Variations of pancyclic graphs

Jill Faudree
University of Alaska, Fairbanks

R. Jasper Faudree
University of Memphis

Ronald J. Gould
Emory University

Michael S. Jacobson
University of Colorado at Denver

Linda Lesniak
Drew University

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Abstract

A graph G of order n is *pancyclic* if it contains a cycle of length ℓ for every ℓ such that $3 \leq \ell \leq n$. If the graph is bipartite, then it contains no cycles of odd length. A balanced bipartite graph G of order 2n is *bipancyclic* if it contains a cycle of length ℓ for every even ℓ, such that $4 \leq \ell \leq 2n$. A graph G of order n is called *k-semipancyclic*, $k \geq 0$, if there is no "gap" of $k+1$ among the cycle lengths in G, i.e., for no $\ell \leq n-k$ is it the case that each of $C_\ell, ..., C_{\ell+k}$ is missing from G. Generalizing this to bipartite graphs, a bipartite graph G of order n is called *k-semibipancyclic*, $k \geq 0$, if there is no "gap" of $k+1$ among the even cycle lengths in G, i.e., for no $\ell \leq n-2k$ is it the case that each of $C_{2\ell}, ..., C_{2\ell+2k}$ is missing from G.

In this paper we generalize a result of Hakimi and Schmiechel in several ways. First to k-semipancyclic, then to bipartite graphs, giving a condition for a hamiltonian bipartite graph to be bipancyclic or one of two exceptional graphs. Finally, we give a condition for a hamiltonian bipartite graph to be k-semibipancyclic or a member of a very special class of hamiltonian bipartite graphs.

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1 Introduction

A graph G of order n is *pancyclic* if G contains a cycle of length ℓ for every ℓ such that $3 \leq \ell \leq n$. If the graph is bipartite, then it contains no cycles of odd length. Generalizing the concept of a pancyclic graph, a balanced bipartite graph G of order $2n$ is *bipancyclic* if it contains a cycle of length ℓ for every even ℓ, such that $4 \leq \ell \leq 2n$.

In this paper, we also consider another such property and its bipartite analogue. A graph G of order n is called *k-semipancyclic*, $k \geq 0$, if there is no "gap" of $k+1$ among the cycle lengths in G, i.e., for no $\ell \leq n-k$ is it the case that each of $C_\ell, ..., C_{\ell+k}$ is missing from G. In addition, a bipartite graph G of order n is called *k-semibipancyclic*, $k \geq 0$, if there is no "gap" of $k+1$ among the even cycle lengths in G, i.e., for no $\ell \leq n-2k$ is it the case that each of $C_{2\ell}, ..., C_{2\ell+2k}$ is missing from G. Thus every pancyclic graph is k-semipancyclic and every bipancyclic graph is k-semibipancyclic for all $k \geq 0$. Note, 0-semipancyclic and 0-semibipancyclic graphs are just pancyclic and bipancyclic graphs, respectively.

In Section 2 we give a generalization of a result by Hakimi and Schmiechel [2] to k-pancyclicity. In Section 3 we give several examples of hamiltonian bipartite graphs which are not bipancyclic, and in certain cases, not k-semibipancyclic. These examples will be the limiting examples of the results presented in Sections 4 and 5. In Section 4 we further generalize a result of Hakimi and Schmiechel to bipartite graphs and give a condition for a hamiltonian bipartite graph to be bipancyclic or one of two exceptional graphs. In Section 5 we will generalize the result of Section 2 to bipartite graphs and give a condition for a hamiltonian bipartite graph to be k-semibipancyclic or an element of a very specific class of hamiltonian bipartite graphs.

2 k-Semipancyclic hamiltonian graphs

In [2] the following result on pancyclic graphs was given.

Theorem 1 *Let G be a graph of order n with $V(G) = \{v_1, ..., v_n\}$ and hamiltonian cycle $v_1, ..., v_n, v_1$. Suppose that* $\deg v_1 + \deg v_n \geq n$. *Then G is either pancyclic, bipartite or missing only an $(n-1)$-cycle.*

Here we establish the following extension for k-semipancyclic graphs, $k \geq 1$.

Theorem 2 *Let G be a graph of order $n \geq 9$ with $V(G) = \{v_1, ..., v_n\}$ and hamiltonian cycle $v_1, ..., v_n, v_1$. Suppose* $\deg v_1 + \deg v_n \geq n - k$ *for some integer k satisfying $1 \leq k \leq n/2$. Then G is k-semipancyclic.*

Proof. Suppose, to the contrary, that G is not k-semipancyclic. Then for some ℓ satisfying $3 \leq \ell \leq n-k$, G contains none of the cycles $C_\ell, ..., C_{\ell+k}$.

Case 1. Suppose that $\ell \leq n/2 - k$. (The case $\ell \geq n/2 + 1$ is handled by a symmetric argument.)

Since $C_\ell, ..., C_{\ell+k} \not\subseteq G$, it follows that v_1 is not adjacent to v_i, for $\ell \leq i \leq \ell + k$ and $n - \ell - k + 2 \leq i \leq n - \ell + 2$. Let $A = \{v_2, ..., v_{\ell-1}\}$, $B = \{v_{\ell+k+1}, ..., v_{n-\ell-k+1}\}$, $C = \{v_{n-\ell+3}, ..., v_{n-1}\}$, $A' = \{v_{\ell-1}, ..., v_{2\ell-4+k}\}$, $B' = \{v_{2\ell+k-2}, ..., v_{n-2}\}$ and $C' = \{v_2, ..., v_{\ell-2+k}\}$. Note that all adjacencies of v_1 are in $A \cup B \cup C$ with the exception of v_n.

Let a, b, and c be the number of adjacencies of v_1 in A, B, C, respectively. Let $v_{i_1}, v_{i_2}, ..., v_{i_a}$ with $2 \leq i_1 < i_2 < ... < i_a \leq \ell - 1$ be the adjacencies of v_1 in A. It follows that $v_n v_{i_1+\ell-3}, v_n v_{i_2+\ell-3}, ...$, and $v_n v_{i_a+\ell-3}$ are not edges of G for otherwise an ℓ - cycle would result. Furthermore, for $t = 1, 2, ..., k$, we see that $v_n v_{i_a+\ell-3+t}$ is not an edge of G since a cycle of length $\ell+t$ would result. Observe that these $a+k$ nonadjacencies of v_n are all in A'. Similarly, the b adjacencies of v_1 in B force $b+k$ nonadjacencies of v_n in B'. Finally, if $v_{i_1}, v_{i_2}, ..., v_{i_c}$ with $n - \ell+3 \leq i_1 < i_2 < ... < i_a \leq n-1$ are the adjacencies of v_1 in C then it follows that $v_n v_{\ell-(n-i_1+1)}$ is not an edge of G since $v_1, v_2, ..., v, v_n, v_{n-1}, ..., v_{i_1}, v_1$ would form an ℓ - cycle. Similarly, $v_n v_{\ell-(n-i_2+1)}, ..., v_n v_{\ell-(n-i_{c-1}+1)}$ and $v_n v_{\ell-(n-i_c+1)}$ are not edges of G for again an ℓ - cycle would result. Additionally, for $t = 1, 2, ..., k$, we see that $v_n v_{i_a+\ell-3+t}$ is not an edge of G since a cycle of length $\ell + t$ would result. Note that these are $c+k$ nonadjacencies of v_n in C'. Now the $\deg v_1 = a + b + c + 1$, and $\deg v_n \leq (n-1) - (a+k+b+k+c+k-m)$, where m is the cardinality of the intersection of A' and C'. Since $\left|A' \cap C'\right| = (\ell+k-2) - (\ell-1) + 1 = k$, we have that $\deg v_n \leq (n-1) - (a+k+b+k+c+k-k) = (n-1) - (\deg v_1 + 2k - 1)$, so that $\deg v_1 + \deg v_n \leq n - 2k$, which is a contradiction since $k \geq 1$.

