Counterexamples to a conjecture of Wang concerning regular 3-partite tournaments

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Abstract

A c-partite tournament is an orientation of a complete c-partite graph. In 1991, Jian-zhong Wang conjectured that every arc of a regular 3-partite tournament D is contained in directed cycles of all lengths $3, 6, 9, \ldots, |V(D)|$.

In this paper we show that this conjecture is completely false. Namely, for each integer t with $3 \leq t \leq |V(D)|$, we present an infinite family of regular 3-partite tournaments D such that there exists an arc in D which is not contained in a directed cycle of length t.

Keywords: Multipartite tournaments; Regular multipartite tournaments

The vertex set of a digraph D is denoted by $V(D)$. If xy is an arc of a digraph D, then we write $x \rightarrow y$ and say x *dominates* y. If X and Y are two disjoint subsets of $V(D)$ such that every vertex of X dominates every vertex of Y, then we say that X *dominates* Y, denoted by $X \rightarrow Y$. The numbers $d^+(x)$ and $d^-(x)$ are the *outdegree* and the *indegree* of the vertex x, respectively. A digraph D is called *regular* or r-*regular* if $d^+(x) = d^-(x) = r$ for all $x \in V(D)$. By a *cycle* or *path* we mean a directed cycle or directed path. A cycle of length m is an m-*cycle*. A path in a digraph D is *Hamiltonian* if it includes all the vertices of D. A c-*partite* or *multipartite tournament* is an orientation of a complete c-partite graph.

In 1991, Jian-zhong Wang [2] posed the following conjecture.

Conjecture 1 (Wang [2] 1991) Every arc of a regular 3-partite tournament D is contained in cycles of all lengths $3, 6, 9, \ldots, |V(D)|$.

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The following examples will show that Conjecture 1 is not valid in general.

Example 2. We start with a well-known regular 3-partite tournament. Let V_1, V_2, and V_3 be the partite sets of the 3-partite tournament D such that $|V_1| = |V_2| = |V_3| = r \geq 2$ and

$$V_1 \rightarrow V_2 \rightarrow V_3 \rightarrow V_1.$$

It is evident that D is r-regular and that D has only cycles of length $3, 6, 9, \ldots, |V(D)| = 3r$.

Next let $u_1u_2u_3u_1$ be a 3-cycle of D such that $u_i \in V_i$ for $i = 1, 2, 3$. If we replace in D the cycle $u_1u_2u_3u_1$ by the cycle $u_1u_3u_2u_1$, then we obtain again an r-regular 3-partite tournament D_1. However, it is easy to see that the arcs u_1u_3, u_3u_2, and u_2u_1 in D_1 are not contained in a cycle of length $6, 9, \ldots, |V(D_1)| = 3r$.

Finally, let $u_1u_2u_3w_1w_2w_3u_1$ be a 6-cycle of D such that $u_i, w_i \in V_i$ for $i = 1, 2, 3$. If we replace in D the cycle $u_1u_2u_3w_1w_2w_3u_1$ by the cycle $u_1w_3w_2w_1u_3u_2u_1$, then we arrive at an r-regular 3-partite tournament D_2. Clearly, the arcs u_1w_3, w_3w_2, w_2w_1, w_1u_3, u_3u_2, and u_2u_1 in D_2 are not contained in a 3-cycle.

Example 2 even shows that for each integer t with $3 \leq t \leq |V(D)|$, there exists an infinite family of regular 3-partite tournaments D such that there are at least three arcs in D which are not contained in a cycle of length t.

Recently, Volkmann and Yeo [1] have proved that every arc of a regular c-partite tournament D is contained in a Hamiltonian path of D.

References

[1] L. Volkmann and A. Yeo, Hamiltonian paths, containing a given path or collection of arcs, in close to regular multipartite tournaments, submitted.

[2] J. Wang, Cycles of all possible lengths in diregular bipartite tournaments, *Ars Combin.* **32** (1991), 279-284.