ON FINITE $\{2,5\}$–SEMIAFFINE LINEAR SPACES

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ABSTRACT. In this paper finite $\{2,t\}$–semiaffine linear spaces are investigated. When $t = 5$ their parameters are determined, and it is also proved that there is a single finite $\{2,5\}$–semiaffine linear space on $v = 20$ points and with constant point degree 7.

1. INTRODUCTION

An interesting problem which has been intensively investigated in finite geometry is to classify or to characterize finite linear spaces fulfilling an arithmetic or a graphic condition on two or more of their parameters.

For instance, once that the fundamental theorem [de Bruijn–Erdős, [6]] appeared, that is in a finite linear space the number b of lines is greater than or equal to the number v of points, and the equality holds if and only if the linear space is a (possibly degenerate) projective plane, a number of papers have been devoted to the classification of finite linear spaces with $b - v = s$, $s \geq 1$, (cf. for example [5, 10, 12, 14, 15, 16, 17]).

In 1955 G. Pickert [13] asked to characterize linear spaces in which for each point–line pair (p,ℓ) with p outside of ℓ the number $\pi(p,\ell)$ of lines containing p and missing ℓ is at most 1. For example projective planes and affine planes fulfill such a property.

In 1962 Dembowski [7] gave a complete description of all such finite linear spaces, which he proceeded to call *(finite) semiaffine planes.* As P. Dembowski mentioned in his 1962 paper, N. Kuiper had about the same time, and independently, proved the same result, that today is known as the Kuiper–Dembowski theorem.

The Kuiper–Dembowski theorem was the starting point for the investigations of finite linear spaces for particular choices of the set H consisting of the numbers $\pi(p,\ell)$ for every point–line pair (p,ℓ), with p outside of ℓ, of the linear space, (*H–semiaffine linear spaces* [1]).

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M. Oehler (1975) characterized what he called *biaffine planes* [11], which are finite linear spaces with $H = \{1, 2\}$. In the literature we can find a number of papers devoted to the characterization of H–semiaffine linear spaces, the interested reader is referred to [1] whose chapter IV is devoted to this question and it contains also the list of references of most of the papers devoted to the "H–semiaffinity condition".

In this paper finite $\{2, t\}$–semiaffine linear spaces are investigated and they are classified when there are points of different degree. If each point has constant degree $n + 1$ it is proved that if $b \neq n^2 + n + 1$ then there is a finite number of finite $\{2, t\}$-semiaffine linear spaces.

Finally the parameters of a finite $\{2, 5\}$–semiaffine linear space with each point on a constant number $n + 1$ of lines are determined, proving that $n \in \{6, 9, 11, 14, 21, 41, 50, 56, 116, 125, 209, 221, 246\}$.

Moreover it is proved that there is a single finite $\{2, 5\}$–semiaffine linear space with constant point degree 7 on $v = 20$ points.

1.1. Definitions and preliminary results. A *finite linear space* on v *points* and with b *lines* is a pair $(\mathcal{P}, \mathcal{L})$, where $\mathcal{P}$ is a finite set of v points and $\mathcal{L}$ is a family of b subsets (the lines) of $\mathcal{P}$ such that: *any two points are on a unique line, each line contains at least two points and there are at least two lines.*

The *degree* of a point p is the number $[p]$ of lines on p and the *length* of a line ℓ is its size $|\ell|$.

Two lines ℓ and ℓ' of a linear space are *parallel* if $\ell = \ell'$ or $\ell \cap \ell' = \emptyset$.

If (p, ℓ) is a point–line pair with $p \notin \ell$, then $\pi(p, \ell) = [p] - |\ell|$ denotes the number of lines passing through p and parallel to ℓ.

We shall denote by $\mathcal{L}_j$, $2 \leq j \leq n+1$, the family of lines of length j and by b_j its size.

The *near–pencil* on v points is the linear space on v points with a line of length $v - 1$.

A (h, k)*–cross*, $3 \leq h \leq k$, is the linear space on $h + k - 1$ points, with a point of degree 2 on which there are two lines of length h and k, respectively.

Let s and t be non–negative integers with $s < t$, $S_{s,t}$ is the linear space with $t + 3$ points and exactly one line of length $t - s + 2$, while every other line has two points.

A *projective* plane is a linear space such that any two lines meet in a point, and every line has at least three points.

An *affine plane* is a linear space such that for every point–line pair (p, ℓ), with $p \notin \ell$, the number of lines on p missing ℓ is 1.

Let $(\mathcal{P}, \mathcal{L})$ be an affine plane, and let ℓ be a line of $(\mathcal{P}, \mathcal{L})$, then the parallel lines to ℓ partition $\mathcal{P}$, so adding a new point ∞ to all the lines parallel to ℓ, we obtain a new linear space $(\mathcal{P}^*, \mathcal{L}^*)$, where

$$\mathcal{P}^* = \mathcal{P} \cup \{\infty\}$$

$$\mathcal{L}^* = \{L \in \mathcal{L} \mid L \text{ is not parallel to } \ell\} \cup \{L \cup \{\infty\} \mid L \in \mathcal{L}\ L \text{ is parallel to } \ell\},$$

called the *affine plane* $(\mathcal{P}, \mathcal{L})$ *with a point at infinity.*

If $(\mathcal{P}, \mathcal{L})$ is a finite linear space and X is a subset of $\mathcal{P}$, such that $\mathcal{P} \setminus X$ contains at least three non-collinear points, the linear space $(\mathcal{P}', \mathcal{L}')$, where

$$\mathcal{P}' = \mathcal{P} \setminus X$$

$$\mathcal{L}' = \{\ell \setminus \{\ell \cap X\} \mid \ell \in \mathcal{L} \text{ and } |\ell \setminus \ell \cap X| \geq 2\},$$

is called the *complement of* X *in* $(\mathcal{P}, \mathcal{L})$ [1].