Case 2. Suppose n is even and $\ell = n/2 - t$ for $0 \leq t \leq k/2 - 1$. (The case $k/2 - 1 < t \leq k - 1$ is handled by a symmetric argument.)

Since $C_\ell, ..., C_{\ell+k} \not\subseteq G$, it follows that v_1 is not adjacent to $v_{n/2-t}, \dots, v_{n/2+t+2}$. Let $A = \{v_2, ..., v_{n/2-t-1}\}, C = \{v_{n/2+t+3}, ..., v_{n-1}\}$, $A' = \{v_{n/2-t-1}, \dots, v_{n-2t-4+k}\}$ and $C' = \{v_2, ..., v_{n/2-t-2+k}\}$. Note that all of the adjacencies of v_1 are in $A \cup C$ with the exception of v_n. Arguing as before, since G is not k-semipancyclic, the a adjacencies of v_1 in A force $a + k$ nonadjacencies of v_n in A'. Likewise, the c adjacencies of v_1 in C force $c + k$ nonadjacencies of v_n in C'. Note that if the adjacencies of v_1 in A are not consecutive, this forces another previously uncounted nonadjacency of v_n. For example, if $v_i v_1 \notin E(G)$ and $v_{i+1} v_1 \in E(G)$, then v_n is adjacent to neither $v_{i+\ell-2}$ nor $v_{i+\ell-1}$.

Define $\Delta_A(\Delta_C)$ to be 1 if the adjacencies of v_1 in A (respectively C) are not consecutive along the hamiltonian cycle and 0 otherwise. Now the $\deg v_1 = a+c+1$, and as above, $\deg v_n \leq (n-1)-(a+k+c+k+\Delta_A+\Delta_C-m)$, where m is the cardinality of the intersection of A' and C'. Since $|A' \cap C'| = (n/2-t-2+k)-(n/2-t-1)+1 = k$, we have that $\deg v_n \leq (n-1)-(a+c+k+\Delta_A+\Delta_C) = (n-1)-(\deg v_1 - 1 + k + \Delta_A + \Delta_C)$. So $\deg v_1 + \deg v_n \leq n-k-(\Delta_A+\Delta_C)$, which is a contradiction unless $\Delta_A = \Delta_C = 0$ and $\deg v_1 + \deg v_n = n-k$. Hence, we may assume the adjacencies of v_1 in A and C are consecutive.

Thus v_1 is adjacent to the vertices $v_2, ..., v_{a+1}$ in A and v_1 is adjacent to the vertices $v_{n-1}, ..., v_{n-c}$ in C. This gives us cycles of lengths $n-(a+c+1)+2$ up to n. Now, we can assume $\deg v_1 \geq (n-k)/2$ and, that $\deg v_n \leq (n-k)/2$. Thus we have cycles of lengths $n-(n-k)/2+2 = n/2+k/2+2$ and larger and this case is complete if $t < k/2-1$ or if $t = k/2-1$ and $\deg v_1 > (n-k)/2$. So we can assume that $t = k/2-1$ and $\deg v_1 = \deg v_n = (n-k)/2$. Note, since $n \geq 9$, it follows that $(n-k)/2 \geq 3$.

Suppose v_1 has all of its adjacencies, except v_n in A. As previously noted, v_1 is adjacent to $v_2, v_3, ..., v_{(n-k)/2}$. This yields cycles of length 3 up to $(n-k)/2$ and from $(n+k+4)/2$ to n. Now consider the adjacencies of v_n. Since $deg\ v_n \geq 3$, it follows that v_n is adjacent to either v_{n-2} or v_2, which implies that G contains a cycle of length $(n+k+2)/2$ or a cycle of length $(n-k+2)/2$, either case being a contradiction. Consequently, we can assume that each of v_1 and v_n have adjacencies to both A and C. Now the cycle $v_1, v_{n-c}, v_{n-c+1}, ..., v_n, v_2, ..., v_1$ has length $(n-k+2)/2$ thus completing this case.

Case 3. Suppose n is odd and $\ell = \lfloor n/2 \rfloor - t$ for $0 \leq t \leq k/2-1$. (The case $k/2-1 < t \leq k-1$ is handled by a symmetric argument.)

Since $C_\ell, ..., C_{\ell+k} \not\subseteq G$, it follows that v_1 is not adjacent to the vertices $v_{\lfloor n/2 \rfloor - t}, ..., v_{\lfloor n/2 \rfloor + t+3}$. Let $A = \{v_2, ..., v_{\lfloor n/2 \rfloor - t-1}\}$, $C = \{v_{\lfloor n/2 \rfloor + t+4}, ..., v_{n-1}\}$, $A' = \{v_{\lfloor n/2 \rfloor - t-1}, ..., v_{n-2t-5+k}\}$ and $C' = \{v_2, ..., v_{\lfloor n/2 \rfloor - t-2+k}\}$. Note that all adjacencies of v_1 are in $A \cup C$ with the exception of v_n. Further, if the adjacencies of v_1 in A are not consecutive, then, as before, this forces another previously uncounted nonadjacency of v_n. Define $\Delta_A(\Delta_C)$ to be 1 if the adjacencies of v_1 in A (respectively in C) are not consecutive on the hamiltonian cycle, and 0 otherwise.

Let a be the number of adjacencies of v_1 in A and let c be the number of adjacencies of v_1 in C. Then, as above, $\deg v_1 = a+c+1$ and $\deg v_n \leq (n-1)-(a+k+c+k+\Delta_A+\Delta_C-m)$, where m is the cardinality of the intersection of A' and C'. Since $|A' \cap C'| = (\lfloor n/2 \rfloor - t-2+k)-(\lfloor n/2 \rfloor - t-1)+1 = k$, we conclude $\deg v_n \leq (n-1)-(a+k+c+k-k+\Delta_A+\Delta_C) = (n-1)-(a+c+k+\Delta_A+\Delta_C) = (n-1)-(\deg v_1 - 1 + k + \Delta_A + \Delta_C)$. So $\deg v_1 + \deg v_n \leq (n-k)-(\Delta_A+\Delta_C)$, which is a contradiction unless $\Delta_A = 0 = \Delta_C$ and $\deg v_1 + \deg v_n = n-k$.

