Maximal Planar (Bipartite) Graphs as Chordal (Bipartite) Graphs

Terry A. McKee
Department of Mathematics & Statistics
Wright State University, Dayton, Ohio 45435 U.S.A.

Abstract

Among the well-studied maximal planar graphs, those having the maximum possible number of 3-cycles are precisely the planar chordal graphs (meaning no induced cycles of lengths greater than three). This motivates a somewhat similar result connecting maximal planar bipartite graphs, 4-cycles, and planar chordal bipartite graphs (meaning bipartite with no induced cycles of lengths greater than four), together with characterizations of planar chordal bipartite graphs as radial graphs of outerplanar multigraphs.

Key words: maximal planar graphs, chordal graphs, maximal planar bipartite graphs, chordal bipartite graphs, radial graphs.

2000 Mathematics Subject Classification: 05C10, 05C62, 05C75.

1 Maximal planar graphs

A *maximal planar graph* is a planar graph such that inserting any additional edge would result in a nonplanar graph. (All graphs in this paper are finite and, unless stated otherwise, simple.) The following results are contained in Sections 1.3 and 2.7 of [5].

Theorem 1 *For a planar graph G with $n \geq 3$ vertices and m edges, the following statements are equivalent*:
(1.1) *G is a maximal planar graph.*
(1.2) *In every plane embedding of G, every face is a facial triangle.*
(1.3) *G has $m = 3n - 6$ edges.*
(1.4) *Every graph formed by inserting a new edge into G contains a homeomorph of K_5.* □

Although maximal planar graphs with n vertices have a fixed number of edges, the number of 3-cycles can still vary since, in addition to the $2n-4$ facial triangles (by Euler's formula, when $n > 3$), there can be additional nonfacial 3-cycles. For instance, both graphs in Figure 1 are maximal planar graphs; the *chordal graph* (meaning no induced cycles of lengths greater than three) on the left has ten 3-cycles, but the octahedron on the right only has eight 3-cycles.

Reference [4] contains any information needed in this paper about chordal graphs. In particular, standard chordal graph theory shows that a maximal planar graph G is chordal if and only if its vertex set has a *perfect elimination ordering*, meaning—in this planar, so K_5-free context—an ordering $\langle v_1, \ldots, v_n \rangle$ of $V(G)$ such that, for each $i \leq n - 4$, $\deg(v_i) = 3$ in the subgraph of G induced by $\{v_i, \ldots, v_n\}$ and v_i is not the center vertex of an induced length-2 path in that subgraph; equivalently, each open neighborhood $N(v_i)$ is isomorphic to K_3 in the subgraph induced by $\{v_i, \ldots, v_n\}$.

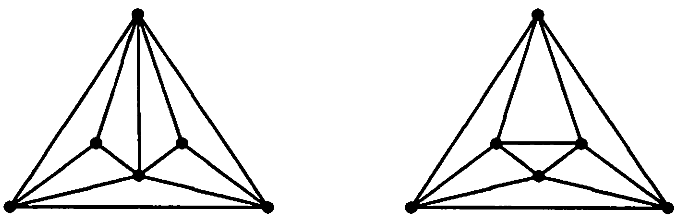

Figure 1: The two maximal planar graphs with 6 vertices. The left graph is chordal with ten 3-cycles; the right graph is not chordal, with eight 3-cycles.

Theorem 2 *Every plane embedding of a maximal planar graph with $n \geq 3$ vertices has at most $3n - 8$ many 3-cycles, with equality if and only if the graph is chordal.*

Proof. First suppose G is maximal planar and chordal with a perfect elimination ordering $\langle v_1, \ldots, v_n \rangle$, where $n \geq 3$. If $n = 3$, then $G \cong K_3$ has only one 3-cycle, and $3 \cdot 3 - 8 = 1$. If $n > 3$, then removing v_1 leaves a graph with, inductively, $3(n-1) - 8$ many 3-cycles, so G has those 3-cycles plus 3 involving v_1, making $3n - 8$ many 3-cycles in all.

