Optimum Allocation of Rooms in Timetables

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Abstract

X-proper edge colourings of bipartite graphs are defined. These colourings arise in timetables where rooms have to be assigned to courses. The objective is to minimize the number of different rooms in which each course must be taught. An optimum assignment is represented by a k-optimum edge colouring of a bipartite graph. Some necessary conditions for a k-optimum colouring are obtained, in terms of forbidden subgraphs. An algorithm based on removing these forbidden subgraphs to obtain improved colourings is described.

1 Introduction

Many kinds of timetabling problems can be formulated in terms of graphs. Both edge colourings of graphs (see Dempster [1]) and node colourings (see Welsh [3] and Wood [5]) have been used in solving these problems.

We consider a timetabling problem where there are n courses (subjects) which may require different numbers of teaching hours. There are m timeslots each of which represents a unit of time (e.g. 1 hour) in which courses can be taught. A timetable consists of an assignment of courses to the timeslots. Such an assignment can be represented by a bipartite graph.

For example, consider the graph shown in Figure 1. The courses are represented by the nodes C_1, C_2, C_3 and the timeslots by the nodes T_1 and T_2. In this timetable all three courses are taught in timeslot T_1, and similarly in timeslot T_2.

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Lemma 1 *The minimum number of colours in an X-proper edge colouring of G is*

$$\Delta(X) = \max_{x \in X} d(x),$$

the maximum valency of nodes in X.

In this paper we will follow the usual graph-theoretic terminology found, for example, in Harary [2]. In particular a *walk* may have repeated nodes and edges, while a *path* has no repeated nodes.

2 Efficient X-proper edge colourings

Definition 2 An X-proper edge colouring $\mathcal{E}$ of G is called *efficient* if it has $\Delta(X)$ colours and $c_{\mathcal{E}}(Y) = |Y|$, i.e. the edges incident with any node in Y have only one colour.

In terms of our initial problem of assigning rooms, if the bipartite graph G has an efficient colouring, then rooms can be assigned to courses in such a way that each course meets in only one room. In seeking to minimize $c_{\mathcal{E}}(Y)$ for an edge colouring $\mathcal{E}$, we are seeking to find the "best" way to assign rooms in the timetable, i.e. we wish to minimize the number of different rooms in which each course is taught.

Certain families of bipartite graphs always have efficient X-proper edge colourings. For example, complete bipartite graphs can always be coloured efficiently, simply by colouring all edges incident with each node in bipartition Y with the same colour. Also acyclic graphs have efficient colourings. The following result can be proved by induction.

Theorem 1 *Let G be a bipartite graph with bipartition (X, Y). If G is acyclic, then G has an efficient X-proper edge colouring with $\Delta(X)$ colours.*

Not all bipartite graphs have efficient X-proper edge colourings. It can easily be verified that the circuit on six nodes and the 3-cube, shown in Figure 2, do not have efficient colourings.

For clarity, we will often draw the bipartite graph embedded in the plane where the $\circ$ denotes a node in X, and the $\bullet$ denotes a node in Y. Thus different colours must be represented at nodes labelled $\circ$. X-proper edge colourings with $\Delta(X)$ colours giving the minimum value of $c_{\mathcal{E}}(Y)$ for the circuit on six nodes and the 3-cube are shown in Figure 2.

The problem of deciding whether a bipartite graph has an efficient X-proper edge colouring can be reduced to a problem of node colourings as

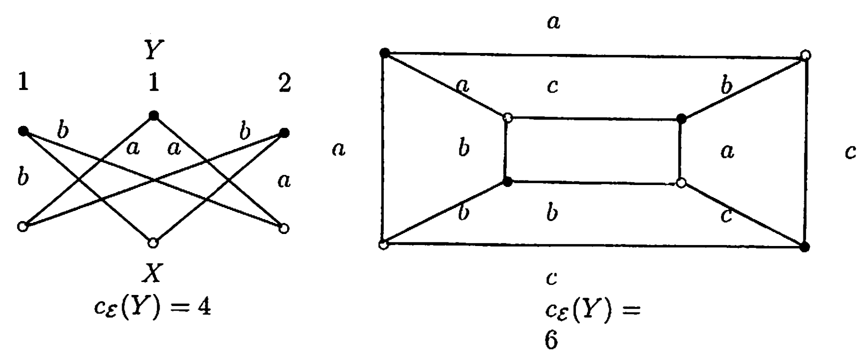

Figure 2:

