On λ-fold coverings with maximum block size four for $\lambda = 3, 4$ and 5

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Abstract

The minimum number of blocks having maximum size precisely four that are required to cover, exactly λ times, all pairs of elements from a set of cardinality v is denoted by $g_{\lambda}^{(4)}(v)$. We present a complete solution to this problem for $\lambda = 3, 4$ and 5.

AMS classifications:
Primary: 05B40, Secondary: 05B05, 05B30.

Keywords:
pairwise balanced design, perfect covering.

1 Introduction

The covering number $g_\lambda^{(k)}(v)$ is defined as the cardinality of the minimal pairwise balanced design on a base set of v points and having largest block size k such that every pair occurs exactly λ times in the design. In this paper we will be concerned with the case $k = 4$, so that the designs may contain pairs, triples and quadruples, with at least one block of the latter type. Because of the requirement for a block of size 4, $g_\lambda^{(4)}(v)$ is only defined if $v \geq 4$. The values of $g_\lambda^{(4)}(v)$ are already known for $\lambda = 1$ and 2, with two possible exceptions. We describe these results and give appropriate references below. In this current paper we determine $g_\lambda^{(4)}(v)$ for $\lambda = 3, 4$ and 5. We remark that the solution for $\lambda = 6$ is trivial since a BIBD with parameters $(v, b, r, k, \lambda) = (v, v(v-1)/2, 2(v-1), 4, 6)$ exists for all v. It is our intention to deal with $\lambda > 6$ in a future paper.

In the course of establishing our results, we will make use of a range of combinatorial designs: pairwise balanced designs (PBDs), balanced incomplete block designs (BIBDs), Steiner triple systems (STSs), Kirkman triple systems (KTSs), near-Kirkman triple systems (NKTSs), group divisible designs (GDDs), mutually orthogonal Latin squares (MOLS) and their resulting transversal designs, one-factors, and one-factorizations of complete graphs. Details of all of these may be found in [1].

A general lower bound for $g_\lambda^{(4)}(v)$ is given by Stanton in [10], namely

$$g_\lambda^{(4)}(v) \geq \left\lceil \frac{v}{12}\left\{6\left\lceil\frac{\lambda(v-1)}{3}\right\rceil - \lambda(v-1)\right\}\right\rceil .$$

We shall refer to the right-hand side of this inequality as the *general lower bound* and denote it by $l_\lambda^{(4)}(v)$.

For $\lambda = 1$, the problem of determining $g_1^{(4)}(v)$ was solved by Stanton and Stinson, [12], apart from three exceptional cases $v = 17, 18, 19$. The general results are summarized in Table 1. For $5 \leq v \leq 10$ we have:

$g_1^{(4)}(5) = 5$ (one quadruple and four pairs),
$g_1^{(4)}(6) = 8$ (one quadruple, one triple and six pairs),
$g_1^{(4)}(7) = 10$ (one quadruple, three triples and six pairs),
$g_1^{(4)}(8) = 11$ (one quadruple, six triples and four pairs),
$g_1^{(4)}(9) = 12$ (two quadruples, seven triples and three pairs),
$g_1^{(4)}(10) = 12$ (three quadruples and nine triples).

The results of [12] show that $g_1^{(4)}(v) \geq 29$ for $v = 17, 18, 19$. For $v = 17$, Seah and Stinson, [6], have given a PBD with 31 blocks comprising 17 quadruples, 10 triples and 4 pairs. The design is listed in [13]. Recently, Stanton, [11], has ruled out the value 29. So $30 \leq g_1^{(4)}(17) \leq 31$.

v	$l_1^{(4)}(v)$	$g_1^{(4)}(v)$	(Pairs, Triples)	Comments
$12s$	$12s^2+s$	$l_1^{(4)}(v)$	$(0,4s)$	
$12s+1$	$12s^2+s$	$l_1^{(4)}(v)$	$(0,0)$	BIBD
$12s+2$	$12s^2+7s+1$	$l_1^{(4)}(v)$	$(1,8s)$	
$12s+3$	$12s^2+7s+1$	$l_1^{(4)}(v)$	$(0,4s+1)$	
$12s+4$	$12s^2+7s+1$	$l_1^{(4)}(v)$	$(0,0)$	BIBD
$12s+5$	$12s^2+13s+4$	$l_1^{(4)}(v)$	$(1,8s+3)$	$v\neq 5,17$
$12s+6$	$12s^2+13s+4$	$l_1^{(4)}(v)$	$(0,4s+3)$	$v\neq 6,18$
$12s+7$	$12s^2+13s+4$	$l_1^{(4)}(v)+3$	$(0,7)$	$v\neq 7,19$
$12s+8$	$12s^2+19s+8$	$l_1^{(4)}(v)$	$(1,8s+5)$	$v\neq 8$
$12s+9$	$12s^2+19s+8$	$l_1^{(4)}(v)$	$(0,4s+4)$	$v\neq 9$
$12s+10$	$12s^2+19s+8$	$l_1^{(4)}(v)+3$	$(0,7)$	$v\neq 10$
$12s+11$	$12s^2+25s+13$	$l_1^{(4)}(v)$	$(1,8s+6)$	

Table 1: $g_1^{(4)}(v)$.

For $v = 18$, Stanton, [9] and [8], has shown that $30 \leq g_1^{(4)}(18) \leq 33$. More recently, Grüttmüller, Roberts and Stanton [4] have improved this to $31 \leq g_1^{(4)}(v) \leq 33$. Finally, Stanton, [7], determined the exact value of $g_1^{(4)}(19)$ as 35 by exhibiting a design with 22 quadruples and 13 triples.

For $\lambda = 2$, the problem of determining $g_2^{(4)}(v)$ was solved in [3]. The general results are summarized in Table 2. For the single exceptional case $v = 8$, $g_2^{(4)}(8) = 13$ (seven quadruples, four triples and two pairs).