The complement of a line in a projective plane is an affine plane.

A *punctured* (*doubly-punctured*) linear space $(\mathcal{P}, \mathcal{L})$ is the complement of a point (two points) in $(\mathcal{P}, \mathcal{L})$.

A finite linear space which has the same parameters as the complement of a set X in a projective plane π is called a *pseudo-complement* of X in π.

Let H be a finite set of non-negative integers and $(\mathcal{P}, \mathcal{L})$ be a finite linear space, then

Definition 1.1. $(\mathcal{P}, \mathcal{L})$ is an H**-semiaffine linear space** if $\pi(p, \ell) \in H$ for any non-incident point-line pair (p, ℓ).

An H-semiaffine linear space is said to be H**-affine** [1] if for any $h \in H$ there exists a pair (p, ℓ) with $p \notin \ell$ such that $\pi(p, \ell) = h$.

Batten and Beutelspacher [1], consider designs as "known" objects, in the sense that in classifying finite linear spaces if we have a design, then we are done. So, since finite $\{s\}$-semiaffine linear spaces, with $s \geq 1$, are precisely 2-$((k+s)(k-1)+1, k, 1)$ block designs, throughout this paper we will assume that $(\mathcal{P}, \mathcal{L})$ is a $\{2, s\}$-affine linear space.

Next we recall the Kuiper–Dembowski theorem and the results on finite $\{2, s\}$-semiaffine linear spaces when $s \leq 4$.

Theorem 1.2 (Kuiper–Dembowski [7]). *If* $(\mathcal{P}, \mathcal{L})$ *is a finite* $\{0, 1\}$*-semiaffine linear space, then it is one of the following:*

(a) *a near-pencil,*
(b) *a projective or an affine plane,*
(c) *a punctured projective plane,*
(d) *an affine plane with one point at infinity.*

The *order* of a finite linear space $(\mathcal{P}, \mathcal{L})$ is the integer n such that $n+1 = max_{p \in \mathcal{P}}[p]$.

Theorem 1.3 (Beutelspacher and Meinhardt 1984 [3]). *If $(\mathcal{P},\mathcal{L})$ is a finite $\{2,3\}$-affine linear space of order n other than $S_{2,3}$, then if $n \geq 6$ $(\mathcal{P},\mathcal{L})$ is the complement of a triangle in a projective plane of order n.*

Theorem 1.4 (Durante and Napolitano 1997 [9], Beutelspacher and Metsch 1986 [4]). *If $(\mathcal{P},\mathcal{L})$ is a finite $\{2,4\}$-affine linear space of order n other than $S_{2,4}$, then $n \in \{5,7,13\}$, moreover:*

- *if $n = 5$ then $v = 15$, $b = 30$, $b_2 = b_4 = 15$,*
- *if $n = 7$, then $v = 33$, $b = 55$, $b_4 = 33$ and $b_6 = 22$*
- *if $n = 13$, then $v = 135$, $b = 183$, $b_{10} = 135$ and $b_{12} = 45$.*

Note. When $n = 5$, there are exactly four non–isomorphic finite $\{2,4\}$–semiaffine linear spaces. In fact, the situation is the same as symmetric configuration 15_4 with missing two blocks added. Since, there exist four symmetric configurations 15_4, (cf. [2], p. 37), the assertion follows. Whereas when $n = 7, 13$ it is not known if there is a finite $\{2,4\}$–semiaffine linear space.

1.2. **Examples of finite $\{2,t\}$–semiaffine linear spaces.** Clearly $S_{2,t}$ is a finite $\{2,t\}$–semiaffine linear space, for every $t \geq 3$.

Let Π a finite projective plane and s,t two non–negative integers, a subset X of points of Π is of *type* (s,t) if each line of Π meets X either in s or t points, and there is a line meeting X in s points and a line meeting X in t points.

Example 1.5. Let π be a projective plane of order 9, and let X be a set of type $\{2,5\}$ in π, then the complement of X in π is a finite $\{2,5\}$–semiaffine linear space. When $\pi = PG(2,9)$, that is the desarguesian projective plane of order nine, then it admits a partition into Baer subplanes, and the union of two disjoint subplanes in π is a set of type $\{2,5\}$ in π.

If π is the Hughes plane of order nine, then there is a set X of type $\{2,5\}$ of size 26 which does not split into two Baer subplanes [8].

Since the projective desarguesian plane $PG(2,q)$, with q square, admits a partition into Baer subplanes, then the previous example can be generalized, and so the complement of two disjoint Baer subplanes in a (desarguesian) projective plane of order $n = (t-2)^2$, $(t \geq 5)$, is a $\{2,t\}$–semiaffine linear space of order n.

Notice that the Example 1.5 shows that a pseudo–complement of two disjoint Baer subplanes may be not a complement.