follows. Let G be a bipartite graph with bipartition (X, Y) representing some timetable. We define a new graph G' with node set $V(G') = Y$ such that $y_1y_2 \in E(G')$ if and only if there is a node $x \in X$ such that $y_1x \in E(G)$ and $xy_2 \in E(G)$. It can easily be verified that G has an efficient X-proper edge colouring with $k = \Delta(X)$ colours if and only if G' is k-node colourable. The colours used to colour the nodes of G' will correspond to the colours assigned to the edges incident with these nodes in the X-proper colouring of G.

For the graphs shown in Figure 2, the corresponding graphs G' will be the complete graphs, K_3 and K_4, which are not 2-node colourable nor 3-node colourable, respectively. Thus the analogy with node colourings shows that the graphs in Figure 2 do not have efficient X-proper colourings.

The problem of determining whether a graph is k-node colourable is known to be NP-hard. We suspect that the problem of deciding whether a bipartite graph has an efficient X-proper colouring is also NP-hard. However, proving this is not the aim of this paper. If a polynomial time algorithm is found for deciding whether a bipartite graph has an efficient X-proper colouring, this would also solve the problem of determining the chromatic number of a large family of graphs (those that can be obtained from bipartite graphs as described above) in polynomial time.

For problems of this complexity it is useful to have "approximate" or heuristic techniques for solving practical problems. Such techniques do not always yield the best solutions but can be executed in polynomial time.

Notice that in the 3-cube of Figure 2, if 4 colours are available we can colour the edges to obtain a lower value $c_{\mathcal{E}}(Y) = 4$. However, in many practical situations the number of colours (corresponding to the number of rooms for the timetable problem) is predetermined, and it is not possible to reduce $c_{\mathcal{E}}(Y)$ simply by increasing this number.

In the next section we look at the minimization of $c_{\mathcal{E}}(Y)$ when the number of colours is fixed. At each step of the recolouring process we maintain an X-proper edge colouring. Unfortunately, it is not possible to carry out a recolouring of nodes of the reduced graph G' that corresponds to this recolouring of edges of G. A colouring of the nodes of G' that is not proper will not correspond to any X-proper colouring of the edges of G. A proper node-colouring of G' will (as described earlier) correspond to an efficient edge-colouring of G. In some sense, the reduced graph G' cannot carry enough 'information' to represent the intermediate states in G of X-proper colourings that are not efficient.

3 k-optimum colourings and bi-alternating subgraphs

Definition 3 An X-proper edge colouring of G is called *k-optimum,* if it has the minimum value of $c_{\mathcal{E}}(Y)$, for all X-proper edge colourings with k colours, $k \geq \Delta(X)$.

The following lemma is obvious.

Lemma 2 *Let G be a bipartite graph with bipartition (X, Y), and let $\mathcal{E}$ be an X-proper k-edge colouring of G. If $\mathcal{E}$ is efficient, then $\mathcal{E}$ is k-optimum.*

The graphs in Figure 2 show a 2-optimum colouring of the circuit on six nodes, and a 3-optimum colouring of the 3-cube, respectively.

Definition 4 Let a and b be two colours in an edge colouring of G. A *bi-alternating walk* with colours a and b is a walk in which the edges are coloured *abaabbaabb....*

Thus, in a bi-alternating walk, the first two edges are coloured differently, and then the colours alternate after a single repetition.