The remainder of the proof of Case 3 is identical to that of Case 2. ■

The requirement that $n \geq 9$ in Theorem 2 and the following two corollaries can be seen to be a necessary condition from the graph C_8, with $k = 4 = n/2$ which fails to be 4-semipancyclic.

Corollary 3 *Let G be a hamiltonian graph of order $n \geq 9$ with $\delta(G) \geq (n-k)/2$ for some k satisfying $1 \leq k \leq n/2$. Then G is k-semipancyclic.*

Let $\sigma_2(G) = \min\{\deg u + \deg v\}$, where the minimum is taken over all pairs u, v of nonadjacent vertices of G.

Corollary 4 *Let G be a hamiltonian graph of order $n \geq 9$ with $\sigma_2 \geq n-k$ for some k satisfying $1 \leq k \leq n/2$. Then G is k-semipancyclic.*

Proof. Let $C : v_1, v_2, ..., v_n, v_1$ be a hamiltonian cycle of G. If any pair of consecutive vertices of C has degree sum at least $n-k$, then the proof is complete by Theorem 2. Fix the pair v_n, v_1 and beginning with v_2, v_3 consider the $\lfloor \frac{n-2}{2} \rfloor$ disjoint consecutive pairs. If for one such pair v_i, v_{i+1} it is the case that $v_n v_{i+1}$ and $v_1 v_i$ are not edges of G (alternately, if $v_n v_i$ and $v_1 v_{i+1}$ are not edges of G) then

$$\begin{aligned} 2(n-k) &\leq (deg\ v_n + deg\ v_{i+1}) + (deg\ v_1 + deg\ v_i) \\ &= (deg\ v_n + deg\ v_1) + (deg\ v_i + deg\ v_{i+1}) < 2(n-k), \end{aligned}$$

a contradiction. Thus, there are at least two edges from v_n, v_1 to v_i, v_{i+1}. Hence, $deg\ v_n + deg\ v_1 \geq \lfloor \frac{n-2}{2} \rfloor + 2 \geq n-1$, and the result follows by Theorem 1. ∎

3 Hamiltonian bipartite graphs that are not k-semibipancyclic

In this section we present several classes of graphs that will be the exceptional cases for results presented in the next two sections. In each of these examples, the resulting graph is bipartite, with partite sets X and Y having $|X| = |Y| = n$, and hamiltonian cycle

$$C = x_1, y_1, x_2, y_2, ..., x_n, y_n, x_1.$$

Example 1 *Consider the family of graphs $\mathcal{F}_n$ containing the graphs $F_n = (X, Y, E)$, with the hamiltonian cycle C, additional edges $x_1 y_3$ and $y_n x_{n-2}$ and for each $i = 4, 5, ..., n-2$, exactly one of the edges $x_1 y_i$ or $y_n x_{i-1}$.*

Each member F_n of $\mathcal{F}_n$ is a hamiltonian bipartite graph with deg x_1 + deg $y_n = n+1$ containing all possible even cycle lengths, with the exception of $2n-2$. To see that F_n doesn't contain a cycle of length $2n-2$, note that if a cycle of length $2n-2$ was contained in any such graph, then the edges x_1y_1, y_1x_2, x_2y_2 and y_2x_3 as well as the edges $y_nx_n, x_ny_{n-1}, y_{n-1}x_{n-1}$ and $x_{n-1}y_{n-2}$ would necessarily be contained on the cycle. But then the only way to leave off exactly two vertices would require that both edges x_1y_i and y_nx_{i-1} were included for some $i = 4, 5, ..., n-2$. To see that F_n contains all other even cycle lengths, let $2 \leq t \leq n-2$ be an integer and we want to exhibit the cycle of length $2t$. If x_1y_t is an edge, then the $2t$-cycle results immediately, hence y_nx_{t-1} must be an edge. Similarly, if y_nx_t is an edge, the $2t$-cycle results immediately, hence x_1y_{t+1} must be an edge. But now looking at the pair x_1y_{t+2} and y_nx_{t+1} we get $y_n, x_{t-1}, y_{t-2} ..., x_1, y_{t+1}, x_{t+1}, y_n$ forming a $2t$-cycle when y_nx_{t+1} is an edge and $x_1, y_{t+2}, x_{t+2} ..., x_4, y_3, x_1$ forming a $2t$-cycle when x_1y_{t+2} is an edge.

Example 2 *Let $H_{n;t} = (X, Y, E)$, be the bipartite graph with the hamiltonian cycle C and additional edges*

$$x_1y_2, x_1y_3, ..., x_1y_{t-1},$$

$$y_nx_2, y_nx_3, ..., y_nx_{t-1}$$

and

$$y_nx_{n-1}, y_nx_{n-2}, ..., y_nx_{2t-1},$$

where t is an integer such that $(n+3)/3 \leq \ t \leq n/2$.

The graph $H_{n;t}$ is a hamiltonian bipartite graph with deg $x_1 = t$ and deg $y_n = n - t + 1$. Consequently, deg x_1 + deg $y_n = n + 1$ and it is easy to see that $H_{n;t}$ contains all possible even cycles lengths, with the exception of $2t$.

Example 3 *Let $I_{n;r,s} = (X, Y, E)$, be the graph with the hamiltonian cycle C and the additional edges*

$$x_1y_2, x_1y_3, ..., x_1y_r,$$

$$x_1y_{n-1}, x_1y_{n-2}, ..., x_1y_{n-s+1},$$

$$y_nx_{n-1}, y_nx_{n-2}, ..., y_nx_{n-s+1},$$

and

$$y_nx_2, y_nx_3, ..., y_nx_{r-1}, y_nx_r,$$

with r and s positive integers and k a non-negative integer, such that $r + s = (n+1-k)/2$.

Example 4 *Let $J_{n;r,s} = (X, Y, E)$, with $X = \{x_1, ..., x_n\}, Y = \{y_1, ..., y_n\}$ be constructed as follows: start with a $K_{r+s-1,r+s}$ between the sets $\{x_1, x_2, ..., x_r, x_n, x_{n-1}, ..., x_{n-s+2}\}$ and $\{y_1, y_2, ..., y_r, y_n, y_{n-1}, ..., y_{n-s+1}\}$. Add the edge $y_n x_{n-s+1}$ and the edges of the path $y_r, x_r, y_{r+1}, x_{r+1}, ..., y_{n-s}, x_{n-s+1}, y_{n-s+1}$.*

The graphs $I_{n;r,s}$ and $J_{n;r,s}$ are hamiltonian bipartite graphs with "short" cycles of length, 4, 6, ..., $2(r+s-1)$ and "long" cycles of length $2n-2(r+s)+4$, $2n-2(r+s)+6$, $2n-2$, $2n$. In the case when $r+s = (n+1)/2$, and n is odd, we note that these graphs contain cycles of all even lengths, with the exception of $n+1$. When $r+s = (n+1)/2 - k$, the graphs do not contain the $k+1$ consecutive even cycle lengths $n-k+1, n-k+3, ..., n+k+1$. We also note that $I_{n;r,s}$ is a subgraph of $J_{n;r,s}$ and for any G with $I_{n;r,s} \subseteq G \subseteq J_{n;r,s}$, then G must also have cycles of lengths as described above.