Example 1.6. Let π be the punctured projective plane of order 4. Breaking up every line of length 4 into an affine plane of order 2 one obtains a finite $\{2,5\}$–semiaffine linear space of order 6, with $v = 20$ points, $b = 46$ lines, $b_2 = 30$ and $b_5 = 16$.

In this paper we prove the following results.

Theorem 1.7. *If $(\mathcal{P}, \mathcal{L})$ is a finite $\{s,t\}$–semiaffine linear space of order n, $(s \geq 1)$, then either $(\mathcal{P}, \mathcal{L})$ is the union of two disjoint lines of length t, or $(\mathcal{P}, \mathcal{L})$ is the linear space $S_{s,t}$ or each point has constant degree.*

Theorem 1.8. *Let $(\mathcal{P}, \mathcal{L})$ be a finite $\{2,t\}$–affine linear space with constant point degree $n+1$ and with $b = n^2+n+1+z$ lines. Then $z \leq t^2-4t+1$. If $n \geq 1/2(t-2)^2+1$ then $z \leq 2t-7$.*

Theorem 1.9. *If $(\mathcal{P}, \mathcal{L})$ is a finite $\{2,t\}$–affine linear space with constant point degree $n+1$ and with $b = n^2+n-1$ lines, then $t \leq 4$.*

Theorem 1.10. *Let $(\mathcal{P}, \mathcal{L})$ be a finite $\{2,t\}$–affine linear space with constant point degree $n+1$ and with $b = n^2+n+1+z$ lines. If $z \neq 0$ then there exist at most finitely many $\{2,t\}$–semiaffine linear spaces. When $z = 0$, then either $(\mathcal{P}, \mathcal{L})$ is the complement of a set of type $\{2,t\}$ in a projective plane of order n, or $n \leq \frac{1}{2}(t^2-1)(t^2+t-2)$.*

Theorem 1.11. *Let $(\mathcal{P}, \mathcal{L})$ be a finite $\{2,5\}$–affine linear space of order n. Then, either $(\mathcal{P}, \mathcal{L})$ is the linear space $S_{2,5}$ or each point has degree $n+1$ and $n \in \{6, 9, 11, 14, 21, 41, 50, 56, 116, 125, 209, 221, 246\}$. Moreover, when $n = 6$ and $v = 20$ the linear space described in Example 1.6 is the unique $\{2,5\}$–semiaffine linear space with such parameters.*

2. Finite $\{s,t\}$–semiaffine linear spaces

In this section $(\mathcal{P}, \mathcal{L})$ is a finite $\{s,t\}$–affine linear space, with $s \geq 1$.

Proposition 2.1. *If there are two lines ℓ and ℓ' such that $\mathcal{P} = \ell \cup \ell'$ then $s = 1$ and $(\mathcal{P}, \mathcal{L})$ is the union of two disjoint lines of length $t+1$.*

PROOF. From $s \geq 1$ there follows that $\ell \cap \ell' = \emptyset$. Thus $s = 1$, otherwise let x be a point of ℓ', a line on x different from ℓ' and parallel to ℓ contains a point outside of $\ell \cup \ell'$. □

In view of the previous proposition we may assume, from now on, that given any two lines there is a point on neither of them.

Proposition 2.2. *Given any two points there is a line through neither of them.*

PROOF. Let p and q be two points of $(\mathcal{P}, \mathcal{L})$. If every line of the linear space pass through one of them, then each point outside of the line pq has degree 2, contradicting the previous assumption. □

Proposition 2.3. *Let $(\mathcal{P}, \mathcal{L})$ be a $\{s,t\}$–semiaffine linear space of order n, then*

(i) $[p] \in \{n+1-t+s, n+1\}$, *for all points $p \in \mathcal{P}$.*

(ii) $|\ell| \in \{n+1+s-2t, n+1-t, n+1-s\}$ *for all lines* $\ell \in \mathcal{L}$.

PROOF. (i) Let p be a point of degree $n+1$, and q be a point different from p. Let ℓ be a line through neither of them, then from $|\ell| = [p] - \pi(p, \ell)$ it follows that $|\ell| \in \{n+1-t, n+1-s\}$. Thus by $[q] = |\ell| + \pi(q, \ell)$ the assertion follows.

(ii) Let ℓ be a line and p be a point not on ℓ. From $|\ell| = [p] - \pi(p, \ell)$ and (i) the assertion follows. □

Proposition 2.4. *There is no line of length* $n+1+s-2t$.

PROOF. Assume to the contrary that there is a line ℓ of length $n+1+s-2t$. Thus $n \geq 2t-s+1$. Let p_0 be a point of degree $n+1$, then $p_0 \in \ell$. Moreover each point outside of ℓ has degree $n+1+s-t$.

Each line not through p_0 *has length* $n+1-t$. Let m be a line not through p_0, then $|m| \neq n+1+s-2t$ and there is a point y outside of $\ell \cup m$. Hence $[y] = n+1+s-t$ and so $|m| = n+1-t$.