In subsequent sections we shall make repeated use of a construction which we now describe. The principal ingredient in this construction is a PBD with index $\lambda = 1$ on $12s + w$ points having all its blocks of size four apart from (when $w \neq 4$) a single block of size w. Such a PBD exists if and only if $s \geq (2w+1)/12$ and $w \equiv 1 \pmod 3$ [5]. By deleting all the points lying in the single block of size w (or lying in any block of size four in the case $w = 4$), we form a PBD with index $\lambda = 1$ on $12s$ points having blocks of sizes three and four, with those of size three forming w parallel classes. Denote this design by $P(s, w)$. If we now have n such designs, $P(s, w_i)$ for $i = 1, 2, \ldots, n$ on a common set of $12s$ points, and for each i we take λ_i copies of $P(s, w_i)$, then we obtain a PBD with index $\lambda = \sum_{i=1}^{n} \lambda_i$ on $12s$ points having block sizes three and four, with those of size three forming $p = \sum_{i=1}^{n} \lambda_i w_i$ parallel classes. We will denote this

v	$l_2^{(4)}(v)$	$g_2^{(4)}(v)$	(Pairs, Triples)	Comments
$12s$	$24s^2+2s$	$l_2^{(4)}(v)$	$(0,8s)$	
$12s+1$	$24s^2+2s$	$l_2^{(4)}(v)$	$(0,0)$	BIBD
$12s+2$	$24s^2+8s+1$	$l_2^{(4)}(v)+1$	$(2,4s)$	
$12s+3$	$24s^2+14s+2$	$l_2^{(4)}(v)$	$(0,8s+2)$	
$12s+4$	$24s^2+14s+2$	$l_2^{(4)}(v)$	$(0,0)$	BIBD
$12s+5$	$24s^2+20s+5$	$l_2^{(4)}(v)+1$	$(2,4s+2)$	
$12s+6$	$24s^2+26s+7$	$l_2^{(4)}(v)$	$(0,8s+4)$	
$12s+7$	$24s^2+26s+7$	$l_2^{(4)}(v)$	$(0,0)$	BIBD
$12s+8$	$24s^2+32s+11$	$l_2^{(4)}(v)+1$	$(2,4s+2)$	$v\neq 8$
$12s+9$	$24s^2+38s+15$	$l_2^{(4)}(v)$	$(0,8s+6)$	
$12s+10$	$24s^2+38s+15$	$l_2^{(4)}(v)$	$(0,0)$	BIBD
$12s+11$	$24s^2+44s+21$	$l_2^{(4)}(v)+1$	$(2,4s+4)$	

Table 2: $g_2^{(4)}(v)$.

design by $Q(s, w_1^{\lambda_1} w_2^{\lambda_2} \cdots w_n^{\lambda_n})$ or by $Q(s, w_i^{\lambda_i})$ for short. The condition for existence is that each $w_i \equiv 1 \pmod 3$ and that $s \geq (2w^* + 1)/12$, where $w^* = \max\{w_1, w_2, \ldots, w_n\}$.

Given a PBD, we shall use g_i to denote the number of blocks of size i. Blocks of sizes 2 and 3 will be described as *small* blocks. For a point x of a PBD, we use $r_i(x)$ to denote the number of blocks of size i in which x appears.

2 The case $\lambda = 3$

When $v = 4t$ or $4t+1$ there exists a BIBD with parameters $(v, v(v-1)/4, v-1, 4, 3)$ [1]. It follows immediately that for $t \geq 1$, $g_3^{(4)}(4t) = t(4t-1) = l_3^{(4)}(4t)$ and $g_3^{(4)}(4t+1) = t(4t+1) = l_3^{(4)}(4t+1)$, and that in both cases the PBD has all its blocks of size four.

In the case $v = 4t+2$, the general lower bound is $l_3^{(4)}(4t+2) = 4t^2+3t+1$. Suppose that there is a solution with this number of blocks. Then $g_2 + g_3 + g_4 = 4t^2+3t+1$, and counting pairs gives $g_2 + 3g_3 + 6g_4 = 24t^2+18t+3$. So we have $5g_2 + 3g_3 = 3$, and the only solution is $(g_2, g_3, g_4) = (0, 1, 4t^2+3t)$. However, if an element x appears in the single triple then $2 + 3r_4(x) = 3(4t+1)$, which is clearly impossible. So we must have $g_3^{(4)}(4t+2) \geq 4t^2+3t+2$. If there is a solution with $4t^2+3t+2$ blocks then repeating the above argument gives $(g_2, g_3, g_4) = (0, 3, 4t^2+3t-1)$ and a point which appears in one triple must appear in all three triples. We will prove that for $v = 4t+2$ and $v \neq 2, 6$, there is indeed a solution of this form having $4t^2+3t+2$ blocks. For $v = 2$ the maximum block size is two. For $v = 6$ it is shown in [10] that an optimal solution has ten blocks. Such a solution is given by $[a, 0, 1, 3]_5$ and $[0, 1, 2]_5$, where, for example, $[a, 0, 1, 3]_5$ denotes the five quadruples obtained by developing the starter block $\{a, 0, 1, 3\}$ cyclically modulo 5 with a as a fixed point. Here and subsequently when using this notation we adopt the convention that literal symbols $(a, b, \ldots)$ are invariant, but numerical symbols $(0, 1, \ldots)$ cycle according to the specified modulus. We also make the convention that the notation indicates distinct blocks so that, for example $[0, 3, 6]_9$ specifies three distinct triples rather than nine triples.