Example 1 In Figure 3, $x_0y_0x_1y_1x_2y_2x_3y_1x_4y_4x_5y_2x_6y_1$ is a bi-alternating walk. The sequence of colours is *abaabbaabb.....*

In Figure 4, $x_0y_0x_1y_1x_2y_2x_3y_1x_1y_0x_4y_4x_5y_1x_6y_2$ is a bi-alternating walk. The sequence of colours is *abaabbaabb.....*

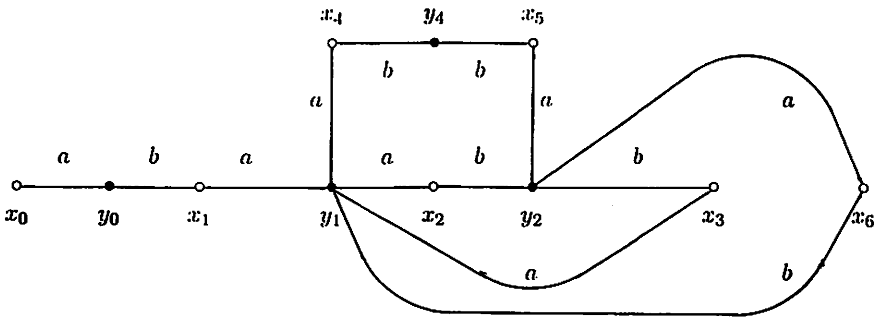

Figure 3:

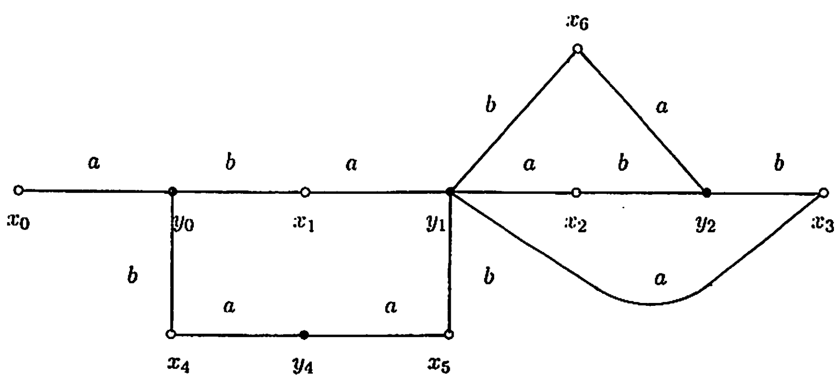

Figure 4:

The initial edge of a bi-alternating walk plays an important role in the following definitions. Although it would seem that a simpler definition of a bi-alternating walk is possible without our initial edge, it would then be more difficult to distinguish between an open and a closed bi-alternating walk because in a closed bi-alternating walk the initial edge x_0y_0 and not just the node y_0 must be repeated.

Definition 5 The *bi-alternating subgraph* of G with colours a and b, and with initial edge x_0y_0, is the union of all bi-alternating walks with colours a and b, and with initial edge x_0y_0.

An *open* bi-alternating walk is a bi-alternating walk in which the initial edge x_0y_0 is not repeated.

A *closed* bi-alternating walk is a bi-alternating walk in which the initial edge x_0y_0 is repeated.

The bi-alternating walks in Example 1 are both open. Figures 5 and 6 show examples of closed bi-alternating walks with initial edge x_0y_0.

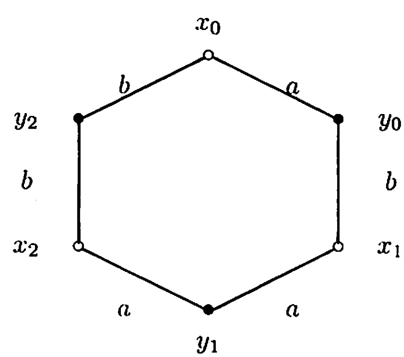

Figure 5:

Definition 6 A *closed* bi-alternating subgraph of G is a bi-alternating subgraph with initial edge x_0y_0, which contains as a subgraph, a closed bi-alternating walk with initial edge x_0y_0.

An *open* bi-alternating subgraph of G is a bi-alternating subgraph which is not closed.

