respectively **highly non-homogeneous**) if $n_i \equiv n_j$ mod 3 (respectively $n_i \not\equiv n_j$ modulo 3) for all $i \neq j$.

**Theorem 1:** All double shells $MS\{n_1, n_2\}$ are 3- equitable for all $n_1, n_2 \geq 4$.

**Proof:** Let $n_j = 3s_j + \beta_j, j = 1, 2$.

**Case 1:** The given multiple shell is homogeneous. Let $\beta_j = \beta, j = 1, 2$.

If $\beta = 1$, use the labeling of type B for one shell and the labling of type C for the second shell. Clearly $v_f(0,1,2) = (s_1 + s_2 + 1, s_1 + s_2, s_1 + s_2), e_f(0,1,2) = (2s_1 + 2s_2, 2s_1 + 2s_2 - 1, 2s_1 + 2s_2 - 1)$, that is, the resulting labeling is 3-equitable.

If $\beta = 2$, use the labeling of type A for one shell and the labeling of type C to the second shell. Clearly $v_f(0,1,2) = (s_1 + s_2 + 1, s_1 + s_2 + 1, s_1 + s_2 + 1), e_f(0,1,2) = (2S_1 + 2s_2, 2s_1 + 2s_2 + 1, 2s_1 + 2s_2 + 1)$, that is, the resulting labeling is 3-equitable.

If $\beta = 0$, use the labeling of type A for one shell and the labeling of type B to the second shell. Clearly $v_f(0,1,2) = (s_1 + s_2, s_1 + s_2 - 1, s_1 + s_2), e_f(0,1,2) = (2s_1 + 2s_2 - 2, 2s_1 + 2s_2 - 2, 2s_1 + 2s_2 - 2)$, that is, the resulting labeling is 3-equitable.

**Case 2:** $\beta_1 \neq \beta_2$, that is $n_1$ is not congruent to $n_2$ modula 3.

If $\beta_1 = 0, \beta_2 = 1$, use the labeling of type A for both the shells. Clearly

$v_f(0,1,2) = (s_1+s_2, s_1+s_2, s_1+s_2), e_f(0,1,2) = (2s_1+2s_2-2, 2s_1+2s_2-1, 2s_1+2s_2-1)$, that is, the resulting labeling is 3-equitable.

If $\beta_1 = 0, \beta_2 = 2$, use the labeling of type A for both the shells. Clearly $v_f(0,1,2) = (s_1+s_2, s_1+s_2, s_1+s_2+1), e_f(0,1,2) = (2s_1+2s_2-1, 2s_1+2s_2-1, 2s_1+2s_2)$, that is, the resulting labeling is 3-equitable.

If $\beta_1 = 1, \beta_2 = 2$, use the labeling of type C for the shell on $n_1$ vertices and labeling of type E for the shell on $n_2$ vertices. Clearly $v_f(0,1,2) = (s_1+s_2, s_1+s_2+1, s_1+s_2+1), e_f(0,1,2) = (2S_1+2s_2, 2s_1+2s_2, 2s_1+2s_2)$, that is, the resulting labeling is 3-equitable.

This covers all the cases and hence any double shell is 3-equitable. If $\beta = 0$, use the labeling of type A for one shell and the labeling of type B to the second shell. Clearly $v_f(0,1,2) = (s_1+s_2, s_1+s_2-1, s_1+s_2), e_f(0,1,2) = (2s_1+2s_2-2, 2s_1+2s_2-2, 2s_1+2s_2-2)$, that is, the resulting labeling is 3-equitable. □

Next we consider triple shells and show that they are 3-equitable. These equitable labelings will then be used to show thal all multiple shells are 3 -equitable.

**Theorem 2:** All triple shells are 3-equitable.

**Proof:** Let $\mathbb{S} = Ms\{n_1, n_2, n_3\}$ be an arbitrary triple shell.

**Case 1:** $\mathbb{S}$ is highly non-homogeneous. With out loss of generality let $n_1 = 3s_1+1, n_2 = 3s_2+2, n_3 = 3s_3$. Use type A labeling for $S_{n_1}$ and $S_{n_3}$ and B type labeling for $S_{n_2}$. One can easily see that $v_f(0,1,2) =$

$(s_1+s_2+s_3+1, s_1+s_2+s_3, s_1+s_2+s_3)$ and $e_f(0,1,2) = (2s_1+2s_2+2s_3-1, 2s_1+2s_2+2s_3-1, 2s_1+2s_2+2s_3-1)$, that is, the resulting labeling is 3-equitable.