Our general solution for $v = 4t + 2$ uses $Q(s, w_i^{\lambda_i})$ designs together with solutions for $v = 10, 14$ and 18. For $v = 10$ and 14, solutions are given in [10] but for completeness we give them here. So, for $v = 10$, take the points $a, b, c, 0, 1, \ldots, 6$ and, as blocks, take the thrice repeated triple $\{a, b, c\}$ along with 21 quadruples $[a, 0, 1, 3]_7$, $[b, 0, 1, 3]_7$, $[c, 0, 1, 3]_7$. For $v = 14$ take the thrice repeated triple $\{a, b, c\}$ along with 44 quadruples $[a, 0, 3, 4]_{11}$, $[b, 0, 4, 5]_{11}$, $[c, 0, 2, 4]_{11}$, $[0, 1, 3, 6]_{11}$. For $v = 18$ suitable quadruples are $[a, 0, 1, 4]_{15}$, $[b, 0, 1, 6]_{15}$, $[c, 0, 2, 8]_{15}$, $[0, 1, 7, 12]_{15}$, $[0, 2, 4, 12]_{15}$.

There exists a $Q(s, 10^3)$ for $s \geq 2$, and such a design has 30 parallel classes of triples on $12s$ points. Now take ten new points, place each such point on the triples of three parallel classes, thereby converting all the existing triples into quadruples, and finally add a solution for $g_3^{(4)}(10)$ on the ten new points. The resulting design on $12s+10$ points has index $\lambda = 3$ and contains a thrice repeated triple with all other blocks being quadruples. Altogether the design has $4t^2 + 3t + 2$ blocks, where $t = 3s + 2$, so that it achieves the bound $g_3^{(4)}(4t+2) = 4t^2 + 3t + 2$ for $t \equiv 2 \pmod 3$ with $t \geq 8$.

There also exists a $Q(s, 13^2 16^1)$ for $s \geq 3$ having 42 parallel classes of triples on $12s$ points. Take 14 new points, place each such point on the triples of three parallel classes, and add a solution for $g_3^{(4)}(14)$ on the 14 new points. Putting $t = 3s + 3$, the resulting design proves that $g_3^{(4)}(4t + 2) = 4t^2 + 3t + 2$ for $t \equiv 0 \pmod 3$ with $t \geq 12$. In a similar fashion, the existence of a $Q(s, 16^1 19^2)$ for $s \geq 4$ having 54 parallel classes of triples gives, with the addition of 18 new points, a design which on putting $t = 3s + 4$ establishes that $g_3^{(4)}(4t + 2) = 4t^2 + 3t + 2$ for $t \equiv 1 \pmod 3$ with $t \geq 16$.

The values of $v \equiv 2 \pmod 4$ which are omitted by the above arguments are 22, 26, 30, 38, 42 and 54. We return to these after consideration of the $v \equiv 3 \pmod 4$ case.

In the case $v = 4t+3$, the general lower bound is $l_3^{(4)}(4t+3) = 4t^2+5t+2$. However, as in the $v = 4t+2$ case, it is easily shown that there is no solution with this number of blocks. Moreover, if there is a solution with $4t^2+5t+3$ blocks then, using the earlier notation, $(g_2, g_3, g_4) = (0, 3, 4t^2+5t)$ and a point which appears in one triple must appear in all three triples. We will prove that for $v = 4t+3$ and $v \neq 7$, there is indeed a solution of this form having $4t^2+5t+3$ blocks. For $v = 7$ it is shown in [10] that an optimal solution has 13 blocks. Such a solution is given by the thrice repeated pair $\{a, b\}$ together with the ten quadruples $[a, 0, 1, 3]_5, [b, 0, 1, 2]_5$.

Our general solution for $v = 4t+3$ again uses $Q(s, w_i^{\lambda_i})$ designs together with solutions for $v = 3, 11$ and 19. For $v = 3$ the maximum block size is three. For $v = 11$ and 19 solutions are given in [10] but again, for completeness, we give them here. So, for $v = 11$ take the points $a, b, c, 0, 1, \ldots, 7$ and, as blocks, take the thrice repeated triple $\{a, b, c\}$ along with the two distinct quadruples $[0, 2, 4, 6]_8$ and the 24 quadruples $[a, 0, 3, 4]_8$, $[b, 0, 1, 3]_8$, $[c, 0, 1, 3]_8$. For $v = 19$ take the thrice repeated triple $\{a, b, c\}$ along with the four distinct quadruples $[0, 4, 8, 12]_{16}$ and the 80 quadruples $[a, 0, 2, 6]_{16}$, $[b, 0, 3, 4]_{16}$, $[c, 0, 2, 5]_{16}$, $[0, 2, 5, 11]_{16}$, $[0, 1, 7, 8]_{16}$.

There exists a $Q(s, 1^1 4^2)$ for $s \geq 1$ having nine parallel classes of triples on $12s$ points. Take three new points, place each such point on the triples of three parallel classes, and add a thrice repeated triple on the three new points. Putting $t = 3s$, the resulting design establishes that $g_3^{(4)}(4t+3) = 4t^2+5t+3$ for $t \equiv 0 \pmod 3$ with $t \geq 3$. Similarly, there exists a $Q(s, 7^1 13^2)$ for $s \geq 3$ having 33 parallel classes on $12s$ points. By adding eleven new points and using a solution for $g_3^{(4)}(11)$ on these points we obtain a design which on putting $t = 3s+2$ proves that $g_3^{(4)}(4t+3) = 4t^2+5t+3$ for $t \equiv 2 \pmod 3$ with $t \geq 11$. The final general case is based on a $Q(s, 19^3)$ which exists for $s \geq 4$ and has 57 parallel classes on $12s$ points. By adding 19 new points, using a solution for $g_3^{(4)}(19)$ on these points, and putting $t = 3s+4$ we prove that $g_3^{(4)}(4t+3) = 4t^2+5t+3$ for $t \equiv 1 \pmod 3$ with $t \geq 16$.