All the lines on p_0*, but* ℓ*, have the same length.* Let h and h' two lines on p_0 different from ℓ. Let $p \in h \setminus \{p_0\}$ and $p' \in h' \setminus \{p_0\}$. Since $p, p' \notin \ell$, one has $[p] = [p'] = n+1+s-t$. Therefore counting v via the lines on p and p', respectively, one gets $|h| = |h'|$.

p_0 *is the unique point of degree* $n+1$. Let q be a point of degree $n+1$ different from p_0. Then $q \in \ell$, and each line different from ℓ has length $n+1-t$. Counting v via the lines on q one has

$$v = n+1+s-2t+n(n-t).$$

Counting v via the lines through a point not on ℓ gives

$$v = 1+(n+1+s-t)(n-t).$$

Comparing these two values of v one obtains $n = t+1$, a contradiction.

So each point different from p_0 has degree $n+1-t+s$.

Let h be a line passing through p_0 and different from ℓ, and let p be a point of $\ell \setminus \{p_0\}$.

Counting v via the lines through p_0 gives

$$v = n+1+s-2t+n(|h|-1),$$

and counting v via the lines through p, we have

$$v = n+s-2t+(n+1+s-t)(n-t).$$

Comparing these two values of v, we obtain a contradiction since $|h| \in \{n+1+s-2t, n+1-t\}$. Thus the assertion is proved. □

Hence

Proposition 2.5. *Let* $(\mathcal{P}, \mathcal{L})$ *be a* $\{s,t\}$*-semiaffine linear space of order* n*, then* $|\ell| \in \{n+1-t, n+1-s\}$ *for all lines* $\ell \in \mathcal{L}$.

2.1. **Classification of finite $\{s,t\}$–semiaffine linear spaces (with $s \geq 1$) and with non–constant point degree.** In this section we are going to show that for every s, t, $s \geq 1$, $S_{s,t}$ is the only $\{s,t\}$–semiaffine linear space with non–constant point degree.

Proposition 2.6. *If $(\mathcal{P},\mathcal{L})$ is a $\{s,t\}$-semiaffine linear space of order n with a point of degree $n+1+s-t$ and with $s \geq 1$, then $(\mathcal{P},\mathcal{L})$ is the linear space $S_{s,t}$.*

PROOF. Let q_0 be a point of degree $n+1+s-t$. Since there are points of different degree then there are lines of both length $n+1-t$ and $n+1-s$.

There is a single line of length $n+1-s$.

Assume to the contrary that there are at least two lines ℓ and ℓ' of length $n+1-s$. Then q_0 is the unique point of degree $n+1-t+s$, and all the lines of lenght $n+1-s$ pass through q_0. Thus through each point different from q_0 there pass one line of length $n+1-s$ and n lines of length $n+1-t$, and so each line passing through q_0 has length $n+1-s$.

Counting v via the lines on q_0 and on a point of degree $n+1$, respectively, and comparing these two values of v we obtain $s = 0$ or $s = t$, a contradiction.

Thus there is a single line of length $n+1-s$. Let ℓ be such a line.

Then $q_0 \in \ell$ and no point of degree $n+1$ is on ℓ.

Let p be a point of degree $n+1$, counting v via the lines on p gives

$$v = 1 + (n+1)(n-t).$$

Counting v via the lines through q_0 gives

$$v = n + 1 - s + (n + s - t)(n - t).$$

Comparing these two values, we have

$$n = t + 1.$$

Therefore the lines different from ℓ have length 2, and ℓ has length $t+2-s$, and so $(\mathcal{P},\mathcal{L})$ is the linear space $S_{s,t}$. □

REMARK. Notice that Proposition 2.5 is true also when $s = 0$. However the above Proposition 2.6 is not true when $s = 0$. In fact, the complement of t concurrent lines in a point p, less the point p, in a finite projective plane of order n is a $\{0,t\}$-semiaffine linear space with non–constant point degree different from $S_{s,t}$.

It is not difficult to prove that a $\{0,t\}$-semiaffine linear space with non–constant point degree is the pseudo–complement of t concurrent lines in a point p, less the point p, in a finite projective plane of order n.

3. Finite $\{2,t\}$–semiaffine linear spaces

Proposition 3.1. *Let $(\mathcal{P},\mathcal{L})$ be a $\{2,t\}$–semiaffine linear space with constant point degree $n+1$. If h is a line of length $n-1$, and $\mathcal{M}$ is the set of lines different from h and parallel to h, then*

$$\sum_{\ell\in\mathcal{M}} |\ell| = 2(v-n+1). \tag{3.1}$$

PROOF. The proof consists in counting incident point line pairs (p,ℓ) with $\ell\in\mathcal{M}$. □

Proposition 3.2. *Let $(\mathcal{P},\mathcal{L})$ be a $\{2,t\}$–semiaffine linear space with constant point degree $n+1$, and with $b=n^2+n+1+z$ lines. If ℓ is a line of length $n-1$, then the number of lines different from ℓ and parallel to ℓ is $2n+z$.*

PROOF. The assertion follows from $b=n^2+n+1+z$ and the fact that ℓ meet $(n-1)n$ other lines. □

Let $(\mathcal{P},\mathcal{L})$ be a $\{2,t\}$–semiaffine linear space with constant point degree $n+1$, if p is a point, define

$$\Lambda_p = \text{ the number of lines of length } n-1 \text{ through } p$$

$$\lambda_p = \text{ the number of lines of length } n+1-t \text{ through } p.$$

It is easy to show that the numbers $\Lambda=\Lambda_p$ and $\lambda=\lambda_p$ do not depend on the point p. And since we are interested in $\{2,t\}$–affine linear spaces, we have $\lambda\geq 1$ and $\Lambda\geq 1$.