Conversely, suppose G is an embedded maximal planar graph with $n \geq 3$ vertices. If $n \leq 4$, then $G \cong K_3$ or K_4 and has $3n - 8$ many 3-cycles. Suppose $n > 4$ (making the minimum vertex degree in G be at least three) and G has the maximum possible number of 3-cycles for n vertices. Observe that G cannot contain an induced wheel consisting of an induced cycle $v_1, v_2, \ldots, v_k, v_1$, $k \geq 4$, with each of $v_1, \ldots, v_k$ adjacent to some vertex w: Otherwise, replacing edges $wv_4, \ldots, wv_k$ with edges $v_1v_3, \ldots, v_1v_{k-1}$ would produce another maximal planar graph with n vertices, but with one more 3-cycle than G (because of the new nonfacial 3-cycle v_1, v_2, v_3, v_1). Also observe that G must contain at least one nonfacial 3-cycle: Otherwise, G would have only the $2n - 4$ facial 3-cycles, which is fewer than the $3n - 8$ many 3-cycles in chordal graphs.

So there must exist a nonfacial 3-cycle $C : a, b, c, a$ that 'surrounds' as few vertices as possible in the embedding. Since G is maximal planar, a must be adjacent to every vertex in a path $b, d_1, \ldots, d_k, c$ with $k \geq 1$ that has all the d_is surrounded by C in the embedding. If $k = 1$, then the 3-cycle $C' : b, c, d_1, b$ will surround fewer vertices than C did, so C' must be a facial triangle and $N(d_1) \cong K_3$; set $v_1 = d_1$ (to begin a perfect elimination ordering). If $k \geq 2$, then since $\{a, b, d_1, \ldots, d_k, c\}$ cannot induce a wheel and G is maximal planar, there must exist an edge $d_{i-1}d_{i+1}$ (letting $d_0 = b$ and $d_{k+1} = c$) so that $\{a, d_{i-1}, d_{i+1}\}$ induces a 3-cycle and $N(d_i) \cong K_3$; set $v_1 = d_i$. Repeating the above for $G - v_1$, then $G - v_2$ and so on, constructs a perfect elimination ordering for G that shows that G is chordal. □

Therefore, the chordal maximal planar graphs are precisely the maximal planar graphs that have the maximum number of 3-cycles. Chordal maximal planar graphs are also easily seen to be precisely the planar 3-trees, where *3-trees* are the graphs defined recursively from K_3 by repeatedly adding new vertices whose open neighborhoods are isomorphic to K_3. (Using the older terminology 'triangulated graphs' for chordal graphs and 'triangulations of the triangle' for maximal planar graphs, the chordal maximal planar graphs are precisely the *triangulated triangulations of the triangle*.)

2 Maximal planar bipartite graphs

A *maximal planar bipartite graph* is a planar bipartite graph such that inserting any additional edge would result in a graph that is either nonplanar or nonbipartite, except that, to avoid uninteresting cases, we shall exclude the stars $K_{1,n-1}$. The following results are contained in Section 3.5 of [5].

Theorem 3 *For a planar bipartite graph $G \not\cong K_{1,n-1}$ with n vertices and m edges, the following statements are equivalent*:
(3.1) *G is a maximal planar bipartite graph.*
(3.2) *In every plane embedding of G, every face is a facial quadrangle.*
(3.3) *G has $m = 2n - 4$ edges.*
(3.4) *Every bipartite graph formed by inserting a new edge into G contains a homeomorph of $K_{3,3}$.* □

A *chordal bipartite graph* [2, 4] is a bipartite graph with no induced cycles of lengths greater than four. Observe that C_4 shows that a chordal bipartite graph need not be chordal (just as a complete bipartite graph need not be complete). For instance, both graphs in Figure 2 are maximal planar bipartite; the graph on the left is chordal bipartite, but the graph G on the right is not (because of the chordless 6-cycle induced by $V(G) - \{a, d\}$). The cube is another maximal planar bipartite graph that is not chordal bipartite.

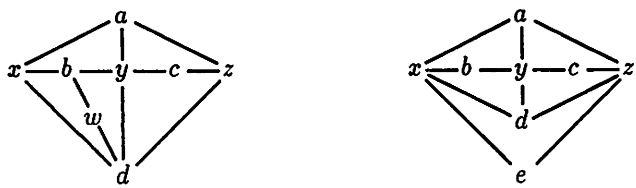

Figure 2: Two maximal planar bipartite graphs with 8 vertices; the left graph is chordal bipartite, but the right graph is not; each has nine 4-cycles.