The values of $v \equiv 3 \pmod 4$ which are omitted by the above arguments are 23, 31, 35, 43 and 55. We now deal with these and the omitted cases for $v \equiv 2 \pmod 4$ listed earlier. We start by giving listings for the lowest four values. Those for 22 and 23 were already given in [10] but we repeat them here for completeness. In each case there is a thrice repeated triple $\{a, b, c\}$ in addition to the quadruples specified.

$v = 22$: $[a, 0, 3, 4]_{19}$, $[b, 0, 2, 9]_{19}$, $[c, 0, 5, 11]_{19}$ together with 57 quadruples forming a BIBD with parameters $(19, 57, 12, 4, 2)$ on the points $0, 1, \ldots, 18$.

$v = 23$: $[0, 5, 10, 15]_{20}$ (5 blocks), $[a, 0, 1, 3]_{20}$, $[b, 0, 2, 6]_{20}$, $[c, 0, 6, 13]_{20}$, $[0, 1, 9, 10]_{20}$, $[0, 3, 7, 15]_{20}$, $[0, 2, 5, 16]_{20}$.

$v = 26$: $[a, 0, 1, 5]_{23}$, $[b, 0, 7, 15]_{23}$, $[c, 0, 2, 9]_{23}$, $[0, 4, 7, 10]_{23}$, $[0, 1, 5, 11]_{23}$, $[0, 2, 5, 11]_{23}$, $[0, 1, 9, 11]_{23}$.

$v = 30:$ $[a,0,1,12]_{27}$, $[b,0,4,8]_{27}$, $[c,0,2,7]_{27}$, $[0,2,7,20]_{27}$, $[0,1,5,18]_{27}$, $[0,3,11,17]_{27}$, $[0,6,12,24]_{27}$, $[0,1,3,11]_{27}$.

To deal with some of the remaining values, we introduce a "v to $3v+1$" construction. Note first that there exists a cyclic BIBD with parameters $(2v+1, v(2v+1), 3v, 3, 3)$ for all $v \geq 1$ ([2], page 104). We may take the points to be $0, 1, \ldots, 2v$ and the triples as $[0,1,2]_{2v+1}$, $[0,2,4]_{2v+1}, \ldots$, $[0,v,2v]_{2v+1}$, with the variation that if $3 \mid (2v+1)$ and $w = (2v+1)/3$ then three copies of the short orbit $[0,w,2w]_{2v+1}$ are included. Secondly, if $v \equiv 2$ or $3 \pmod 4$ and if $v \neq 2, 6$ or 7, we can take a solution for $g_3^{(4)}(v)$, that is a PBD of index $\lambda = 3$ having three triples and all other blocks as quadruples, on a disjoint set of points $x_1, x_2, \ldots, x_v$. Now, for each i, place x_i on all $2v+1$ triples $[0,i,2i]_{2v+1}$ to form quadruples $[x_i,0,i,2i]_{2v+1}$ and combine these quadruples with the blocks of the PBD to give a new PBD of index $\lambda = 3$ on the $3v+1$ points $0, 1, \ldots, 2v, x_1, x_2, \ldots, x_v$ having three triples and all other blocks as quadruples. This new PBD establishes the value of $g_3^{(4)}(3v+1)$. Applying this construction in turn to $v = 10, 14$ and 18 gives the values $g_3^{(4)}(31) = 234, g_3^{(4)}(43) = 453$ and $g_3^{(4)}(55) = 744$.

For those orders which still remain, namely $v = 35, 38, 42$ and 54, we use methods based on GDDs.

There exists a 4-GDD of index $\lambda = 3$ and type 8^4 [1]. Take such a design with groups $\{0_i, 1_i, \ldots, 7_i\}$ for $i = 1, 2, 3, 4$. Add three new points a, b, c and place on each set of eleven points $\{a, b, c, 0_i, 1_i, \ldots, 7_i\}$ a PBD giving a solution for $g_3^{(4)}(11)$ and having the thrice repeated triple $\{a, b, c\}$. Removing the excess copies of the triple $\{a, b, c\}$ so that only three copies remain gives a PBD of index $\lambda = 3$ on 35 points which establishes that $g_3^{(4)}(35) = 299$.

There also exists a 4-GDD of index $\lambda = 3$ and type 7^5 [1]. Applying the previous technique and using a solution for $g_3^{(4)}(10)$ gives a PBD of index $\lambda = 3$ on 38 points which establishes that $g_3^{(4)}(38) = 353$.

To obtain a solution for $v = 42$ take three MOLS of order 8 and form a 5-GDD of index $\lambda = 1$ and type 8^5. Removing a single point from one group gives a GDD of index $\lambda = 1$, with blocks of sizes four and five, of type $7^1 8^4$. Triplicate each 4-block and replace each 5-block with the five 4-blocks obtained from it by deleting each element in turn. This gives a 4-GDD with index $\lambda = 3$. As before, add three new points a, b, c and place solutions for $g_3^{(4)}(10)$ or $g_3^{(4)}(11)$ having repeated triple $\{a, b, c\}$ on the union