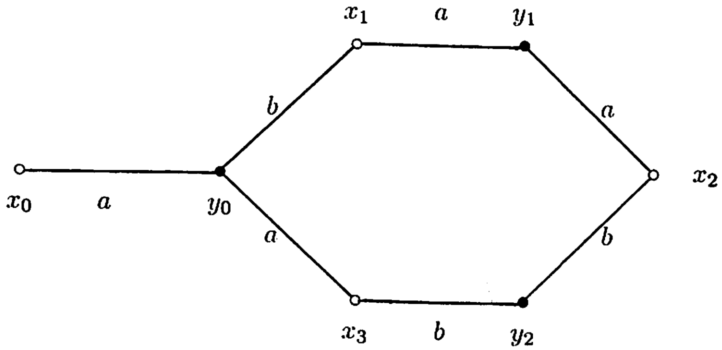

Figure 6:

Thus in an open bi-alternating subgraph there is no bi-alternating walk that begins and ends with the initial edge.

4 Necessary conditions for k-optimum colourings

We will proceed to show that open bi-alternating subgraphs are "forbidden subgraphs" for k-optimum colourings. This is because the colours in an open bi-alternating subgraph (except for the initial edge) can be switched to produce an improved colouring. We will explain why it is necessary to use the entire bi-alternating subgraph and not only a path contained in it.

We will need the following lemmas.

Lemma 3 *Let G be a bipartite graph with X-proper k-edge colouring $\mathcal{E}$. If S is an open bi-alternating subgraph with colours a and b, and with initial edge x_0y_0, then every edge xy_0 $(x \neq x_0)$ in S has colour b.*

Proof. Assume that there is an edge xy_0 of S, with colour a, $x \neq x_0$. Then, xy_0 belongs to some bi-alternating walk with initial edge x_0y_0. Since xy_0 and y_0x_0 have the same colour a, this walk can be extended to include the edge y_0x_0. Thus the initial edge is repeated. This contradicts the fact that S is open. Hence, x_0y_0 is the only edge of S incident with y_0 that has colour a.

■

Lemma 4 *Let G be a bipartite graph with X-proper k-edge colouring $\mathcal{E}$. If S is an open bi-alternating subgraph with colours a and b, and with initial edge x_0y_0, then x_0y_0 is the only edge of S incident with x_0.*

Proof. Assume that there is an edge of S, yx_0 $(y \neq y_0)$. Then yx_0 cannot have colour a, since $\mathcal{E}$ is an X-proper edge colouring and x_0y_0 already has colour a. Therefore, yx_0 has colour b. But yx_0 belongs to some bi-alternating walk with initial edge x_0y_0. Since yx_0 has colour b and x_0y_0 has colour a, this walk can be extended to include the edge x_0y_0. This contradicts the fact that S is open. Hence, x_0y_0 is the only edge of S incident with x_0.

■

The following theorem gives the desired necessary condition for a k-optimum colouring.

Theorem 2 *Let G be a bipartite graph with X-proper k-edge colouring $\mathcal{E}$. If $\mathcal{E}$ is k-optimum then G has no open bi-alternating subgraph.*

Proof. (We prove the contrapositive)

Suppose G has an open bi-alternating subgraph, say S, with colours a and b, and with initial edge x_0y_0. Then x_0y_0 has colour a.

We can recolour the edges of S as follows. Leave x_0y_0 coloured a. For every other edge of S, replace colour a by colour b, or replace colour b by colour a. We call this new colouring $\mathcal{E}'$.

$\mathcal{E}'$ is also an X-proper edge colouring of G because

1. By Lemma 4, x_0y_0 is the only edge of S incident with x_0, and therefore the colours of the edges incident with x_0 remain unchanged.

2. If $x \in X$, $(x \neq x_0)$ is in S, then there is at most one edge incident with x with colour a and at most one edge incident with x with colour b. Interchanging the colours of these edges results in another X-proper edge colouring of G.

Let xy be an edge of S, $(y \neq y_0)$. If xy has colour a, then every other edge with colour a incident with y is also in S. (This is so because any bi-alternating walk with the edge xy can be extended using such an edge.) Therefore, replacing colour a with colour b cannot increase the number of colours represented at y. Similarly, if xy has colour b, replacing colour b with colour a cannot increase the number of colours represented at y.