Using a characterization of chordal bipartite graphs from [3], a maximal planar bipartite graph $G \not\cong K_{1,n-1}$ is chordal bipartite if and only if its vertex set $V(G)$ has a *bipartite vertex elimination ordering*, meaning—in this planar, so $K_{3,3}$-free context—an ordering $\langle v_1, \ldots, v_n \rangle$ of $V(G)$ such that, for each $i \leq n-3$, $\deg(v_i) = 2$ in the subgraph of G induced by $\{v_i, \ldots, v_n\}$ and v_i is not the center vertex of an induced length-4 path in that subgraph; equivalently, each open neighborhood $N(v_i)$ is a vertex set $\{a, b\}$ with $N(a) \subseteq N(b)$ in the subgraph induced by $\{v_i, \ldots, v_n\}$. (For instance, $\langle c, w, b, ..., \rangle$ begins one bipartite vertex elimination ordering in the left graph in Figure 2.)

Although maximal planar bipartite graphs with n vertices other than $K_{1,n-1}$ have a fixed number of edges, the number of 4-cycles can still vary since, in addition to the $n-2$ facial quadrangles (by Euler's formula, when $n > 4$), there can be additional nonfacial 4-cycles. For instance, both graphs in Figure 2 are maximal planar bipartite graphs that have nine 4-cycles, while the cube has only six 4-cycles. Therefore, whether a maximal planar bipartite graph is chordal bipartite cannot be determined simply by counting 4-cycles. But section 3 will present characterizations in terms of being 'radial graphs'; based on these, condition (7.3) of Theorem 7 is a more practical characterization in terms of a type of vertex elimination ordering and how the $K_{2,3}$ subgraphs are embedded.

3 Maximal planar bipartite graphs as radial graphs

Suppose M is any embedded planar *multi*graph with vertex set $V(M)$, edge set $E(M)$, and face set $F(M)$ (loops are not allowed). The *radial graph* [5] is the embedded planar bipartite graph $\mathcal{R}(M)$ having vertex set $V(M) \cup F(M)$ and edges that correspond to incident vertex-face pairs in M. An example is shown in Figure 3. Observe that a face with two edges (a face between parallel edges) in M will produce a degree-2 vertex in $\mathcal{R}(M)$, and that the edges of M correspond precisely to the faces of $\mathcal{R}(M)$, each of which is a facial quadrangle.

Each of the two multigraphs in Figure 4 has the corresponding maximal planar bipartite graph in Figure 2 as its radial graph.

The following result is proved in [5, §3.5].

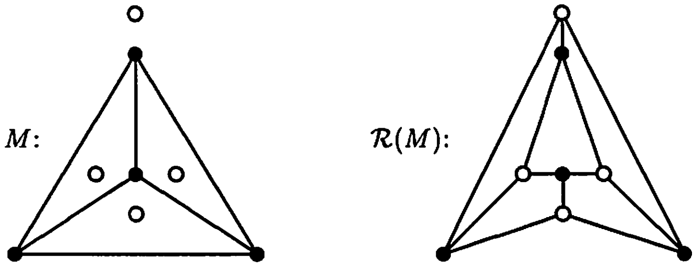

Figure 3: The radial graph of the tetrahedron (on the left, faces marked with hollow circles) is the cube (on the right).

Figure 4: Multigraphs of which the two graphs in Figure 2 are the radial graphs.

Lemma 4 *A graph $G \not\cong K_{1,n-1}$ is a maximal planar bipartite graph if and only if G is the radial graph of an embedded planar multigraph.* □

Notice how either of the color classes of a planar embedding of the bipartite graph $\mathcal{R}(M)$ can be taken to correspond to vertices and the other color class to faces so as to reconstruct both a multigraph M and its *geometric dual* multigraph M^* (meaning that M^* and M are obtained from each other by interchanging the role of vertices and faces). Let ${}^2K_3$ denote the 'doubled triangle' multigraph that consists of three vertices, each two joined by two parallel edges, embedded in the plane with no vertex between two parallel edges. Observe that $\mathcal{R}({}^2K_3)$ is the radial graph both of ${}^2K_3$ (which is *outerplanar*, meaning that it has a plane embedding in which all the vertices are on a common face) and also of its dual graph $K_{2,3}$ (which is not outerplanar). (Reference [1] contains any information needed in this paper about outerplanar graphs.)

Theorem 5 *A maximal planar bipartite graph $G \not\cong K_{1,n-1}$ is chordal bipartite if and only if G is the radial graph* only *of outerplanar multigraphs.*

Proof. Suppose G is a plane embedding of any maximal planar bipartite graph other than $K_{1,n-1}$. By Lemma 4, G is the radial graph of (only) some multigraph M and its geometric dual M^*. Since a planar multigraph is outerplanar if and only if it contains no submultigraph homeomorphic to K_4 or $K_{2,3}$, G is the radial graph only of outerplanar multigraphs if and only if M contains no submultigraph homeomorphic to K_4 (which is self-dual) or to $K_{2,3}$ or ${}^2K_3$ (which are duals of each other).