These examples will be exceptions for the conditions given in the results presented in Sections 4 and 5 and will occur if the graph is in fact not bipancyclic or $k-$semibipancyclic.

4 Hamiltonian graphs that are bipancyclic

In [2] the following result on pancyclic graphs was given.

Theorem 5 *Let G be a graph of order n with $V(G) = \{v_1, ..., v_n\}$ and hamiltonian cycle $v_1, ..., v_n, v_1$. Suppose that $\deg v_1 + \deg v_n \geq n$. Then G is either pancyclic, bipartite or missing only an $(n-1)$-cycle.*

In [1], Amar gives the following generalization for bipartite graphs.

Theorem 6 *Let G be a bipartite hamiltonian graph of order $2n$ with two vertices v_1 and v_2 which lie a distance two apart on a hamiltonian cycle of G, with $\deg v_1 + \deg v_2 \geq n+1$. Then G is either bipancyclic or one of several special graphs.*

Here we establish the following version for hamiltonian bipartite graphs which considers the degree sum of consecutive vertices of a hamiltonin cycle, as did Hakimi and Schmeichel.

Theorem 7 *Let $G = (X, Y, E)$ be a bipartite graph with $X = \{x_1, ..., x_n\}$, $Y = \{y_1, ..., y_n\}$ and hamiltonian cycle $x_1, y_1, x_2, y_2, ..., x_n, y_n, x_1$. If $\deg x_1 + \deg y_n \geq n+1$ then either* *i.* *G is bipancyclic,*

ii. $F_n \subseteq G$ *for some* $F_n \in \mathcal{F}_n$*, thus G is missing at most the* $2n-2$ *cycle,*

iii. $H_{n;t} \subseteq G$*, thus G is missing at most the* $2t$ *cycle or,*

iv. $I_{n;r,s} \subseteq G$ *with* $2(r+s) = n+1$ *and n odd, thus G is missing at most an* $(n+1)$*-cycle.*

Proof. We proceed by induction on n. It is clear that if $n = 2$ then G is bipancyclic. When $n = 3$, then G is either a C_6, and thus $G = I_{3;1,1}$ or G is bipancyclic. When $n = 4$ it is again clear that G is bipancyclic, while when $n = 5$, it follows that G is bipancyclic, $G = H_{5;3}$, $G = I_{5;2,1}$ or $G = I_{5;1,2}$. Let G be a bipartite graph, with partite sets X and Y having $|X| = |Y| = n$, and hamiltonian cycle

$$C_{2n} = x_1, y_1, x_2, y_2, \ldots, x_n, y_n, x_1.$$

Furthermore, assume that $\deg x_1 + \deg y_n \geq n+1$. We define the ℓ *−pairing* of possible edges from x_1 and y_n as $x_1 y_{n-(\ell-i)}$ is paired with $y_n x_i$ if $1 \leq i \leq \ell$, and $x_1 y_{i-(\ell-1)}$ is paired with $y_n x_i$ if $\ell + 1 \leq i \leq n-1$. Observe that a cycle of length 2ℓ results if for some i both edges of the ℓ − pairing are edges of G. Suppose that G is not bipancyclic. It follows that for some ℓ satisfying $4 \leq 2\ell \leq 2(n-1)$, the graph G does not contain a cycle of length 2ℓ and thus not both pairs from the ℓ − pairing can be edges.

Claim 1 *If* $\deg x_1 + \deg y_n \geq n+2$ *then G is bipancyclic.*

Proof. Suppose that $\deg x_1 + \deg y_n \geq n+2$. By the ℓ − pairing, it must be the case that for each i, at least one of those pairs are not an edge of G. But this implies that

$$\begin{aligned} deg\ x_1 &\leq n - (deg\ y_n - 1) \\ deg\ x_1 + deg\ y_n &\leq n+1, \end{aligned}$$

which contradicts the assumption. ∎

Hence we may assume that the $deg\ x_1 + deg\ y_n = n+1$. This also implies that if G is not bipancyclic and does not contain a cycle of length 2ℓ then, for each $1 \leq i \leq n-1$, exactly one of the edges in the ℓ − pairing must be an edge in G.

The proof of Theorem 7 will be completed by considering the possible cases for the inclusion of the edges $x_1 y_{n-1}$ and $y_n x_2$ in G.

Claim 2 *If G is not bipancyclic and both* $x_1 y_{n-1}$ *and* $y_n x_2$ *are edges of G, then* $I_{n;r,s} \subseteq G$ *with* $2(r+s) = n+1$.

Proof. Suppose G is not bipancyclic. As noted above, since G does not contain a cycle of length 2ℓ, for some ℓ then it follows that exactly one of the edges in the ℓ – pairing must be an edge in G. Also note that $\ell < n-1$ since having $y_n x_2$ in G implies that a $(2n-2)$ – cycle is contained in G.

Case 1. Suppose $\ell < \frac{n}{3}$.

Since G contains no cycles of length 2ℓ, it follows that $y_n x_{n-\ell+1}$ is not an edge of G. By the ℓ – pairing, this implies that $x_1 y_{n-2\ell+2}$ is an edge of G since $\ell \leq \frac{n}{2}$. Now if $y_n x_{n-\ell}$ were an edge of G then

$$y_n, x_{n-\ell}, y_{n-\ell-1}, x_{n-\ell-1}, \ldots, y_{n-2\ell+2}, x_1, y_1, x_2, y_n$$

would be a cycle of length 2ℓ. Thus $x_1 y_{n-2\ell+1}$ is an edge of G. Continuing this argument, it must be the case that $y_n x_{n-\ell-1}, \ldots, y_n x_{\ell+1}$ are not edges of G, hence it must be that $x_1 y_{n-2\ell}, \ldots, x_1 y_2$ are edges of G. Consequently, it follows that $x_1 y_\ell$ is an edge, since $\ell < \frac{n}{3}$, which implies that G contains a cycle of length 2ℓ, a contradiction. Thus we may assume that $\frac{n}{3} \leq \ell$.

Case 2. Suppose $\frac{n}{3} \leq \ell \leq \frac{n}{2}$.