Every edge with colour b incident with y_0 is in S. Since (by Lemma 3) x_0y_0 is the only edge of S incident with y_0 with colour a, it is clear that if we replace colour b with colour a at all the edges incident with y_0 and leave x_0y_0 unchanged in colour, the number of colours represented at y_0 is reduced by 1. Thus,

$$c_{\mathcal{E}'}(Y) < c_{\mathcal{E}}(Y)$$

and $\mathcal{E}$ is not k-optimum. ■

It should be clear from the above proof that every edge (except the initial edge) in the bi-alternating subgraph must have its colour switched in order to reduce $c_{\mathcal{E}}(Y)$. Consider an edge $xy \neq x_0y_0$ in S. If xy is changed in colour then any other edge of S incident with x must also be changed in colour, otherwise the two edges incident with x will have the same colour, contradicting the X-proper edge colouring of the graph. If xy is changed in colour then any other edge of the same colour (before the change) incident with y must also be changed in colour else the number of colours represented at y may not be reduced. Since bi-alternating subgraphs are connected, once we switch the colour of one edge ($\neq x_0y_0$) we must switch the colour of all other such edges to reduce $c_{\mathcal{E}}(Y)$.

Theorem 2 gives a necessary condition for an X-proper k-edge colouring to be k-optimum. Using the following results, which involve arguments on the lengths of bi-alternating cycles, we are able to find a family of bipartite graphs for which the necessary condition of Theorem 2 is also sufficient for a k-optimum edge colouring.

Definition 7 A *bi-alternating cycle* with initial edge x_0y_0 in a bipartite graph G, is a closed bi-alternating walk

$$x_0y_0x_1y_1\cdots y_nx_0y_0$$

in which no nodes except x_0 and y_0 are repeated.

Theorem 3 *Every closed bi-alternating walk contains a bi-alternating cycle.*

Proof. We prove this result by induction on the number of nodes of the bi-alternating walk. The closed bi-alternating walk with the smallest number of nodes is the walk on 6 nodes shown if Figure 5. Clearly, this is itself a bi-alternating cycle.

Assume that the result is true for bi-alternating walks on less than m nodes ($m > 6$), and let W be a closed bi-alternating walk on m nodes. Without loss of generality, we will assume W is minimal, in the sense that the initial edge appears exactly twice, as the first and last edges. (Every closed bi-alternating walk must contain such a minimal one.) Thus W is either

$$\text{(a)}\ \ x_0y_0x_1y_1\dots x_ny_0x_0 \qquad x_n \neq x_0,$$

or

$$\text{(b)}\ \ x_0y_0x_1y_1\cdots y_{n-1}x_0y_0$$

with colours a and b.

Suppose (a) that y_0x_0 is the last edge. Then x_ny_0 and y_0x_0 both have the same colour a. Then the walk

$$x_ny_0x_1y_1\cdots y_{n-1}x_ny_0$$

has colours *abaabb...bb* and is a closed bi-alternating walk with initial edge x_ny_0 on $m-1$ nodes. By the induction hypothesis this walk (and hence W itself) contains a bi-alternating cycle.

Now consider (b) the case where W is

$$x_0y_0x_1y_1\cdots y_{n-1}x_0y_0$$

with colours a and b. $W-\{y_0\}$ is the walk $x_1y_1\dots x_{n-1}y_{n-1}x_0$ which must contain a path (no repeated nodes)

$$P = x_1y_1x'_2y'_2\dots x'_{k-1}y'_{k-1}x_0$$

where $y'_{k-1} = y_{n-1}$. If the edges $x'_iy'_i$ and $y'_ix'_{i+1}$ have the same colour for $i = 1,\dots,k-1$ (where $x'_1 = x_1$, $y'_1 = y_1$, $x'_k = x_0$) then the cycle

$$x_0y_0x_1y_1x'_2y'_2\dots x'_{k-1}y'_{k-1}x_0$$

is a bi-alternating cycle with colours a and b and the inductive step is proved. Otherwise assume that for some j $(1 \leq j \leq k-1)$, $x'_jy'_j$ and $y'_jx'_{j+1}$ have different colours, say a and b respectively.. Then these cannot be consecutive edges along W. We will assume, without loss of generality, that $x'_jy'_j$ precedes $y'_jx'_{j+1}$ along W. Then there is a subwalk of W