**Case 2:** Let $\mathbb{S}$ be homogeneous. Let $n_i = 3s_i + \beta, 1 \leq i \leq 3$. Use labelings of type A, B, C for these three shells.

If $\beta = 0$, one can check that $v_f(0,1,2) = (s_1+s_2+s_3, s_1+s_2+s_3-1, s_1+s_2+s_3-1)$ and $e_f(0,1,2) = (2s_1+2s_2+2s_3-3, 2s_1+2s_2+2s_3-3, 2s_1+2s_2+2s_3-3)$.

If $\beta = 1$, one can check that $v_f(0,1,2) = (s_1+s_2+s_3+1, s_1+s_2+s_3, s_1+s_2+s_3)$ and $e_f(0,1,2) = (2s_1+2s_2+2s_3-1, 2s_1+2s_2+2s_3-1, 2s_1+2s_2+2s_3-1)$.

If $\beta = 2$, $v_f(0,1,2) = (s_1+s_2+s_3+2, s_1+s_2+s_3+1, s_1+s_2+s_3+1)$ and

$e_f(0,1,2) = (2s_1+2s_2+2s_3+1, 2s_1, 2s_2, 2s_3+1, 2s_1+2s_2+2s_3+1)$.

This shows that every homogeneous triple shell is 3-equitable.

**Case 3:** The triple shell is neither homogeneous nor highly non-homogeneous. Let $n_i = 3s_i + \beta_i, 1 \leq i \leq 3$. Without loss of generality, let $\beta_1 = \beta_2 = \beta \neq \beta_3$.

**I:** $\beta = 0, \beta_3 = 1$. Label all the shells using labeling of type A. One can easily check that $v_f(0,1,2) = (s_1+s_2+s_3-1, s_1+s_2+s_3, s_1+s_2+s_3)$ and

$e_f(0,1,2) = (2s_1+2s_2+2s_3-3, 2s_1+2s_2+2s_3-2, 2s_1+2s_2+2s_3-2)$.

**II:** $\beta = 0, \beta_3 = 2$. Label $S_{n_1}, S_{n_3}$ using labeling of type A and assign labeling of type C to $S_{n_2}$. Clearly, $v_f(0,1,2) = (s_1+s_2+s_3, s_1+s_2+s_3, s_1+s_2+s_3)$ and

$e_f(0,1,2) = (2s_1+2s_2+2s_3-2, 2s_1+2s_2+2s_3-2, 2s_1+2s_2+2s_3-1)$.

**III:** $\beta = 1, \beta_3 = 0$. Label $S_{n_3}$ using labeling of type A and assign labeling of type B to $S_{n_1}$ and type C to $S_{n_2}$. Clearly, $v_f(0,1,2) = (s_1+s_2+s_3, s_1+s_2+s_3, s_1+s_2+s_3)$ and

$e_f(0,1,2) = (2s_1+2s_2+2s_3-1, 2s_1+2s_2+2s_3-2, 2s_1+2s_2+2s_3-2)$.

**IV:** $\beta = 1, \beta_3 = 2$. Label $S_{n_2}, S_{n_3}$ using labeling of type C and assign labeling of type B to $S_{n_1}$. One can easily check that $v_f(0,1,2) = (s_1+s_2+s_3+1, s_1+s_2+s_3+1, s_1+s_2+s_3)$ and $e_f(0,1,2) = (2s_1+2s_2+2s_3, 2s_1+2s_2+2s_3, 2s_1+2s_2+2s_3-1)$.

**V:** $\beta = 2, \beta_3 = 0$. Label $S_{n_1}, S_{n_3}$ using labeling of type A and assign labeling of type B to $S_{n_2}$. One can easily check that $v_f(0,1,2) = (s_1+s_2+s_3+1, s_1+s_2+s_3, s_1+s_2+s_3+1)$ and $e_f(0,1,2) = (2s_1+2s_2+2s_3, 2s_1+2s_2+2s_3-1, 2s_1+2s_2+2s_3)$.

**VI:** $\beta = 2, \beta_3 = 1$. Label $S_{n_1}$ using labeling of type D and assign labeling of type E to $S_{n_2}$ and labeling of type A to $S_{n_3}$. One can easily check that $v_f(0,1,2) = (s_1+s_2+s_3+1, s_1+s_2+s_3+1, s_1+s_2+s_3+1)$ and

$e_f(0,1,2) = (2s_1+2s_2+2s_3+1, 2s_1+2s_2+2s_3, 2s_1+2s_2+2s_3)$.