v	$l_3^{(4)}(v)$	$g_3^{(4)}(v)$	(Pairs, Triples)	Comments
$12s$	$36s^2 - 3s$	$l_3^{(4)}(v)$	$(0,0)$	BIBD
$12s+1$	$36s^2 + 3s$	$l_3^{(4)}(v)$	$(0,0)$	BIBD
$12s+2$	$36s^2 + 9s + 1$	$l_3^{(4)}(v) + 1$	$(0,3)$	
$12s+3$	$36s^2 + 15s + 2$	$l_3^{(4)}(v) + 1$	$(0,3)$	
$12s+4$	$36s^2 + 21s + 3$	$l_3^{(4)}(v)$	$(0,0)$	BIBD
$12s+5$	$36s^2 + 27s + 5$	$l_3^{(4)}(v)$	$(0,0)$	BIBD
$12s+6$	$36s^2 + 33s + 8$	$l_3^{(4)}(v) + 1$	$(0,3)$	$v \neq 6$
$12s+7$	$36s^2 + 39s + 11$	$l_3^{(4)}(v) + 1$	$(0,3)$	$v \neq 7$
$12s+8$	$36s^2 + 45s + 14$	$l_3^{(4)}(v)$	$(0,0)$	BIBD
$12s+9$	$36s^2 + 51s + 18$	$l_3^{(4)}(v)$	$(0,0)$	BIBD
$12s+10$	$36s^2 + 57s + 23$	$l_3^{(4)}(v) + 1$	$(0,3)$	
$12s+11$	$36s^2 + 63s + 28$	$l_3^{(4)}(v) + 1$	$(0,3)$	

Table 3: $g_3^{(4)}(v)$.

of each group with $\{a,b,c\}$. We obtain a PBD of index $\lambda = 3$ on 42 points which establishes that $g_3^{(4)}(42) = 432$.

A solution for $v = 54$ may be similarly obtained by taking three MOLS of order 11, forming a 5-GDD of index $\lambda = 1$ and type 11^5, and removing four points from one group to give a GDD of index $\lambda = 1$, with blocks of sizes four and five, of type $7^1 11^4$. Triplicate the 4-blocks and expand the 5-blocks as previously to get a 4-GDD of index $\lambda = 3$. Finally add three new points a, b, c and place solutions for $g_3^{(4)}(10)$ or $g_3^{(4)}(14)$ having repeated triple $\{a,b,c\}$ on the union of each group with $\{a,b,c\}$. We obtain a PBD of index $\lambda = 3$ on 54 points which establishes that $g_3^{(4)}(54) = 717$.

The results of this section are summarized in Table 3. The exceptional values are $g_3^{(4)}(6) = 10$ and $g_3^{(4)}(7) = 13$.

3 The case $\lambda = 4$

We start by considering the profile of possible solutions in $l_4^{(4)}(v)$ blocks in terms of the numbers of pairs and triples which must be present. First we observe that there exists a BIBD with parameters $(3t+1, t(3t+1), 4t, 4, 4)$ for all $t \geq 1$. Hence, for $v \equiv 1 \pmod 3$ any solution in $l_4^{(4)}(v)$ blocks is without any pairs or triples.

For $v = 3t$, counting the occurrences of each point x with other points in blocks of different sizes gives $r_2(x)+2r_3(x)+3r_4(x) = 4(3t-1) = 12t-4$. So we cannot have $r_2(x) = r_3(x) = 0$, i.e. every point must appear in some small block. It follows that there are at least $3t/3 = t$ small blocks, and therefore $g_2 + g_3 \geq t$. The usual analysis shows that a solution on $l_4^{(4)}(v)$ blocks must satisfy $g_2+g_3+g_4 = 3t^2 - \lfloor \frac{t}{2} \rfloor$ and $g_2+3g_3+6g_4 = 18t^2-6t$, so that $5g_2+3g_3 = 6t-6\lfloor \frac{t}{2} \rfloor = 6\lceil \frac{t}{2} \rceil$. The only solution of this consistent with the previous inequality is $(g_2, g_3) = (0, 2\lceil \frac{t}{2} \rceil)$. But if $g_2 = 0$, then $r_2(x) = 0$ for every x and we get $2r_3(x) + 3r_4(x) = 12t - 4$ for every x. From this we deduce that $r_3(x) \equiv 1 \pmod 3$, so that $r_3(x)$ has possible values $1, 4, 7, \ldots$. However, $\sum_x r_3(x) = 3g_3 = 6\lceil \frac{t}{2} \rceil$. If t is even this evaluates to $3t$ and so $r_3(x) = 1$ for every point x, while if t is odd it evaluates to $3t + 3$ so precisely one point a must have $r_3(a) = 4$ and all other points x have $r_3(x) = 1$. Thus for even t we must have no pairs and t triples with each point appearing in one triple, while for odd t we must have no pairs and $t + 1$ triples with one point appearing in four triples, and each other point appearing in one triple.

In the case $v = 3t+2$, for each point x we have $r_2(x)+2r_3(x)+3r_4(x) = 4(3t+1) = 12t+4$. From this we deduce that $(r_2(x), r_3(x)) \equiv (0,2)$, $(1,0)$ or $(2,1) \pmod 3$. So, for all solutions we have $2r_2(x)+r_3(x) \geq 2$. Summing over all points x gives $4g_2+3g_3 \geq 6t+4$. But a solution in $l_4^{(4)}(v)$ blocks must satisfy $g_2+g_3+g_4 = 3t^2+4t+2$ and $g_2+3g_3+6g_4 = 18t^2+18t+4$, so that $5g_2+3g_3 = 6t+8$. Combining this with the previous inequality gives $g_2 \leq 4$. Thus the only solutions for (g_2, g_3) are $(1, 2t+1)$ and $(4, 2t-4)$. In the case $g_2 = 4$ the inequalities above must be equalities and so $2r_2(x) + r_3(x) = 2$ for every x, giving $(r_2(x), r_3(x)) = (0,2)$ or $(1,0)$. Thus, if $g_2 = 4$, there are eight points which each appear in one pair and no triples, and all the remaining $3t - 6$ points appear in no pairs and in two of the $2t - 4$ triples. The case when $g_2 = 1$ may be analysed similarly; there are two points, say a, b which appear in the single pair. If $2r_2(x) + r_3(x) > 5$ for any point x then repeating the initial argument gives first $4g_2 + 3g_3 > 6t + 7$, and this leads to $g_2 < 1$, a contradiction. Hence $2r_2(x)+r_3(x) \leq 5$ for all x. Bearing in mind that for $g_2 = 1$ we have $r_2(x) = 0$ or 1, the only possibilities are $(r_2(x), r_3(x)) = (0,2)$, $(0,5)$, $(1,0)$ or $(1,3)$, and in order to achieve $g_2 = 1$ we require $4g_2 + 3g_3 = 6t + 7$ and hence precisely one point z to have $(r_2(z), r_3(z)) = (0,5)$ or $(1,3)$. Thus there are two possible structures for a solution when $g_2 = 1$, namely

(i) A pair ab, three triples containing a, no triples containing b, and a further $2t - 2$ triples. All points $x \neq a, b$ appear in precisely two triples. All remaining blocks are quadruples.