$$x_ry_rx_{r+1}y_{r+1}\dots x_{s-1}y_sx_s$$

such that $y_r = y_s = y'_j$, either $x_r = x'_j$ or $x_{r+1} = x'_j$, and either $x_{s-1} = x'_{j+1}$ or $x_s = x'_{j+1}$. Then the walk

$$x_{s-1}y_rx_{r+1}y_{r+1}\dots x_{s-1}y_s$$

is a closed bi-alternating walk with colours b and a. Clearly this walk has fewer nodes than W (since it must be missing either nodes x_1 or x_0) and by the induction hypothesis, this bi-alternating walk (and hence W) contains a bi-alternating cycle. This completes the inductive step and the result follows.

■

Lemma 5 *Let $\mathcal{E}$ be an X-proper k-edge colouring of G. Then, any bi-alternating cycle in G has length $\equiv 2 \pmod 4$.*

Proof.

Let C be the bi-alternating cycle $x_0y_0x_1y_1 \dots x_ky_kx_0y_0$ with colours a and b. The edges are coloured $abaabb \dots bba$.

At nodes $y_1, y_3, \dots, y_{k-1}$ only colour a is represented.

At nodes $y_2, y_4, \dots, y_k$ only colour b is represented.

At y_0 both colours a and b are represented. Clearly, k is even and the number of nodes of C is $2(k+1) = 2k+2 \equiv 2 \pmod 4$.

■

Corollary 1 *Every closed bi-alternating walk contains a cycle of length $\equiv 2 \pmod 4$.*

Proof.

By Theorem 3 every closed bi-alternating walk contains a bi-alternating cycle. By Lemma 5 this must have length $\equiv 2 \pmod 4$.

■

The following results give a family of bipartite graphs for which the necessary condition in Theorem 2 is also sufficient for a k-optimum edge colouring.

Theorem 4 *Let G be a bipartite graph with bipartition (X, Y), and let G have cycles only of length $\equiv 0 \pmod 4$. Also, let $\mathcal{E}$ be an X-proper k-edge colouring of G. If G has no open bi-alternating subgraph, then $\mathcal{E}$ is k-optimum.*

Proof.

We will assume that G has no open bi-alternating subgraph.

Let x_0y_0 and y_0x_1 be edges in G ($x_0, x_1 \in X, y_0 \in Y$ such that x_0y_0 and y_0x_1 have different colours, say a and b. Consider the bi-alternating subgraph with colours a and b, and with initial edge x_0y_0. Since G has no cycles of length $\equiv 2 \pmod 4$, it follows from Corollary 1 that G has no closed bi-alternating walk. The bi-alternating subgraph with initial edge x_0y_0 must, therefore, be open. This is a contradiction to the hypothesis that G has no open bi-alternating subgraph. Hence, there cannot exist edges x_0y_0 and y_0x_1 incident with a node $y_0 \in Y$, which have two different colours. $\mathcal{E}$ must, therefore, be an efficient edge colouring of G and hence, by Lemma 2 is k-optimum.

■

The following corollary is immediate from Theorem 4.

Corollary 2 *If G is a bipartite graph with no cycles of length $\equiv 2 \pmod 4$, then a k-edge colouring $\mathcal{E}$ of G is k-optimum if and only if it has no open bi-alternating subgraph.*

5 Algorithm and Computer Implementation

The necessary condition given in Theorem 2, for k-optimum colourings, can be used in a practical situation to show that a given assignment of rooms in a timetable is not optimum. A computer search for open bi-alternating subgraphs can be done, and if one is found, the subgraph can be removed by switching colours of the edges of this subgraph (except for the initial edge). This results in an improved colouring (assignment of rooms). This technique will result in a bipartite graph with no open bi-alternating subgraph. We state the algorithm more formally as follows.

Algorithm 1

Input: A bipartite graph G with an X-proper edge colouring.

Iteration:
Repeat

{ $k = 0$ [A counter for the number of open bi-alternating subgraph found.]
For each edge x_0y_0 (with colour a)
For each colour b ($\neq a$)
{ Find the bi-alternating subgraph S with colours a and b and initial edge x_0y_0.
If S is non-trivial and open
{ Interchange the colours a with b for all edges of S except x_0y_0.
$k \longleftarrow k + 1$
}
}
}

Until $k = 0$ [i.e. Stop when all the edges are searched with no open bi-alternating subgraph being found.]