Again since G contains no cycles of length 2ℓ, it follows that $y_n x_{n-\ell+1}$ is not an edge of G. By the ℓ – pairing, this implies that $x_1 y_{n-2\ell+2}$ is an edge of G since $\ell \leq \frac{n}{2}$. Arguing as in the previous case, it follows that $y_n x_{n-\ell}$ is not an edge in G while $x_1 y_{n-2\ell+1}$ is an edge in G. Hence it follows that x_1 is adjacent to $y_1, y_2, \ldots, y_{n-2\ell+2}$. Since $\ell \leq \frac{n}{2}$ implies that $2 \leq n-2\ell+2$, the edge $y_n x_{n-\ell+2}$ is not an edge of G for otherwise

$$y_n, x_{n-\ell+2}, y_{n-\ell+2}, x_{n-\ell+3}, \ldots y_{n-1}, x_1, y_2, x_2, y_n$$

would form a cycle of length 2ℓ. But $y_n x_{n-\ell+2}$ not an edge implies that $x_1 y_{n-2\ell+3}$ is an edge of G. Continuing to argue in this fashion, it follow that y_n is not adjacent to $x_{n-(\ell-1)}, x_{n-(\ell-2)}, x_{n-(\ell-3)}, \ldots, x_{n-2}$, implying that x_1 is adjacent to $y_{n-2\ell+2}, y_{n-2\ell+3}, \ldots, y_{n-\ell}$, for otherwise a 2ℓ – cycle results. Since $n-\ell \geq \ell$, it follows that $x_1 y_\ell$ is an edge and thus a 2ℓ – cycle results, a contradiction. Thus we may assume that $\ell > \frac{n}{2}$.

Case 3. Suppose $\frac{n+1}{2} = \ell$.

In this case clearly y_n is not adjacent to $x_{(n+1)/2}$ and $x_{(n-1)/2}$. Since $y_n x_2$ is an edge then $y_n x_{(n+3)/2}$, is not an edge, for otherwise a 2ℓ-cycle would result. This non-edge implies that $x_1 y_2$ is an edge. Hence we may assume that $x_1 y_j$ is an edge for $j = 1, 2, \ldots r$ and $x_1 y_{r+1}$ is not an edge for some $r \leq (n-1)/2$. By the ℓ – pairing we get that y_n is not adjacent to $x_{(n+1)/2}, x_{(n+3)/2}, \ldots, x_{r+(n-1)/2}$ and is adjacent to $x_{r+(n+1)/2}$. Now if $x_1 y_{r+2}$ were an edge, the cycle

$$x_1, y_{r+2}, x_{r+3}, y_{r+3}, \ldots, x_{r+(n+1)/2}, y_n, x_2, y_1, x_1$$

would be a 2ℓ – cycle. Now using ℓ-pairings and the edge y_nx_2 it follows that x_1 is not adjacent to $y_{r+1}, y_{r+2}, ..., y_{(n+2)/2}$ and consequently y_n would be adjacent to $x_{r+1+n/2}, x_{r+2+n/2}, ..., x_n$. If either x_1 or y_n had an adjacency among the collection of vertices $y_{r+1}, x_{r+2}, y_{r+2}, x_{r+2}, ..., x_{r+n/2}, y_{r+n/2}$ then a 2ℓ – cycle results. Since deg x_1 + deg $y_n = n+1$, it must be the case that x_1 is also adjacent to $y_{r+1+n/2}, y_{r+2+n/2}, ..., y_n$ and y_n is also adjacent to $x_1, x_2, ..., x_r$. Thus $I_{n;r,s} \subseteq G$ with $2(r+s) = n+1$ and n odd.

Case 4. Suppose $\ell = \frac{n+2}{2}$.

In this case it follows that y_n is not adjacent to $x_{(n+2)/2}, x_{n/2}$, and $x_{(n+4)/2}$, the later non-edge implying that x_1y_2 is an edge. Hence we may assume that x_1y_j is an edge for $j = 1, 2, ...r$ and x_1y_{r+1} is not an edge for some $r \leq (n-2)/2$. By the ℓ – pairing we get that y_n is not adjacent to $x_{(n+4)/2}, x_{(n+6)/2}, ..., x_{r+n/2}$ and is adjacent to $x_{r+1+n/2}$. Now if x_1y_{r+2} were an edge, the cycle

$$x_1, y_{r+2}, x_{r+3}, y_{r+3}, ..., x_{r+1+n/2}, y_n, x_2, y_1, x_1$$

would be a 2ℓ – cycle, thus x_1 is not adjacent to $y_{r+1}, y_{r+2}, ..., y_{(n+2)/2}$ and y_n would be adjacent to $x_{r+1+n/2}, x_{r+2+n/2}, ..., x_n$. If either x_1 or y_n had an adjacency among the collection of vertices $y_{r+1}, x_{r+2}, y_{r+2}, x_{r+2}, ..., x_{r+n/2}, y_{r+n/2}$ then a 2ℓ – cycle results. But this implies that deg x_1 + deg $y_n \leq n$, a contradiction.

Case 5. Suppose $\frac{n+3}{2} \leq \ell \leq \frac{2n}{3}$.

Since G contains no cycles of length 2ℓ, it follows that x_1y_ℓ is not an edge of G. By the ℓ – pairing, this implies that $y_nx_{2\ell-n}$ is an edge of G. As in the previous cases, if $x_1y_{\ell-1}$ were an edge a 2ℓ – cycle would result. Hence, y_n is adjacent to $x_{2\ell-n}, x_{2\ell-n-1}, ..., x_2$ and x_1. But since it is the case that $\frac{n+3}{2} \leq \ell$, it follows that y_n is adjacent to x_3, but this implies that $x_1y_{\ell-1}$ is not an edge, and thus $y_nx_{2\ell-n+1}$ is an edge. Consequently, $y_nx_{n-\ell+1}$ is an edge, resulting in a 2ℓ – cycle, again a contradiction.

Case 6. Suppose $\frac{2n+1}{3} \leq \ell \leq n-2$.

Since G contains no cycles of length 2ℓ, it follows that x_1y_ℓ is not an edge of G. By the ℓ – pairing, this implies that $y_nx_{2\ell-n}$ is an edge of G. As in the previous cases, if $x_1y_{\ell-1}$ were an edge a 2ℓ – cycle would result. Hence, y_n is adjacent to $x_{2\ell-n}, x_{2\ell-n-1}, ..., x_2$ and x_1. But since it is the case that $\frac{2n+1}{3} \leq \ell$, it follows that $y_nx_{n-\ell+1}$ is an edge of G, which results in G containing a cycle of length 2ℓ, a contradiction.

With all cases considered the claim follows. ■

Claim 3 *If G is not bipancyclic and exactly one of x_1y_{n-1} and y_nx_2 is an edge of G then G contains $H_{n;\ell}$ for $n/3 \leq \ell \leq n/2$, thus G is missing at most the 2ℓ cycle.*