(ii) A pair ab, no triples containing a or b, a point $c \neq a, b$ lying in precisely five triples, and a further $2t-4$ triples. All points $x \neq a, b, c$ appear in precisely two triples. All remaining blocks are quadruples.

We shall examine $g_4^{(4)}(v)$ in more detail by splitting v into residue classes modulo 12. Table 4 gives the general lower bound $l_4^{(4)}(v)$ in each class. Apart from the classes $v \equiv 2, 3, 6$ and 11 (mod 12), and with a few small exceptions, the general lower bound is achieved either by combining solutions for $g_1^{(4)}(v)$ and $g_3^{(4)}(v)$ or by using an appropriate BIBD. These solutions are noted in Table 4 with the range of validity. We shall show subsequently that the general lower bound is also achieved in the cases $v \equiv 2, 3, 6$ and 11 (mod 12).

v	$l_4^{(4)}(v)$	$g_4^{(4)}(v)$	Comments
$12s$	$48s^2 - 2s$	$l_4^{(4)}(v)$	$g_1^{(4)}(v) + g_3^{(4)}(v)$
$12s+1$	$48s^2 + 4s$	$l_4^{(4)}(v)$	BIBD
$12s+2$	$48s^2 + 16s + 2$	$l_4^{(4)}(v)$	See subsequent text
$12s+3$	$48s^2 + 22s + 3$	$l_4^{(4)}(v)$	See subsequent text
$12s+4$	$48s^2 + 28s + 4$	$l_4^{(4)}(v)$	BIBD
$12s+5$	$48s^2 + 40s + 9$	$l_4^{(4)}(v)$	$g_1^{(4)}(v) + g_3^{(4)}(v)$, $s \geq 2$
$12s+6$	$48s^2 + 46s + 11$	$l_4^{(4)}(v)$	See subsequent text
$12s+7$	$48s^2 + 52s + 14$	$l_4^{(4)}(v)$	BIBD
$12s+8$	$48s^2 + 64s + 22$	$l_4^{(4)}(v)$	$g_1^{(4)}(v) + g_3^{(4)}(v)$, $s \geq 1$
$12s+9$	$48s^2 + 70s + 26$	$l_4^{(4)}(v)$	$g_1^{(4)}(v) + g_3^{(4)}(v)$, $s \geq 1$
$12s+10$	$48s^2 + 76s + 30$	$l_4^{(4)}(v)$	BIBD
$12s+11$	$48s^2 + 88s + 41$	$l_4^{(4)}(v)$	See subsequent text

Table 4: $g_4^{(4)}(v)$.

For $v < 4$ there can be no quadruples. For $g_4^{(4)}(5)$ a solution in $l_4^{(4)}(5)$ blocks is given by

$$ab, b01, b02, b12, a012, a012, ab01, ab02, ab12.$$

For $g_4^{(4)}(8)$ a solution in $l_4^{(4)}(8)$ blocks is given by

$$ab, c01, c03, c12, c24, c34, ca24, ca13, cab0, cab1, cb34, cb02,$$
$$ab23, a034, a124, a014, a023, b024, b134, b014, b123, 0123.$$

For $g_4^{(4)}(9)$ a solution in $l_4^{(4)}(9)$ blocks is given by a BIBD with parameters $(8, 14, 7, 4, 3)$ on the point set $\{0, 1, \ldots, 7\}$ along with the four distinct triples $[a, 0, 4]_8$ and the eight quadruples $[a, 0, 1, 3]_8$.

For $g_4^{(4)}(17)$ a solution in $l_4^{(4)}(17)$ blocks can be obtained by taking a resolvable BIBD with parameters $(12, 44, 11, 3, 2)$ [1] and duplicating all the blocks to give a $(12, 88, 22, 3, 4)$-BIBD having 22 parallel classes of triples. Then take five new points, place each such point on the triples of four parallel classes, and add a solution for $g_4^{(4)}(5)$ on the five new points.

Consider now the case $v = 12s+2$. In order to obtain a general solution for this residue class in $l_4^{(4)}(v)$ blocks, we need a solution for $v = 14$ in $l_4^{(4)}(14) = 66$ blocks. The blocks of such a solution, having 4 pairs, 4 triples and 58 quadruples are as follows:

01,	23,	45,	67,	*abc*,	*abc*,	*def*,	*def*,
0347,	0356,	0247,	0246,	0256,	1247,	1257,	1346,
1356,	1357,	*ab*01,	*ab*23,	*ac*45,	*ac*67,	*bc*05,	*bc*16,
*de*01,	*de*23,	*df*45,	*df*67,	*ef*05,	*ef*16,	*be*34,	*be*46,
*be*57,	*be*27,	*cf*02,	*cf*24,	*cf*13,	*cf*37,		

together with 28 further quadruples obtained by placing each of the seven pairs *ad*, *bd*, *cd*, *ae*, *ce*, *af*, *bf* onto one of the seven one-factors $\{01, 25, 36, 47\}$, $\{03, 14, 26, 57\}$, $\{03, 17, 26, 45\}$, $\{04, 12, 35, 67\}$, $\{06, 14, 25, 37\}$, $\{07, 12, 34, 56\}$, $\{07, 15, 23, 46\}$.