Counting in double way the point–line pairs (p,ℓ), with $p\in\ell$ and $|\ell|=n+1-t$, and $|\ell|=n-1$ respectively, one obtains

$$b_{n+1-t}=\frac{v\lambda}{n+1-t}, \text{ and } b_{n-1}=\frac{v\Lambda}{n-1}=\frac{v(n+1-\lambda)}{n-1}. \tag{3.2}$$

If $(\mathcal{P},\mathcal{L})$ is a finite $\{2,t\}$–affine linear space with constant point degree $n+1$, then $\Lambda\geq 2$. Otherwise, let $\Lambda=1$, then from Equation (3.2) it follows that

$$b_{n-1}=\frac{v}{n-1}=\frac{n^2-(t-1)n-1}{n-1}=n-(t-2)-\frac{t-1}{n-1},$$

and so

$$n\leq t,$$

contradicting the fact that $n+1-t\geq 2$, since each line has length at least 2.

Hence $\Lambda \geq 2$.

As a consequence of this property we have the following result.

Proposition 3.3. *If $(\mathcal{P}, \mathcal{L})$ is a finite $\{2,t\}$-affine linear space with constant point degree $n+1$, then $b \geq n^2+n-3$.*

PROOF. Since $\Lambda \geq 2$ there are two lines ℓ and ℓ' of length $n-1$ which intersect in a point, so counting the lines meeting ℓ or ℓ' one has $b \geq (n-2)^2+4(n-2)+n+1 = n^2+n-3$. □

Now we show that also $\lambda = 1$ is not possible. Assume $\lambda = 1$, then the second of the two equations (3.2) becomes $b_{n-1} = \dfrac{vn}{n-1}$, and so from $v = n^2-n-1-(t-2)\lambda = n^2-n+1-t$ there follows

$$b_{n-1} = n^2+1-t-\frac{t-1}{n-1}$$

that is not possible.

Let $(\mathcal{P}, \mathcal{L})$ be a finite $\{2,t\}$-affine linear space with constant point degree $n+1$, put $b = n^2+n+1+z$, then a line ℓ of length $n-1$ admits $2n+z$ parallel lines different from ℓ, let u denote the number of lines of length $n-1$ parallel to ℓ. Since $v = n^2-n-1-(t-2)\lambda$, Equation (3.1) gives

$$2n^2-4n-2(t-2)\lambda = u(n-1)+(2n+z-u)(n+1-t)$$

and so

$$(3.3) \qquad 2(t-2)n = 2(t-2)\lambda + u(t-2) - z(t-2) + zn + 2n - z.$$

There follows

$$(3.4) \qquad t-2||zn+2n-z|$$

and

$$(3.5) \qquad u = 2n-2\lambda+z-\frac{zn+2n-z}{t-2}.$$

As a consequence we have the following result.

Proposition 3.4. *If $(\mathcal{P}, \mathcal{L})$ is a finite $\{2,t\}$-affine linear space with constant point degree $n+1$ and $b = n^2+n-1$ lines then $t \leq 4$.*

PROOF. In such a case $z = -2$, and so the assertion easily follows from Equation (3.4). □

Using $v = n^2-n-1-(t-2)\lambda$ we can write (3.2) in the following way

$$(3.6) \qquad b_{n+1-t} = \lambda n + \lambda(t-2) + \frac{(t^2-3t+1)\lambda - (t-2)\lambda^2}{n+1-t},$$

and

$$(3.7) \quad b_{n-1} = n^2 - (\lambda-1)n - (t-2)\lambda - 1 + \frac{\lambda^2(t-2) - (2t-5)\lambda - 2}{n-1}.$$

So

$$b = n^2 + n - 1 + \frac{\lambda^2(t-2) - (2t-5)\lambda - 2}{n-1} + \frac{(t^2-3t+1)\lambda - (t-2)\lambda^2}{n+1-t.}$$

Let ℓ be a line of length $n-1$, then ℓ meets $(n-1)(n-\lambda)$ other lines, and so

(3.8)

$$u = b_{n-1} - (n-1)(n-\lambda) - 1 = 2n - (t-1)\lambda - 2 + \frac{\lambda^2(t-2) - (2t-5)\lambda - 2}{n-1}.$$

By Equation (3.7) one obtains $\lambda \leq n-2$, hence

$$2 \leq \lambda \leq n-2.$$

From Equation (3.3) it follows that

$$z(n+1-t) = 2(t-3)n - 2(t-2)\lambda - u(t-2)$$

and so

$$(3.9) \qquad z = 2(t-3) + \frac{2(t-3)(t-1) - (2\lambda+u)(t-2)}{n+1-t}.$$

Clearly $z \neq 2t-6$. Actually, if $z = 2t-6$ then $(2(t-3)(t-1) = (2\lambda+u)(t-2)$. Thus $t = 4$, and so $3 = 2\lambda + u$, a contradiction since $\lambda \geq 2$.

Proposition 3.5. $z \leq t^2 - 4t + 1$.

Proof. From Equation (3.9) and $\lambda \geq 2$ and $n+1-t \geq 2$ it follows that

$$z \leq 2(t-3) + \frac{2(t-3)(t-1) - 4(t-2)}{2}$$

and so the assertion easily follows. □

Proposition 3.6. *If* $n \geq \frac{1}{2}(t-2)^2 + 1$ *then* $z \leq 2t-7$.