This covers all the possibilities and shows that all the triple shells are 3-equitable.

**Remark:** If a triple shell is homogeneous and $n_i = 3s_i + \beta, i = 1,2,3$, the total number of vertices is $3(s_1 + s_2 + s_3) + 3\beta - 2$ and hence is equivalent to 1 modula three. The number of edges is $6(s_1 + s_2 + s_3) + 6\beta - 9$, which is a multiple of three. In Theorem 2, we saw that such a triple shell has a 3-equitable labeling with $v_f(0,1,2) = (N+1, N, N)$ and $e_f(0,1,2) = (M, M, M)$, where $3N + 1$ is the number of vertices and $3M$ is the number of edges.

If a triple shell is highly non-homogeneous, the total number of vertices is again of the form $3N + 1$ and the number of edges is of the form $3M$. Thus, by Theorem 2, we have a 3-equitable labeling with $v_f(0,1,2) = (N+1, N, N)$ and $e_f(0,1,2) = (M, M, M)$.

**Theorem 3:** All multiple shells $MS\{n_1^{t_1}, \cdots, n_r^{t_r}\}$ are 3-equitable.

**Proof:** We give a 3-equitable labeling of $\mathbb{S} = MS\{n_1^{t_1}, \cdots, n_r^{t_r}\}$ in two stages.

**Stage 1:** let $\mathbb{C}_i$ = the class of all the shells in $\mathbb{S}$ whose sizes are equivalent to $i$ modula 3, $0 \leq i \leq 2$. In each $\mathbb{C}_i$, first form as many triple shells as possible and label them equitably as in the above remark. At the end of this process all the labels 0, 1, 2 are received by same number of non-apex vertices as well as edges. Now form as many as possible highly non-homogeneous triple shells and again label them as in the above remark. We note that the maximum number of highly non-homogeneous triple shells possible is 2. This is the end of stage one. At this stage, apart from the apex, which has label zero, all the labels have been received by equal number of vertices as well as edges.

**Stage 2:** After the first stage we can treat the labeled shells in the three classes $\mathbb{C}_0, \mathbb{C}_1, \mathbb{C}_2$ as non-existant. Now the residue of at least one clss is empty and the multishell $\mathbb{S}$ is 3-equitable if and only if the residual multishell is 3-equitable. If there are at most three shells are remaining then we have already shown it to be 3-equitable. Hence it remains to show that a 4-tuple shell formed by two homogeneous double shells of two different classes is 3-equitable.

Let the four shells be $\mathbb{S}_1, \cdots, \mathbb{S}_4$ of sizes $n_i = 3s_i + \beta_i, 1 \leq i \leq 4$ respectively. Let $s = s_1 + s_2 + s_3 + s_4$. Suppose $\beta_1 = \beta_2$ and $\beta_3 = \beta_4$.

**I:** $\beta_1 = 0, \beta_3 = 1$ Assign labeling A and B to $\mathbb{S}_1, \mathbb{S}_2$ respectively and labeling B, C to $\mathbb{S}_3, \mathbb{S}_4$ respectively. We then have $v_f(0, 1.2) = (s, s - 1, s), e_f(0, 1, 2) = (2s - 2.2s - 3, 2s - 3)$.

**II:** $\beta_1 = 0, \beta_3 = 2$ Assign labeling A and B to $\mathbb{S}_1, \mathbb{S}_2$ respectively and labeling A, C to $\mathbb{S}_3, \mathbb{S}_4$ respectively. We then have $v_f(0, 1, 2) = (s, s, s + 1), e_f(0, 1, 2) = (2s - 2, 2s - 1, 2s - 1)$.

**III:** $\beta_1 = 1, \beta_3 = 2$ Assign labeling C and B to $\mathbb{S}_1, \mathbb{S}_2$ respectively and labeling A, C to $\mathbb{S}_3, \mathbb{S}_4$ respectively. We then have $v_f(0, 1, 2) = (s + 1, s + 1, s + 1), e_f(0, 1, 2) = (2s, 2s, 2s)$.

In all these case the resulting labeling is 3-equitable and hence any multiple shell $\mathbb{S}$ is 3-equitable.

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