Covering 2-paths with 4-paths

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Abstract

In this paper we solve the existence problem for covering the 2-paths of K_n with 4-paths. This also settles the spectrum of 3-path systems of the line graph of K_n. The proof technique allows the embedding problem for (4,2)-path coverings to be settled.

1 Introduction

There has been considerable interest in finding for some graphs G and H whether or not there exists a partition of the edges of H, each element of which induces a copy of G; this is known as a G-decomposition of H. For example, necessary and sufficient conditions have been found for the existence of a G-decomposition of K_n in the cases where G is a cycle [1, 18, 11], a path [20], a star [19] and a small graph [3]; see also [17] for a survey. Block designs also fall into this category, being K_k-decompositions of λK_v (where λH is the multigraph formed from H by taking each pair of adjacent vertices and joining them with λ edges). Allowing H to be a multigraph is also common [19, 20], and directed versions also exist (see [5] for example). Sometimes additional structure is also required of the decomposition (see [7] for example).

Let a k-path denote a path of length k. In this paper we consider the existence of a collection S of 4-paths in K_n with the property that each 2-path in K_n occurs in exactly one 4-path in S (see Theorem 2.4).

This extends results in the literature that solve the problem of finding sets of 3-paths, 4-cycles and hamilton cycles that cover the 2-paths in K_n [8, 9, 13, 15]. The problem of finding k-paths that cover all the $k-1$-paths in K_n has also been solved [15].

As a corollary of Theorem 2.4, we settle the existence problem for 3-path decompositions of the line graph of K_n (see Corollary 2.5). This is a companion result to that of Colby, Heinrich, Nonay and Rodger [6, 9] that finds the integers n for which there exists a 4-cycle decomposition of the line graph of K_n.

It has also been a focal point in this area to take partial or complete decompostions and to embed them in larger complete decompositions of K_n. One classic unsolved result in this area is the embedding problem for partial triple systems (K_3-decompositions of K_n) [2, 10], but many other decompositions have also been considered (see [12, 14] for example). Recently the embedding problem for 3-path coverings of the 2-paths in K_n was settled [16]; the proof of Theorem 2.4 allows this result to be extended here (see Corollary 2.6).

More generally, for any simple graph G let $\lambda T(G)$ be the multiset in which each 2-path in G occurs λ times; if $\lambda = 1$ then denote this simply by $T(G)$. Define a $(4,2)$-*path covering* of G of *index* λ to be a multiset $F(G)$ of 4-paths in G that satisfies

$$\{(a,b,c),(b,c,d),(c,d,e) \mid (a,b,c,d,e) \in F(G)\} = \lambda T(G)$$

(So each 2-path in G is a subgraph of exactly λ 4-paths in $F(G)$.)

2 Existence of the 4-path covering

The proof of the main result relies on the existence of certain quasigroups (a quasigroup is an ordered pair $(V,\circ)$, where V is a set and $\circ$ is a binary operation defined on V such that for each $a,b \in V$ there exist unique elements c and d in V for which $a \circ c = b$ and $d \circ a = b$; so they are equivalent to latin squares). This quasigroup is said to be *idempotent* if $a \circ a = a$ for all $a \in V$ and is said to be *antisymmetric* if $a \circ b \neq b \circ a$ for all $a \neq b$ in V.

Theorem 2.1 *There exists an idempotent antisymmetric latin square of order v for all $v \geq 4$.*

Proof:

It is not hard to construct such latin squares recursively. However, it suffices to note that if $v \neq 6$ then there exists a self-orthogonal latin square of order v [4]. Such a latin square L square is necessarily antisymmetric.

and each symbol must appear exactly once on the diagonal. So, the symbols in L can be permuted to make the resulting latin square L' idempotent; clearly L' is also antisymmetric. The following is a solution when $v = 6$.

$\circ$	1	2	3	4	5	6
1	1	4	2	5	6	3
2	5	2	4	6	3	1
3	6	5	3	1	4	2
4	3	1	6	4	2	5
5	2	6	1	3	5	4
6	4	3	5	2	1	6

□

Let $T(v,u)$ be the following set of 2-paths defined on the vertex set $(\mathbb{Z}_v \times \{0\}) \cup (\mathbb{Z}_u \times \{1\})$:

$$T(v,u) = \{((a,0),(b,1),(c,0)),((c,0),(a,0),(b,1)),((a,0),(c,0),(b,1)) \mid \{a,c\} \subseteq \mathbb{Z}_v, a < c, b \in \mathbb{Z}_u\}.$$

Then $T(v,u)$ contains exactly $3((v(v-1)/2)(u)$ 2-paths.