First suppose $G \cong \mathcal{R}(M)$ is chordal bipartite and M is an embedded planar graph that is not outerplanar (arguing toward a contradiction). Then M must contain a submultigraph M^- that is homeomorphic to one of K_4, $K_{2,3}$, or ${}^2K_3$. In the $M^- \cong K_4$ case, let v, x, y, and z be the four degree-3 vertices in M^-. Let each of a, b, and c be, respectively, the first vertex met starting from v and going toward x, y, and z in M^- (so a, b, and c are in a common color class). Let C be a cycle in $G \cong \mathcal{R}(M)$ that consists of an induced path from a to b whose non-endpoints are 'surrounded' by the cycle through x, v, and y in the embedding of M^-, followed by an induced path from b to c whose non-endpoints are surrounded by the cycle through y, v, and z in M^-, followed by an induced path from c to a whose non-endpoints are surrounded by the cycle through z, v, and x in M^-. This cycle C will be an induced cycle in G of length at least six, contradicting that G is chordal bipartite.

The $M^- \cong K_{2,3}$ case is similar, where v is either of the degree-3 vertices and x, y, and z are the degree-3 vertices.

For the $M^- \cong {}^2K_3$ case, let x, y, and z be the three degree-4 vertices in M^-. Let C be a cycle in G that consists of an induced path from x to y whose non-endpoints are surrounded by the cycle through x and y (but not z) in the embedding of M^-, followed by an induced path from y to z whose non-endpoints are surrounded by the cycle through y and z (but not x) in M^-, followed by an induced path from z to x whose non-endpoints are surrounded by the cycle through z and x (but not y) in M^-. This cycle C will be an induced cycle in G of length at least six, again contradicting that G is chordal bipartite.

Conversely, suppose $G \cong \mathcal{R}(M)$ is not chordal bipartite; say G has an induced cycle $C : v_1, f_1, v_2, f_2, \ldots, v_k, f_k, v_1$ of length $2k \geq 6$ where $v_1, \ldots, v_k$ correspond to vertices of M and $f_1, \ldots, f_k$ correspond to faces of M. Notice each vertex f_i of G is either 'inside' (surrounded by) or 'outside' the cycle $v_1, v_2, \ldots, v_k$, v_1 in the embedding of G. Let C_i [and C_o] be [respectively] a minimal length cycle in G that passes through $v_1, \ldots, v_k$ with all the vertices of C_i [and C_o] other than $v_1, v_2, \ldots, v_k$ inside [outside] C. If neither C_i nor C_o is the boundary of a face in G, then M will contain a submultigraph homeomorphic to $K_{2,k}$, and so one homeomorphic to $K_{2,3}$. If one but not the other of C_i and C_o is the boundary of a face in the embedding of G, then M will contain a submultigraph homeomorphic to K_4. If both C_i and C_o are the boundaries of faces in G, then M will contain a submultigraph homeomorphic to ${}^2K_3$. In any case, G will not be the radial graph only of outerplanar multigraphs. □

Call $\langle v_1, \ldots, v_n \rangle$ a *degree-2 vertex elimination ordering* of a maximal planar bipartite graph G on the vertex set $\{v_1, \ldots, v_n\}$ if and only if, for each $i \leq n-3$, $\deg(v_i) = 2$ in the subgraph of G induced by $\{v_i, \ldots, v_n\}$ (so bipartite vertex elimination orderings are special degree-2 vertex elimination orderings). Both graphs in Figure 2 have degree-2 vertex elimination orderings, although only the one on the left has a bipartite vertex elimination ordering. The cube has no degree-2 vertex elimination ordering.

A multigraph is *series-parallel* if and only if it contains no subgraph that is homeomorphic to K_4. (Reference [1] contains any information needed in this paper about series-parallel graphs.) This is also equivalent to the multigraph being reducible to a single edge by performing a sequence of the following two operations:

- Contracting either of the edges incident with a degree-2 vertex.
- Deleting either of a pair of parallel edges.

The class of series-parallel multigraphs lies between the classes of outerplanar multigraphs and planar multigraphs, and $K_{2,3}$ is the smallest series-parallel graph that is not outerplanar.