There exists a $Q(s, 13^2 16^2)$ for $s \geq 3$ having 58 parallel classes of triples on $12s$ points. Take 14 new points, place each such point on the triples of four parallel classes, and add a solution for $g_4^{(4)}(14)$ on the 14 new points. The resulting design on $12s + 14$ points establishes that $g_4^{(4)}(v) = l_4^{(4)}(v)$ for $v \equiv 2 \pmod{12}$ with $v \geq 50$. The values of $v \equiv 2 \pmod{12}$ omitted by this argument are $v = 26$ and 38; we return to these later.

We next turn our attention to the case $v = 12s + 3$. In order to obtain a general solution for this residue class in $l_4^{(4)}(v)$ blocks, we need a solution for $v = 15$ in $l_4^{(4)}(15) = 73$ blocks having 6 triples and 67 quadruples. Such a design is listed below.

$x01$,	$x23$,	$x45$,	$x67$,	abc,	def,
$xabf$,	$xace$,	$xbcd$,	$xaef$,	$xbdf$,	$xcde$,
$xad3$,	$xbe5$,	$xcf7$,	$x025$,	$x036$,	$x047$,
$x134$,	$x127$,	$x156$,	$x246$,	$ad57$,	$be37$,
$cf35$,	$a345$,	$a023$,	$a167$,	$b134$,	$b257$,
$b056$,	$c367$,	$c125$,	$c047$,	$d356$,	$d347$,
$d012$,	$e014$,	$e567$,	$e235$,	$f237$,	$f457$,
$f016$,	$ab46$,	$cd04$,	$ef24$,	$ab07$,	$cd05$,
$ef03$,	$ac26$,	$bf06$,	$de46$,	$ac05$,	$bf03$,
$de07$,	$ad24$,	$be26$,	$cf46$,	$ad36$,	$be45$,
$cf27$,	$af14$,	$ce16$,	$bd12$,	$af15$,	$ce13$,
$bd17$,	$ae02$,	$df26$,	$bc24$,	$ae17$,	$df15$,
$bc13$.					

There exists a $Q(s, 13^1 16^3)$ for $s \geq 3$ having 61 parallel classes of triples on $12s$ points. Take 15 new points, place each such point on the triples of four parallel classes, and add a solution for $g_4^{(4)}(15)$ on the 15 new points. The resulting design on $12s + 15$ points establishes that $g_4^{(4)}(v) = l_4^{(4)}(v)$ for $v \equiv 3 \pmod{12}$ with $v \geq 51$. The values of $v \equiv 3 \pmod{12}$ omitted by this argument are $v = 27$ and 39; we return to these later.

In the case $v \equiv 6 \pmod{12}$, we start by giving a solution on $v = 6$ points which establishes that $g_4^{(4)}(6) = l_4^{(6)}(v) = 11$. We may take the blocks to be $[0, 2, 4]_6$ (two distinct triples), $[0, 1, 3, 4]_6$ (three distinct quadruples) and $[0, 1, 4, 5]_6$ (six quadruples). To deal with the general case, take a 4-GDD of index $\lambda = 1$ and type 6^{2s+1}. Such a design exists for $s \geq 2$ [1]. Take four copies of each quadruple and place a solution for $g_4^{(4)}(6)$ on each group. This gives a PBD of index $\lambda = 4$ on $12s + 6$ points having $4s + 2$ triples and all other blocks quadruples. This design establishes that $g_4^{(4)}(v) = l_4^{(4)}(v)$ for $v \equiv 6 \pmod{12}$ with $v \geq 30$. The only unresolved value of $v \equiv 6 \pmod{12}$ is $v = 18$; we return to this later.

In the case $v \equiv 11 \pmod{12}$, we note first that a PBD which corresponds to a solution for $g_2^{(4)}(12s + 11)$ has 2 pairs and $4s + 4$ triples, with all remaining blocks as quadruples. Furthermore, it is shown in [3] that for $s \neq 0$, there is such a PBD whose blocks include two pairs xa, xb and two triples abp and abq. Now combine two such PBDs on a common set of points where the first PBD has the pairs and triples specified and the second has pairs xp, xq and triples pqa, pqb. Replace the pairs xa, xb, xp and the triple abp by the quadruple $xabp$. The result is PBD of index $\lambda = 4$ on $12s + 11$ points having one pair, $8s + 7$ triples, and all remaining blocks as quadruples. This establishes that $g_4^{(4)}(v) = l_4^{(4)}(v)$ for $v \equiv 11 \pmod{12}$ with $v \geq 23$. The only unresolved value in this case is $v = 11$.

All that remains in this section is to discuss the outstanding values $v = 11, 18, 26, 27, 38$ and 39. For $v = 11$ we give a solution in $l_4^{(4)}(11) = 41$ blocks having four pairs, two triples and 35 quadruples. Initially, take six sets each comprising a pair of complementary triples on a set of six points and also take eight one-factors on the same points, taking care to ensure that these triples and pairs give a fourfold covering of pairs. An example of this is provided by the pairs of triples $P_1 = \{012, 345\}$, $P_2 = \{013, 245\}$, $P_3 = \{024, 135\}$, $P_4 = \{025, 134\}$, $P_5 = \{034, 125\}$, $P_6 = \{035, 124\}$ with the one-factors $F_1 = \{01, 23, 45\}$, $F_2 = \{01, 23, 45\}$, $F_3 = \{02, 14, 35\}$, $F_4 = \{03, 15, 24\}$, $F_5 = \{04, 13, 25\}$, $F_6 = \{04, 15, 23\}$, $F_7 = \{05, 12, 34\}$, $F_8 = \{05, 14, 23\}$. Then take five new points a, b, c, d, e. Take the two triples as abc, abc and the four pairs as de together with pairs from, say, F_1. For the quadruples take $abcd, abce$, together with the 21 blocks obtained by placing $ad, ae, bd, be, cd, ce, de$ on F_2 to F_8 in any order, and finally those obtained by placing each of a, b, c on any two of P_1 to P_6.