Output: Final X-proper edge colouring.

The necessary condition of Theorem 2 is not sufficient for a k-optimum colouring and the above algorithm may not produce a k-optimum colouring. The graph with colouring shown in Figure 7 (a) has no open bi-alternating subgraph but this colouring is not 2-optimum. A 2-optimum colouring is shown in Figure 7 (b). This shows that there are small examples for which the algorithm will not yield an optimal colouring.

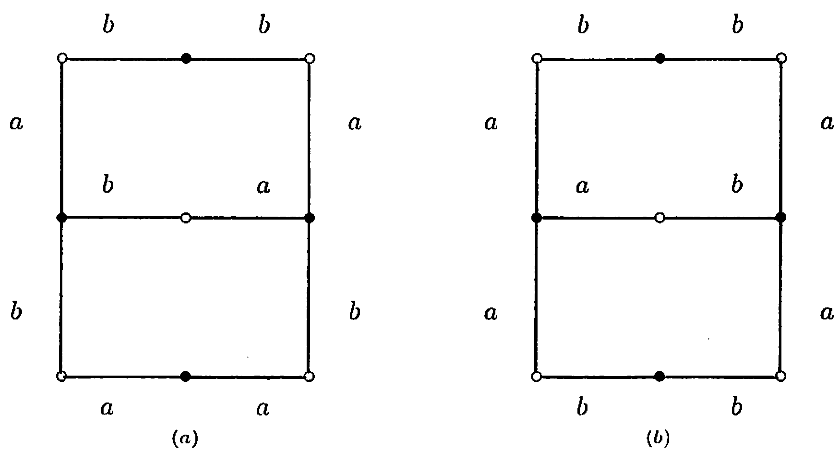

Figure 7:

Our technique of searching for bi-alternating subgraphs will have complexity of $O(n^3)$ where n is the number of nodes of the graph. (See West [4] for complexity of alternating path algorithms, for example.) By using appropriate data structures, more specific to bipartite graphs, it might be possible to obtain lower bounds than this. To perform the search for a bi-alternating subgraph with initial edge x_0y_0, a breadth first search can be performed. Successive steps will yield sets of nodes in the subgraphs at distances on more than in the previous step. If the initial edge x_0y_0 is repeated at any step, the search can be terminated because it will yield a closed bi-alternating subgraph.

The algorithm based on removing bi-alternating subgraphs has been implemented in a computer program used to generate timetables for the Faculty of Agriculture and Natural Science at the University of the West Indies. We will not discuss here the programming details which are mostly routine. In a typical example there were 187 courses, 39 rooms and 50 timeslots. Even for timetables of this size the searching for bi-alternating subgraphs is rapid enough for the technique to be used in an interactive manner. We think it is best to use the technique in an interactive manner since it gives the user a choice of either removing or not removing the bi-alternating subgraph found. In most timetables there are many considerations of which reducing the number of rooms for each course is only one of these. Using the algorithm in an interactive mode allows the flexibility of taking other considerations into account. Our program will display, for example, the following.

Timeslot 5	BL38H	
Rooms	114	113
Timeslot 4	M12B	BL38H
Timeslot 6	M12B	BL38H

Here the course BL38H is taught in timeslot 5 in room 114, and in timeslots 4 and 6 in room 113. The user can then prompt the computer to make an interchange of rooms, in which case the courses M12B and BL38H will interchange rooms in timeslots 4 and 6, but BL38H will remain in the same room (114) in timeslot 5. This results in an improvement for course BL38H as shown below.

Timeslot 5	BL38H	
Rooms	114	113
Timeslot 4	BL38H	M12B
Timeslot 6	BL38H	M12B

Notice here that the course BL38H in timeslot 5 corresponds to the initial edge of a bi-alternating subgraph, illustrating the usefulness of the initial edge in our definition of a bi-alternating subgraph.

Although this study of bi-alternating subgraphs was motivated by timetabling problems, it is quite possible that they will have applications in other problems of an operational research character.

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