Lemma 2.2 *If $v \geq 5$ and $u = 1$ then there exists a set F of 4-paths such that the multiset of 2-paths that occur in 4-paths in F is precisely $T(v,u)$*

Proof: Let $(\mathbb{Z}_v, \circ)$ be an antisymmetric idempotent quasigroup. Define a set F of 4-paths as follows.

$$F = \{((a \circ b, 0), (a,0), (0,1), (b,0), (b \circ a, 0)) \mid \{a,b\} \subset \mathbb{Z}_v, a < b\}.$$

Notice that since $(\mathbb{Z}_v, \circ)$ is idempotent and antisymmetric, each 4-path in F does indeed contain 5 distinct vertices. It is easy to check that each 2-path in $T(v,1)$ is in at most one 4-path in F. Clearly F contains $v(v-1)/2$ 4-paths each of which contains exactly 3 2-paths. Also, $3v(v-1)/2 = 3uv(v-1)/2$, where $u = 1$, which is the number of 2-paths in $T(v,u)$. So the result follows. □

Proposition 2.3 *Suppose $n \geq 5$. If there exists a $(4,2)$-path covering of K_n then there exists a $(4,2)$-path covering of K_{n+1}.*

Proof: Let F_1 be a $(4,2)$-path covering of K_n on the vertex set N. Let M be a set of size 1 with $M \cap N = \emptyset$. There exists a $(4,2)$-path covering F_2 of $T(n,1)$ formed by using Lemma 2.2 and then renaming the vertices $(\mathbb{Z}_n \times \{0\})$ and $\{0\} \times \{1\}$ with N and M as respectively. Then $F_1 \cup F_2$ is a $(4,2)$-path covering of K_{n+1} on the vertex set $M \cup N$. □

Theorem 2.4 *There exists a* $(4,2)$*-path covering of* K_n *if and only if* $n \notin \{3,4\}$.

Proof: To prove necessity, note that K_3 and K_4 contain 2-paths, but no 4-paths.

To prove sufficiency, we begin by noting that K_1 and K_2 contain no 2-paths, so the result follows vacuously. So, we can assume that $n \geq 5$. We begin by finding a $(4,2)$-path covering of K_5. Define a $(4,2)$-path covering of K_5 with the vertex set $V(K_5) = \{0,1,2,3,4\}$ and defining $F(K_5) = \{(4,0,2,1,3),(0,3,2,4,1),(0,4,1,2,3),(4,3,1,0,2),(3,4,0,1,2),$ $(4,2,0,3,1),(3,0,4,2,1),(0,1,4,3,2),(0,2,3,1,4),(1,0,3,4,2)\}$.

Now, Suppose that $n \geq 6$ and that for any $z < n$ there exists a $(4,2)$-path covering of K_z. In particular, there exists a $(4,2)$-path covering of K_{n-1}, so by Proposition 2.3 there exists a $(4,2)$-path covering of K_n. So the result follows by induction. □

We can now obtain two corollaries that supplement results in the literature. As described in the introduction, there has been considerable interest in finding for some graphs G and H a G-decomposition of H. H is often taken to be one of a family of graphs, such as $K_v, K_{v,w}$ or, as in the following case, the line graph $L(K_v)$ of K_v.

Corollary 2.5 *There exists a 3-path decompostion of* $L(K_v)$ *if and only if* $v \neq 3$.

Proof: Since the line graph of K_3 contains 2-paths but no 3-paths, the necessity follows. Since $L(K_v)$ contains no 2-paths when $v \in \{1,2\}$, the result follows vacuously in these two cases.

If $v = 4$ then the required decomposition is provided by the following set of 3-paths: $\{(\{0,2\},\{0,1\},\{1,3\},\{2,3\}),(\{0,3\},\{0,1\},\{1,2\},\{2,3\}),$ $(\{0,2\},\{2,3\},\{0,3\},\{1,3\}),(\{0,3\},\{0,2\},\{1,2\},\{1,3\})\}$.

For $v \geq 5$ the result follows from Theorem 2.4 by taking the following set of line graphs of 4-paths: set:

$\{L(p) \mid p \in F, F$ is a $(4,2)$-path covering of $K_v\}$.

(This works because the line graph of a 2-path is an edge, and the line graph of a 4-path is a 3-path.) □

A second focus of attention in the literature has been on embedding decompositions of various sorts into similar larger structures. The following result follows others detailed in the introduction.

Corollary 2.6 *For all* $v, w \geq 1$, *any* $(4,2)$*-path covering of* K_v *can be embedded into a* $(4,2)$*-path covering of* K_{v+w} *if and only if* $v + w \notin \{3,4\}$ *whenever* $v \in \{1,2\}$.

Proof: If $v \in \{1,2\}$ then the result follows from Theorem 2.4, since any $(4,2)$-path covering of K_{n+m} provides the required embedding.

If $v \in \{3,4\}$ then the result follows vacuously.

If $v \geq 5$ then begin with a $(4,2)$-path covering of K_v and recursively apply Proposition 2.3 w times. Then clearly, we have a $(4,2)$-path covering of K_{v+w} that contains the given covering. □

It is also worth mentioning the following corollary. A G-decomposition of H of *index* λ is actually just a G-decomposition of λH. Such objects are also considered, often because in many graph decomposition problems there may be no G-decomposition of H, yet there is one of λH for some value of λ.

Corollary 2.7 *There exists a* $(4,2)$*-path covering of* λK_n *if and only if* $n \notin \{3,4\}$.

Proof: This follows immediately from Theorem 2.4 by taking λ copies of each 4-path, and by noting that λK_n contains 2-paths but no 4-paths when $n \in \{3,4\}$. □

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