Proof. Without loss of generality let $y_n x_2$ be an edge of G, while $x_1 y_{n-1}$ is not. If G is not bipancyclic, then we may assume that G contains no cycle of length 2ℓ. Arguing as in the previous claim, for all cases with the exception of Case 2 and Case 3, the edge $x_1 y_{n-1}$ is unused, thus the proofs for those cases follow as above. When $\ell = \frac{n+1}{2}$ a contradiction arises with the inclusion of the edges to x_1 as in Case 3. Thus we only need to consider the case $\frac{n}{3} \le \ell \le \frac{n}{2}$. Observe that $x_1 y_{n-1}$ not an edge of G implies that $y_n x_{\ell-1}$ is an edge of G, by the ℓ – pairing. If $y_n x_{\ell+1} \in E$ then $y_n, x_{\ell+1}, y_\ell, ..., x_2, y_n$ would form a cycle of length 2ℓ, thus $y_n x_{\ell+1} \notin E$ and hence by the ℓ - pairing have that $x_1 y_2 \in E$. If $x_1 y_{n-2} \in E$ then the 2ℓ – cycle;

$$x_1, y_{n-2}, x_{n-1}, y_{n-1}, x_n, y_n, x_{\ell-1}, y_{\ell-2}, x_{\ell-2}, ..., x_3, y_2, x_1$$

results. Consequently, we may assume that $x_1 y_{n-2} \notin E$ and thus that $y_n x_{\ell-2} \in E$. Continuing in this fashion, we get that x_1 is not adjacent to $y_{n-1}, y_{n-2}, ..., y_{n-\ell}$ and $y_{n-\ell+1}$ and that y_n is adjacent to $x_1, x_2, ..., x_{\ell-1}$. Since $y_n, x_{\ell-1}, y_{\ell-1}, x_\ell, ..., y_{2\ell-3}, x_1, y_n$ would form a cycle of length 2ℓ it follows that $x_1 y_{2\ell-3} \notin E$. Arguing in a similar fashion, we get that x_1 is not adjacent to $y_{2\ell-4}, y_{2\ell-5}, ..., y_\ell$ and by the ℓ – pairing that $y_n x_{2\ell-1-t} \in E$ for $t = 0, 1, ..., n-2\ell+1$. Also observe that $y_n x_{2\ell-t} \notin E$ for $t = 1, ..., \ell-1$ for otherwise $y_n, x_{2\ell-t}, y_{2\ell-t-1}, ..., x_{\ell-1-t}, y_n$ would form a cycle of length 2ℓ. By the ℓ-pairing it follows that x_1 is adjacent to $y_{\ell-1}, y_{\ell-2}, ..., y_1$. With all pairs exhausted, it follows that $deg\ x_1 = \ell$ and the $deg\ y_n = n + 1 - \ell$ and thus $H_{n;\ell} \subseteq G$ and the claim follows. ∎

So we may assume that neither $x_1 y_{n-1}$ nor $y_n x_2$ are edges of G.

Claim 4 *If G is not bipancyclic and neither $x_1 y_{n-1}$ nor $y_n x_2$ are edges of G then for some $F \in \mathcal{F}_n, F \subseteq G$, thus G is missing at most the $2n-2$ cycle.*

Proof. By the ℓ – pairing, since neither $x_1 y_{n-1}$ nor $y_n x_2$ are edges of G, it follows that $y_n x_{\ell-1}$ and $x_1 y_{n-\ell+2}$ are edges of G.

Case 1. Suppose $\ell \le \frac{n+2}{2}$.

By the ℓ – pairing exactly one of $x_1 y_{\ell-1}$ or $y_n x_{2\ell-2}$ is an edge of G. In the former case,

$$x_1 y_{n-\ell+2} x_{n-\ell+3} y_{n-\ell+3} \dots x_n y_n x_{\ell-1} y_{\ell-1} x_1$$

would form a cycle of length 2ℓ, while in the later case,

$$y_n x_{\ell-1} y_{\ell-1} x_\ell \dots y_{2\ell-3} x_{2\ell-2} y_n$$

would form a cycle of length 2ℓ, a contradiction.

Case 2. Suppose $\frac{n+3}{2} \le \ell \le \frac{2n}{3}$.

By the ℓ – pairing, the edge $y_n x_{2\ell-n}$ is in G since the edge $x_1 y_\ell$ is clearly not in G. Also, by the ℓ – pairing and the range of ℓ for this case, exactly one of $x_1 y_{2\ell-n}$ and $y_n x_{3\ell-n-1}$ is an edge of G. In the former case,

$$x_1 y_{n-\ell+2} x_{n-\ell+3} y_{n-\ell+3} \cdots x_n y_n x_{2\ell-n} y_{2\ell-n} x_1$$

would form a cycle of length 2ℓ, while in the later case,

$$y_n x_{2\ell-n} y_{2\ell-n} x_{2\ell-n+1} \cdots y_{3\ell-n-2} x_{3\ell-n-1} y_n$$

would form a cycle of length 2ℓ, a contradiction.

Case 3. Suppose $\frac{2n+1}{3} \leq \ell \leq n-2$.

Suppose for some $1 \leq \iota \leq n-1-\ell$, that $y_n x_{2+\iota} \in E$. Since

$$x_1 y_{n-\ell+2} x_{n-\ell+3} \cdots y_n x_{2+\iota} y_{2+\iota} x_1$$

would form a cycle of length 2ℓ, it follows that $y_{2+\iota} x_1 \notin E$. By the $\ell-$ pairing, it follows that $y_n x_{\ell+\iota+2}$ is an edge of G, but this gives the 2ℓ cycle

$$y_n x_{2+\iota} y_{2+\iota} \cdots x_{\ell+\iota+2} y_n.$$

Consequently, for each $1 \leq \iota \leq n-1-\ell$, it must be the case that $y_n x_{2+\iota} \notin E$, and by the $\ell-$ pairing, it follows that $x_1 y_{n-\ell+2+\iota} \in E$. A symmetric argument shows that for $2 \leq \iota \leq n-\ell, x_1 y_{n-\iota} \notin E$ while $y_n x_{\ell-\iota} \in E$.

Suppose $y_n x_{n-1} \in E$. Since y_n is adjacent to the vertices $x_{\ell-1}, x_{\ell-2}, \ldots x_{2\ell-n}$ and y_n is not adjacent to $x_{n-\ell+1}$, there is a first place in this range, say r, such that $y_n x_r \in E$ and $y_n x_{r-1} \notin E$. By the $\ell-$ pairing we get that $x_1 y_{n-\ell+r-1} \in E$. But now

$$y_n x_r y_{r-1} x_{r-1} \cdots x_1 y_{n-\ell+r-1} x_{n-\ell+r} \cdots x_{n-1} y_n$$

forms a cycle of length 2ℓ. Thus we may conclude that $y_n x_{n-1} \notin E$ and that $x_1 y_{n-\ell} \in E$. Note since $\ell \leq n-2$ and y_n is not adjacent to $x_{n-\ell}$ there is a first place say r, such that $y_n x_r \in E$ and $y_n x_{r-1}, y_n x_{r-2} \notin E$. Arguing in a similar fashion, it follows that $y_n x_{n-2} \notin E$.

For each neighbor of y_n, say x_ι, with $n-\ell+2 \leq \iota \leq n-3$, it follows that x_1 is not a neighbor of $y_{\iota+1}$ since

$$x_1 y_{n-\ell} x_{n-\ell+1} \cdots x_\iota y_n x_n \cdots y_{\iota+1} x_1$$

would form a cycle of length 2ℓ. Also note that x_1 is not adjacent to y_{n-1} and $y_{n-\ell+1}$, both not excluded by the previous argument and that all of the neighbors of y_n lie in the range of ι above, with the exception of x_1 and x_n. Hence we get

$$\deg\ x_1 \leq n-2-(\deg\ y_n - 2)$$

which implies that

$$\deg x_1 + \deg y_n \leq n,$$

a contradiction, thus leaving only the possibility of $\ell = n - 1$.