Lemma 6 *A maximal planar bipartite graph $G \ncong K_{1,n-1}$ has a degree-2 vertex elimination ordering if and only if G is the radial graph* only *of series-parallel multigraphs.*

Proof. Suppose $G \cong \mathcal{R}(M)$ is a maximal planar bipartite graph other than $K_{1,n-1}$. If G has a degree-2 vertex elimination ordering, then removing each degree-2 vertex in that ordering will correspond to either contracting an edge incident to a degree-2 vertex of M (effectively removing that vertex from M) or deleting an edge of a 2-sided face of M (effectively removing that face from M), and so M will be series-parallel.

Conversely, if $G \cong \mathcal{R}(M)$ is the radial graph only of series-parallel multigraphs, then M is series-parallel and so either contracting an edge incident to a degree-2 vertex of M or deleting an edge of a 2-sided face of M will correspond to removing a degree-2 vertex of G. Repeating this will produce a degree-2 vertex elimination ordering for G. □

Theorem 7 *For every plane embedding of a maximal planar bipartite graph $G \ncong K_{1,n-1}$, the following statements are equivalent:*

(7.1) *G is a maximal planar bipartite graph that is chordal bipartite.*

(7.2) *G is the radial graph* only *of series-parallel multigraphs* and *the degree-3 vertices of each $K_{2,3}$ subgraph of G are on a common face of G.*

(7.3) *G has a degree-2 vertex elimination ordering* and *the degree-3 vertices of each $K_{2,3}$ subgraph of G are on a common face of G.*

Proof. Suppose G is a plane embedding of a maximal planar bipartite graph and $G \cong \mathcal{R}(M) \ncong K_{1,n-1}$. Notice that G being the radial graph only of series-parallel multigraphs is equivalent to M containing no submultigraph homeomorphic to K_4. Only the equivalence of (7.1) and (7.2) needs to be shown, since (7.2) and (7.3) are equivalent by Lemma 6.

First suppose G is chordal bipartite. Since every outerplanar graph is series-parallel, Theorem 5 shows that G is the radial graph only of series-parallel multigraphs. Suppose a and b are the degree-3 vertices of a $K_{2,3}$ subgraph of G, and x, y, and z are the degree-2 vertices. Suppose a and b are not on a common face (arguing toward a contradiction). Let C be a cycle in G that consists of an induced path from x to y whose non-endpoints are 'surrounded' by the cycle a, x, b, y, a in the embedding of G, followed by a similar induced path from y to z surrounded by the cycle a, y, b, z, a in G, followed by a similar induced path from z to x surrounded by the cycle a, z, b, y, a in G. But C would be an induced cycle of length at least six that would contradict G being chordal bipartite.

Conversely (as in the proof of Theorem 5), suppose G is not a chordal bipartite graph, say with an induced cycle $C : v_1, f_1, v_2, f_1, \ldots, v_k, f_k, v_1$ of length $2k \geq 6$ where $v_1, \ldots, v_k$ correspond to vertices of M and $f_1, \ldots, f_k$ correspond to faces of M. Therefore $f_1, \ldots, f_k$ correspond to vertices in the geometric dual multigraph M^* and $v_1, \ldots, v_k$ correspond to faces of M^*. Let C_i [and C_o] be [respectively] a minimal length cycle in G that passes through $v_1, \ldots, v_k$ with all the vertices of C_i [and C_o] other than $v_1, v_2, \ldots, v_k$ inside [outside] C. If neither C_i nor C_o is the boundary of a face in G, then M will contain a submultigraph homeomorphic to $K_{2,3}$ that corresponds to a $K_{2,3}$ subgraph of G having degree-3 vertices that are not be on a common face of G. If one but not the other of C_i and C_o is the boundary of a face in the embedding of G, then M will contain a submultigraph homeomorphic to K_4 and so G would not be the radial graph only of series-parallel multigraphs. If both C_i and C_o are the boundaries of faces in G, then M will contain a submultigraph homeomorphic to ${}^2K_3$, and so M^* will contain a submultigraph homeomorphic to $K_{2,3}$ that corresponds to a $K_{2,3}$ subgraph of G having degree-3 vertices that are not be on a common face of G. □

To illustrate condition (7.3) of Theorem 7, although both graphs in Figure 2 have degree-2 vertex elimination orderings, the right graph has a $K_{2,3}$ subgraph in which a and d are the degree-3 vertices, yet a and d are not on a common face (whereas, in the left graph, the two vertices in each of the pairs $\{x, y\}$, $\{y, z\}$, and $\{x, z\}$ of degree-3 vertices of $K_{2,3}$ subgraphs are on a common face).

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