PROOF. If $2(t-3)(t-1)-2(t-2)\lambda-u(t-2)\leq 0$ then $z\leq 2t-7$. If $(2\lambda+u)(t-2)<2(t-3)(t-1)$, then $2\lambda+u<2t-5$, and so $\lambda<t-3$. Equation (3.8) gives

$$u+2\lambda=2n-(t-3)\lambda-2+\frac{\lambda^2(t-2)-(2t-5)\lambda-2}{n-1}$$

and so

$$2t-5>2n-(t-3)\lambda-2+\frac{\lambda^2(t-2)-(2t-5)\lambda-2}{n-1}.$$

There follows, being $\lambda<t-3$,

$$2t-5>2n-(t-3)^2-2+\frac{\lambda^2(t-2)-(2t-5)\lambda-2}{n-1},$$

and so, since from $\lambda\geq 2$ there follows $\lambda^2(t-2)-(2t-5)\lambda-2\geq 0$, we have

$$2t-5>2n-(t-3)^2-2$$

and so

$$n<\frac{1}{2}(t-2)^2+1. \qquad \square$$

Consider now the case $t=3$. From Equation (3.9) it follows that $z=-\frac{u+2\lambda}{n-2}$, and so $z\leq -1$. Moreover Equation (3.5) becomes $u=2z-2\lambda-zn$, and so comparing with Equation (3.8) one has

$$2z-zn=2n-2+\frac{\lambda^2-\lambda-2}{n-1}.$$

It follows that $z=-3$ and $\lambda=n-2$, and so $(\mathcal{P},\mathcal{L})$ is the pseudo-complement of a triangle in a finite projective plane of order n (see [3]). Thus by the results of Beutelspacher and Meinhardt [3], for $n\geq 6$ $(\mathcal{P},\mathcal{L})$ is the complement of a triangle in a projective plane of order n.

3.0.1. *The case $\lambda=2$.* In this section we consider the case $\lambda=2$. This case is interesting since when $\lambda=2$ the number v of points of the linear spaces is maximum and since some interesting examples, such as the complement of two disjoint Baer subplanes in a finite projective plane, fulfill $\lambda=2$.

Proposition 3.7. *Let $(\mathcal{P},\mathcal{L})$ be a finite $\{2,t\}$-semiaffine linear space with constant point degree $n+1$, with exactly two lines of length $n+1-t$ on each point and with $b=n^2+n+1+z$ lines. Then $z=-2+\frac{2(t^2-5t+5)}{n+1-t}$ and either $n=2t^2-9t+9$ and $z=-1$, or $n\leq(t-2)^2$.*

PROOF. Let $\lambda=2$ then Equation (3.8) gives $u=2n-2t$, and so Equation (3.9) becomes

$$z=-2+\frac{2(t^2-5t+5)}{n+1-t}. \tag{3.10}$$

and the assertion easily follows. □

From the previous proposition it follows that $n = (t-2)^2$ if, and only if, $z = 0$, and so $(\mathcal{P}, \mathcal{L})$ is the pseudo-complement of two disjoint Baer subplanes in a projective plane of order n.

When $t = 4$, then $z = -1$ $n = 5$ and there are exactly four such finite linear spaces, (see the note after Theorem 1.4).

3.1. On the finiteness of finite $\{2,t\}$–semiaffine linear spaces, $t \geq 5$. Since the classification of finite H–semiaffine linear spaces with constant point degree is a difficult task, an interesting question in the investigations on them is to get information about their number for a given H, (see for example [3]).

In this section we study such question for $\{2,t\}$–semiaffine linear spaces.

Since the case $t \leq 4$ has been already studied in [9, 3], we consider the cases $t \geq 5$.

We start by recalling the following two useful theorems.

Theorem 3.8 (Beutelspacher and Metsch, [4]). *Suppose $(\mathcal{P}, \mathcal{L})$ is a finite $\{2,t\}$–affine linear space with constant point degree $n+1$, with $b \leq n^2+n+1$ and $n \geq \frac{1}{2}(t^2-1)(t^2+t-2)$. Then $(\mathcal{P}, \mathcal{L})$ is embeddable into a projective plane of order n.*

Theorem 3.9 (Beutelspacher and Metsch, [4]). *Suppose $(\mathcal{P}, \mathcal{L})$ is a finite $\{2,t\}$–affine linear space with constant point degree $n+1$ and more than n^2+n+1 lines. Then $2n \leq (t^2-1)(t^2+t-2)+2t(t-1)(t-3)$.*

Thus, Theorem 3.8 shows that if $b > n^2+n+1$ then there is a finite number of such linear spaces.

Next we study the cases $b \leq n^2+n+1$.

In view of Propositions 3.3 and 3.4 we have to study the cases

$$b \in \{n^2+n-3, n^2+n-2, n^2+n, n^2+n+1\}.$$

From Theorem 3.8 it follows that in order to prove the finiteness of such linear spaces one has to prove that they are not embeddable in a finite projective plane of order n.

If $b = n^2+n+1$, by Theorem 3.8 either $(\mathcal{P}, \mathcal{L})$ is the complement[1] of a set of type $\{2,t\}$ in a finite projective plane of order n, or $n \leq \dfrac{1}{2}(t^2-1)(t^2+t-2)$.