Case 4. Suppose $\ell = n - 1$.

Since $\ell = n - 1$, it follows that both x_1y_3 and $y_nx_{n-2} \in E$. In addition, by the ℓ – pairing exactly one of x_1y_i or y_nx_{i-1} is an edge of G. This results in the desired conclusion, that for some $F \in \mathcal{F}_n$, $F \subseteq G$, thus G is missing at most the $2n - 2$ cycle. ∎

Consequently, with all cases exhausted, the theorem follows. ∎

5 Hamiltonian graphs that are k-semibipancyclic

In this section we establish the following version of Theorem 2 for bipartite graphs:

Theorem 8 *Let $G = (X, Y, E)$ be a bipartite graph with $X = \{x_1, ..., x_n\}$, $Y = \{y_1, ..., y_n\}$, $n \geq 7$, with hamiltonian cycle $x_1, y_1, x_2, ..., x_n, y_n, x_1$. Suppose $\deg x_1 + \deg y_n \geq n + 1 - k$ for some integer k satisfying $1 \leq k \leq n/2$. Then either G is k-semibipancyclic or $I_{n;r,s} \subseteq G \subseteq J_{n;r,s}$ with $2(r + s) = n + 1 - k$.*

Proof. Let G be a bipartite graph, with partite sets X and Y having $|X| = |Y| = n$, and hamiltonian cycle

$$C_{2n} = x_1, y_1, x_2, y_2, ..., x_n, y_n, x_1.$$

Furthermore, assume that $\deg x_1 + \deg y_n \geq n + 1 - k$. Suppose that G is not k-semibipancyclic. It follows that for some ℓ satisfying $4 \leq 2\ell \leq 2(n - k)$, the graph G does not contain any cycles of length $2\ell, 2\ell + 2, ..., 2(\ell + k)$.

Case 1. Suppose that $\ell \leq n/2 - k$. (The case $\ell \geq n/2 + 1$ is handled by a symmetric argument.)

Since G does not contain any cycles of length $2\ell, 2\ell + 2, ..., 2(\ell + k)$, it follows that x_1 is not adjacent to y_i, for $\ell \leq i \leq \ell + k$ and for $n - \ell - k + 2 \leq i \leq n - \ell + 2$. Let $A = \{y_1, ..., y_{\ell-1}\}$, $B = \{y_{\ell+k+1}, ..., y_{n-\ell-k}\}$, $C = \{y_{n-\ell+2}, ..., y_{n-1}\}$, $A' = \{x_\ell, ..., x_{2\ell+k-1}\}$, $B' = \{x_{2\ell+k+1}, ..., x_{n-1}\}$ and $C' = \{x_2, ..., x_{\ell+k-1}\}$. Note that all adjacencies of x_1 are in $A \cup B \cup C$ with the exception of y_n.

Let a, b, and c be the number of adjacencies of x_1 in A, B, and C, respectively. Let $y_{i_1}, y_{i_2}, ..., y_{i_a}$ with $2 \leq i_1 < i_2 < ... < i_a \leq \ell - 1$ be the adjacencies of x_1 in A. It follows that $y_nx_{i_1+\ell-1}, y_nx_{i_2+\ell-1}, ..., y_nx_{i_a+\ell-1}$ are not edges of G for otherwise a cycle of length 2ℓ would result. Furthermore, for $t = 1, 2, ..., k$, we see that $y_nx_{i_a+\ell-1+t}$ is not an edge of G since a cycle of length $2(\ell + t)$

would result. Observe that these $a+k$ nonadjacencies of y_n are all in A'. Similarly, the b adjacencies of x_1 in B force $b+k$ nonadjacencies of y_n in B'. Finally, suppose that $y_{i_1}, y_{i_2}, ..., y_{i_c}$ with $n-\ell+3 \leq i_1 < i_2 < ... < i_c \leq n-1$ are the adjacencies of x_1 in C. It follows that $y_n x_{\ell-(n-i_1)}$ is not an edge of G since $x_1, y_1, x_2, y_2 ..., x_{\ell-(n-i_1)}, y_n, x_n, y_{n-1}, ..., y_{i_1}, x_1$ would form a cycle of length 2ℓ. Similarly, $y_n x_{\ell-(n-i_2)}, ..., y_n x_{\ell-(n-i_{c-1})}$ and $y_n x_{\ell-(n-i_c)}$ are not edges of G for again a 2ℓ - cycle would result. Additionally, for $t=1,2,...,k$, we see that $y_n x_{\ell-(n-i_c)+t}$ is not an edge of G since a cycle of length $2(\ell+t)$ would result. Note that these are $c+k$ nonadjacencies of y_n in C'. Now the $\deg x_1 = a+b+c+1$, and $\deg y_n \leq n-(a+k+b+k+c+k-m)$, where m is the cardinality of the intersection of A' and C'. Note that B' does not intersect either A' or C'. Since $\left|A' \cap C'\right| = (\ell+k-2)-(\ell-1)+1 = k$, we have that $\deg y_n \leq n-(a+k+b+k+c+k-k) = n-(\deg x_1 - 1 + 2k)$, so that $\deg x_1 + \deg y_n \leq (n+1)-2k$, which is a contradiction since $k \geq 1$.

Case 2. Suppose n is even and $\ell = n/2 - t$ for $0 \leq t \leq k/2-1$. (The case $k/2-1 < t \leq k-1$ is handled by a symmetric argument.)

Since G does not contain any cycles of length $2\ell, 2\ell+2, ..., 2(\ell+k)$, it follows that x_1 is not adjacent to $y_{n/2-t}, ..., y_{n/2+t+1}$. Let $A = \{y_1, ..., y_{n/2-t-1}\}$, $C = \{y_{n/2+t+2}, ..., y_{n-1}\}$, $A' = \{v_{n/2-t}, ..., v_{n-2t+k-2}\}$ and $C' = \{y_2, ..., y_{n/2-t+k-2}\}$. Note that all of the adjacencies of x_1 are in $A \cup C$ with the exception of y_n. Arguing as before, since G is not k-semibipancyclic, containing no cycles of length $2\ell, 2\ell+2, ..., 2(\ell+k)$, the a adjacencies of x_1 in A force $a+k$ nonadjacencies of y_n in A'. Likewise, the c adjacencies of x_1 in C force $c+k$ nonadjacencies of y_n in C'. Note that if the adjacencies of x_1 in A are not consecutive, this forces another previously uncounted nonadjacency of y_n. For example, if $x_1 y_{i+1} \notin E$ and $x_1 y_i \in E$, and y_i is not the last adjacency of x_1 in A then it is clear that y_n is adjacent to neither $x_{i+\ell}$ nor $x_{i+\ell+1}$.

Define $\Delta_A (\Delta_C)$ to be 1 if the adjacencies of x_1 in A (respectively C) are not consecutive on the hamiltonian cycle and 0 otherwise. Now the $\deg x_1 = a+c+1$, and as above, $\deg y_n \leq n-(a+k+c+k+\Delta_A+\Delta_C-m)$, where m is the cardinality of the intersection of A' and C'. Since $|A' \cap C'| = (n/2-t+k-1)-(n/2-t)+1 = k$, we have that $\deg y_n \leq n-(a+c+k+\Delta_A+\Delta_C) = (n+1)-(\deg x_1 + k + \Delta_A + \Delta_C)$. So $\deg x_1 + \deg y_n \leq n+1-k-(\Delta_A+\Delta_C)$, which is a contradiction unless $\Delta_A = \Delta_C = 0$ and $\deg v_1 + \deg v_n = n+1-k$.

Hence, we may assume the adjacencies of x_1 in A and C are consecutive vertices of Y on the hamiltonian cycle and further that the adjacencies of y_n are consecutive vertices of X on the hamiltonian cycle. Without loss of generality we can assume that $\deg x_1 \geq (n+1-k)/2$ and, that $\deg y_n \leq (n+1-k)/2$. Suppose x_1 has no adjacenies in C, that is that $c=0$. Also assume that x_1 is adjacent to the vertices $y_1, ..., y_a$ in A. Thus, G contains even cycles of length $4, 6, ..., 2a$ as well as $2n-2a+2, ..., 2n$. If the $\deg x_1 > (n+1-k)/2$, then it follows that G is missing at most k consecutive even cycle lengths, and thus it would follow that G is k − semibipancyclic. Hence, we may assume that $\deg x_1 = (n+1-k)/2$ and, consequently, $\deg y_n = (n+1-k)/2$, and that G is missing at most even cycles of length $n+1-k$ through $n+1+k$. Since $n \geq 7$, it follows that $\deg y_n \geq 3$. If y_n is adjacent to x_{n-1}, then a cycle of length $n+1+k$ results, and G would be $k-$ semibipancyclic. Thus it must be the case that y_n is adjacent to $x_n, x_1, x_2, x_3, \ldots$, and x_a. Clearly it follows then that $I_{n;a,1} \subseteq G \subseteq J_{n;a,1}$ since any additional edge in the bipartite graph G between a vertex in $y_n, x_1, y_1, \ldots, x_a, y_a$ and a vertex in $x_{a+1}, y_{a+1}, \ldots, y_{n-1}, x_n$ would give a cycle in the range $n+1-k$ to $n+1+k$.

A similar argument results if the only adjacency of x_1 in A is y_1. If y_n is adjacent to x_2 then a cycle of length $n+1-k$ results and it would follow that G is k − semibipancyclic. Again, it would be the case that $I_{n;a,1} \subseteq G \subseteq J_{n;a,1}$. Thus we may assume that $a \geq 2$ and that $c \geq 1$.

Let x_1 be adjacent to the vertices $y_1, \ldots, y_a$ in A and $y_{n-1}, \ldots, y_{n-c}$ in C. If y_n is adjacent to x_2 then even cycles of length $4, 6, \ldots, 2(a+c)$ as well as $2n-2(a+c+1)+4, \ldots, 2n$ occur in G. Thus, as above, the only even cycle lengths that are possibly not in G are $n+1-k$, $n+1-k+2, \ldots$, and $n+1+k$. It follows that $2\ell = n-2t = n+1-k$, which implies that $k-1 = 2t$, but this contradicts the assumption that $t \leq k/2-1$. Consequently, y_n cannot be adjacent to x_2, thus it must be the case that y_n is adjacent to $x_1, x_n, x_{n-1}, \ldots x_{n-d+1}$. Hence we have cycles of even lengths between $2n-(n+1-k)+4 = n+k+3$ and 2n and this case is complete if $t < k/2-1$ or if $t = k/2-1$ and $\deg x_1 > (n+1-k)/2$. So we can assume that $t = k/2-1$ and $\deg x_1 = \deg y_n = (n+1-k)/2$. Note this implies, that since n is even, that k must be odd.

Thus it follows that G is missing precisely $k+1$ even cycle lengths. Now consider the adjacencies of y_n. Since $deg\ y_n = deg\ x_1$, it follows that either y_n is adjacent to x_{n-2}, which yields an $n+k+1-$cycle or y_n is adjacent to $x_2, x_3, \ldots, x_{(n-1-k)/2}$, which implies that G contains $I_{n;r,s}$, with $r = (n-1-k)/2$ and $s = 1$. If x_1 has adjacencies to both A and C then it is easy to see that the graph would necessarily contain cycles whose lengths are between $n-k+1$ and $n+k+1$. This completes the proof of this case.

Case 3. Suppose n is odd and $\ell = \lfloor n/2 \rfloor - t$ for $0 \leq t \leq k/2-1$. (Again, the case $k/2-1 < t \leq k-1$ is handled by a symmetric argument.)

Since $C_{2\ell}, \ldots, C_{2(\ell+k)} \not\subseteq G$, it follows that x_1 is not adjacent to

$$y_{\lfloor n/2 \rfloor - t}, \ldots, y_{\lfloor n/2 \rfloor + t + 1}.$$

Let $A = \{y_1, \ldots, y_{\lfloor n/2 \rfloor - t - 1}\}$, $C = \{y_{\lfloor n/2 \rfloor + t + 2}, \ldots, y_{n-1}\}$, $A' = \{x_{\lfloor n/2 \rfloor - t}, \ldots, x_{n-2t-2+k}\}$ and $C' = \{x_2, \ldots, x_{\lfloor n/2 \rfloor - t - 1 + k}\}$. Note that all adjacencies of x_1 are in $A \cup C$ with the exception of y_n. Further if the adjacencies of x_1 in A are not consecutive, then, as before, this forces another previously uncounted nonadjacency of y_n. Define $\Delta_A(\Delta_C)$ to be 1 if the adjacencies of v_1 in A (respectively in C) are not consecutive on the hamiltonian cycle, and 0 otherwise.

Let a be the number of adjacencies of x_1 in A and let c be the number of adjacencies of x_1 in C. Then, as above, $\deg x_1 = a + c + 1$ and $\deg y_n \leq n-(a+k+c+k+\Delta_A+\Delta_C-m)$, where m is the cardinality of the intersection of A' and C'. Since $|A' \cap C'| = (\lfloor n/2 \rfloor - t - 1 + k) - (\lfloor n/2 \rfloor - t) + 1 = k$, we conclude $\deg y_n \leq n-(a+k+c+k-k+\Delta_A+\Delta_C) = n-(a+c+k+\Delta_A+\Delta_C) = n-(\deg x_1 - 1 + k + \Delta_A + \Delta_C)$. So $\deg x_1 + \deg y_n \leq (n+1-k) - (\Delta_A + \Delta_C)$, which is a contradiction unless $\Delta_A = 0 = \Delta_C$ and $\deg x_1 + \deg y_n = n+1-k$.

The remainder of the proof of Case 3 is identical to that of Case 2. ■

References

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