From now on we assume that $(\mathcal{P}, \mathcal{L})$ is a finite $\{2,t\}$–affine linear space embeddable in a finite projective plane π_n with $b \leq n^2+n$. So $(\mathcal{P}, \mathcal{L})$ is obtained from π_n by deleting a set of points X containing at least a line.

[1]For example the complement of two disjoint Baer subplanes in a projective plane of square order n.

$b = n^2 + n - 3$. In such a case X contains four lines. We have three possibilities, the four lines are concurrent, only three of them are concurrent, they form a quadrangle. In every case $(\mathcal{P}, \mathcal{L})$ should have a line of length neither $n-1$ nor $n+1-t$.

$b = n^2 + n - 2$. Also in this case, since X contains three lines, $(\mathcal{P}, \mathcal{L})$ would have lines of length different from $n-1$ and $n+1-t$.

$b = n^2 + n$. Each line of length $n-1$ has $2n-1$ parallel lines, which form two parallel classes. Let ℓ an ℓ' two lines of length $n-1$ meeting in a point p, they have three common parallel lines and at most two of these parallel lines are in a parallel class of either ℓ or ℓ'. These three lines form neither a triangle nor are pairwise parallel. Let Π_ℓ the parallel class of ℓ containing exactly one of these parallel, and u the number of lines of length $n-1$ in Π_ℓ, then

$$n^2 - n - 1 - (t-2)\lambda = u(n-1) + (n-u)(n+1-t),$$

and so

$$(t-2)n + (t-2)\lambda = 1 + (t-2)u,$$

a contradiction since $t \geq 5$.

Thus, summarizing these results and using Proposition 3.3, there follows

Proposition 3.10. *Let $(\mathcal{P}, \mathcal{L})$ be a finite $\{2,t\}$–affine linear space with constant point degree $n+1$ and with $b = n^2+n+1+z$ lines. If $z \neq 0$ there are at most finitely many $\{2,t\}$–semiaffine linear spaces. When $z = 0$, either $(\mathcal{P}, \mathcal{L})$ is the complement of a set of type $\{2,t\}$ in a finite projective plane of order n, or $n \leq \frac{1}{2}(t^2-1)(t^2+t-2) - 1$*

Finally,

Proposition 3.11. *Let $(\mathcal{P}, \mathcal{L})$ be a finite $\{2,t\}$–affine linear space with constant point degree $n+1$ and with $b \leq n^2+n+1$ lines. Then,*

$b = n^2 + n - 3 \Rightarrow t - 2 | 2n - 4,$
$b = n^2 + n - 2 \Rightarrow t - 2 | n - 3,$
$b = n^2 + n \Rightarrow t - 2 | n + 1,$
$b = n^2 + n + 1 \Rightarrow t - 2 | 2n.$

4. Finite $\{2,5\}$–affine linear spaces with constant point degree

In this section $(\mathcal{P}, \mathcal{L})$ is a finite $\{2,5\}$–affine linear space with constant point degree $n+1$.

So each line has length either $n-4$ or $n-1$, $v = n^2 - n - 1 - 3\lambda$, $b \geq n^2 + n - 3$, and

$$b_{n-4} = \frac{v\lambda}{n-4} = \lambda(n+3) + \frac{11\lambda - 3\lambda^2}{n-4}, \tag{4.1}$$

(4.2) $b_{n-1} = \dfrac{v\Lambda}{n-1} = \dfrac{v(n+1-\lambda)}{n-1} = n^2+n-1-\lambda(n+3)+\dfrac{3\lambda^2-5\lambda-2}{n-1}.$

Hence

$$b = n^2 + n - 1 + f, \tag{4.3}$$

with

$$f = \frac{11\lambda - 3\lambda^2}{n-4} + \frac{3\lambda^2 - 5\lambda - 2}{n-1}, \tag{4.4}$$

and

$$\lambda = \frac{1}{6}(2n + 3 \mp \sqrt{\Delta}), \tag{4.5}$$

where

$$\Delta = 4(1-f)n^2 + 4(1+5f)n - 16f + 41. \tag{4.6}$$

From Propositions 3.3, 3.4 and 3.6, and from Equations (4.4) and (4.6) it follows that

$$-2 \le f \le 5, \text{ and } f \ne 0.$$

Consider first the cases $f \ge 2$.

$\boxed{f = 2, b = n^2 + n + 1, \Delta = -4n^2 + 44n + 9}$

Then $\Delta \ge 0$ gives $n \le 11$. It is easy to see that only the cases $n = 9$ and $\lambda = 5$ or $\lambda = 2$ are possible.

When $n = 9$ and $\lambda = 2$ the pseudo–complement of two disjoint Baer subplanes in a projective plane of order 9 is an example of such a linear space, (see Example 1.5).

$\boxed{f = 3, b = n^2 + n + 2, \Delta = -8n^2 + 64n - 7}$

Since Δ is a square it follows that $n = 7$.
But $n = 7$ gives $\lambda = 4$ and so by Equation (4.1) it follows a contradiction.

$\boxed{f = 4, b = n^2 + n + 3, \Delta = -12n^2 + 84n - 23}$

Also this case is not possible.

$\boxed{f = 5, b = n^2 + n + 4, \Delta = -16n^2 + 104n - 39}$

In this case Δ is non–negative if $n = 6$. And so $\lambda = 2, 3$.

The linear space of Example 1.6 has the parameters of the previous case for $\lambda = 3$.

We are going to show that it is the only one with these parameters.

In order to prove this fact we need the notion of dual of a linear space.

REMARK. If $(\mathcal{P}, \mathcal{L})$ is a finite linear space in which any two lines of maximal length meet in a point, and on each point there are at least two lines of maximal length, then the pair $(\mathcal{P}^*, \mathcal{L}^*)$, with

$$\mathcal{P}^* = \{ \text{ lines of maximal length in } (\mathcal{P}, \mathcal{L})\}$$

$$\mathcal{L}^* = \mathcal{P}$$

is a linear space called the *dual* of $(\mathcal{P}, \mathcal{L})$.

Proposition 4.1. *The linear space of Example 1.6 is the only finite $\{2,5\}$-semiaffi-ne linear space of order $n = 6$, with $v = 20$, $b = 46$ and $\lambda = 3$.*

PROOF. Let $(\mathcal{P}, \mathcal{L})$ be a finite $\{2,5\}$–semiaffine linear space with $n = 6$ $\lambda = 3$ $v = 20$ and $b = n^2 + n + 4 = 46$. Then any two lines of length $n - 1$ intersect in a point. So the dual of $(\mathcal{P}, \mathcal{L})$, that is the linear space whose points are the lines of length 5 and whose lines are points of $(\mathcal{P}, \mathcal{L})$, is the affine plane of order 4.

Thus each point of ℓ is the vertex of a configuration of four points and six lines of length 2, that is the affine plane of order 2. There are exactly five of such configurations, since two meeting lines of length 5 get the same five configurations. Let us denote by C_i, $i = 1, \dots, 5$, these affine planes of order 2, then they contain all the lines of length 2 of $(\mathcal{P}, \mathcal{L})$.

Consider the linear space whose points are those of $(\mathcal{P}, \mathcal{L})$, whose lines are the lines of length 5 of $(\mathcal{P}, \mathcal{L})$ and the " five lines" C_i, then it is a linear space of order 5, with 20 points and 21 lines, and the lines C_i are a parallel class, so it is the punctured projective plane of order 4. It follows that $(\mathcal{P}, \mathcal{L})$ is obtained from the punctured projective plane of order 4 by deforming the lines of the parallel class in five affine planes of order 2. So the assertion follows. □

Consider now the cases $f \in \{-2, -1, 1\}$. In such cases Equation (4.6) always gives $\Delta \geq 0$.

In view of the results of the previous section we have $n \leq 336$.

Hence, running a computer programme, Equations (4.5) and (4.6) give the following possibilities

$$\boxed{f = -2, b = n^2 + n - 3, n = 41}$$

$$\boxed{f = -1, b = n^2 + n - 2, n \in \{21, 246\}}$$

$$\boxed{f = 1, b = n^2 + n, n \in \{11, 14, 50, 56, 116, 125, 209, 221\}}$$

Hence

Proposition 4.2. *If $(\mathcal{P}, \mathcal{L})$ is a finite $\{2,5\}$-affine linear space of order n, then*

$$b \in \{n^2 + n - 3, n^2 + n - 2, n^2 + n, n^2 + n + 1, n^2 + n + 4\},$$

and

$$n \in \{6, 9, 11, 14, 21, 41, 50, 56, 116, 125, 209, 221, 246\}.$$

We end with an appendix containing the list of possible parameter cases for a $\{2,5\}$–affine linear space.

Parameter cases of finite $\{2,5\}$–affine linear spaces

$n = 6$:	$v = 23$, $b_2 = 23$, $b_5 = 23$
	$v = 20$, $b_2 = 30$, $b_5 = 16$
$n = 9$:	$v = 65$, $b_5 = 26$, $b_8 = 65$
	$v = 56$, $b_5 = 56$, $b_8 = 35$
$n = 11$:	$v = 88$, $b_7 = 88$, $b_{10} = 44$
$n = 14$:	$v = 175$, $b_{10} = 35$, $b_{13} = 175$
$n = 21$:	$v = 368$, $b_{17} = 368$, $b_{20} = 92$
$n = 41$:	$v = 1528$, $b_{37} = 1528$, $b_{40} = 191$
$n = 50$:	$v = 2380$, $b_{46} = 1190$, $b_{49} = 1360$
$n = 56$:	$v = 3040$, $b_{52} = 760$, $b_{55} = 2432$
$n = 116$:	$v = 13.195$, $b_{112} = 5655$, $b_{115} = 7917$
$n = 125$:	$v = 15.400$, $b_{121} = 4200$, $b_{124} = 11.550$
$n = 209$:	$v = 43.225$, $b_{205} = 17.290$, $b_{208} = 26.600$
$n = 221$:	$v = 48433$, $b_{217} = 13.838$, $b_{220} = 35.224$
$n = 246$:	$v = 59.675$, $b_{242} = 48.825$, $b_{245} = 11.935$.

REMARK. A very short computer run shows that for the parameter case $n = 6$, $v = 23$, there exists exactly one $\{2,5\}$–semiaffine linear space. Its automorphism group is the cyclic group C_{23}.

The authors do not know, whether the structures considered in Proposition 4.2 exist in the cases different from $n = 6, 9$.

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