For $v = 18$ a solution in $l_4^{(4)}(18) = 105$ blocks is given by the blocks $[0, 6, 12]_{18}$ (six distinct triples), $[0, 3, 9, 12]_{18}$ (nine distinct quadruples), and the 90 quadruples $[0, 1, 10, 11]_{18}, [0, 1, 13, 15]_{18}, [0, 2, 4, 5]_{18}, [0, 5, 7, 13]_{18}, [0, 3, 7, 11]_{18}$.

For $v = 26$, initially take two copies of a KTS(21) on the points $0, 1, \ldots, 20$; these resolve into 20 parallel classes of triples. Next take five new points, place each such point on the triples of four parallel classes to give 140 quadruples, and add a solution for $g_4^{(4)}(5)$ on the five new points. Complete the design with two copies of each of the seven distinct triples obtained from $[0, 7, 14]_{21}$ and the 63 quadruples $[0, 2, 3, 5]_{21}, [0, 6, 10, 16]_{21}, [0, 8, 9, 17]_{21}$. This gives a total of $226 = l_4^{(4)}(26)$ blocks.

A solution for $v = 38$ may be obtained somewhat similarly by taking a KTS(33) on the points $0, 1, \ldots, 32$ which resolves into 16 parallel classes of triples. Next take five new points a, b, c, d, e, place each of a, b, c, d on the triples of four parallel classes, add the the 33 quadruples $[e, 0, 1, 9]_{33}$, the 11 distinct quadruples obtained from $[e, 0, 11, 22]_{33}$ and a solution for $g_4^{(4)}(5)$ on $\{a, b, c, d, e\}$. Then add 22 blocks comprising two copies of each of the 11 distinct triples obtained from $[0, 11, 22]_{33}$, and finally add the 231 quadruples $[0, 1, 15, 16]_{33}, [0, 2, 12, 16]_{33}, [0, 2, 12, 15]_{33}, [0, 6, 9, 13]_{33}, [0, 5, 8, 13]_{33}, [0, 7, 9, 14]_{33}, [0, 4, 10, 16]_{33}$. This gives a total of $482 = l_4^{(4)}(38)$ blocks.

For $v = 27$ we give a solution in $l_4^{(4)}(27) = 239$ blocks having ten triples and 229 quadruples. Take the point set to be $\{x\} \cup A \cup B$ where $A = \{0, 1, \ldots, 7\}$ and $B = \{0', 1', \ldots, 17'\}$. Two main ingredients are required. The first is obtained from a 4-GDD of type $5^1 2^9$ [1] by deleting the group of size five. Take the resulting GDD, say $\mathcal{G}$, of type 2^9 on the point set B; we may assume that the nine groups of size two are $0'9', 1'10', \ldots, 8'17'$. This GDD has for its blocks 30 triples arranged into five parallel classes, and a further nine quadruples. The second main ingredient is obtained from a NKTS(18) on the point set B. This gives a PBD, say $\mathcal{D}$, having 48 triples arranged into eight parallel classes of triples together with nine pairs forming a single parallel class of pairs which we may take to be $\{0'9', 1'10', \ldots, 8'17'\}$. The triples contributing to the solution are the four distinct triples $[x, 0, 4]_8$ plus the six triples forming one of the parallel classes of $\mathcal{G}$. The remaining blocks containing x are $[x, 0, 1, 3]_8$ (eight quadruples) together with 24 quadruples obtained by placing x on each triple of the remaining four parallel classes of $\mathcal{G}$. A further nine quadruples are those obtained directly from $\mathcal{G}$ and eight more are obtained from $[0, 2, 3, 7]_8$. By this stage, x appears with every other point four times, every pair from B (apart from the pairs $0'9', 1'10', \ldots, 8'17'$) appears once, the pairs $[0, 1]_8, [0, 2]_8, [0, 3]_8, [0, 4]_8$ (four distinct pairs) appear respectively 3, 2, 3, 3 times, and there are no pairs ab with $a \in A$ and $b \in B$. To complete the design, take three copies of $\mathcal{D}$ together with an additional parallel class of pairs $\{0'9', 1'10', \ldots, 8'17'\}$. Place each point from A onto the triples of three of the 24 parallel classes of triples arising from $\mathcal{D}$ to give 144 quadruples. Then add the 32 quadruples

0 1 0′ 9′	1 2 1′ 10′	...	7 0 7′ 16′
2 4 0′ 9′	3 5 1′ 10′	...	1 3 7′ 16′
5 7 0′ 9′	6 0 1′ 10′	...	4 6 7′ 16′
3 6 0′ 9′	4 7 1′ 10′	...	2 5 7′ 16′

and a final four quadruples 0 4 8′ 17′, 1 5 8′ 17′, 2 6 8′ 17′, 3 7 8′ 17′.

For $v = 39$ we give a solution in $l_4^{(4)}(39) = 501$ blocks having 14 triples and 487 quadruples. Take the point set to be $\{x\} \cup A \cup B \cup C$ where $A = \{0, 1, \ldots, 7\}$, $B = \{0', 1', \ldots, 14'\}$ and $C = \{0'', 1'', \ldots, 14''\}$. The first four triples contributing to the solution are the four distinct triples $[x, 0, 4]_8$. The first eight quadruples of the solution are given by $[x, 0, 1, 3]_8$ and a further four quadruples are given by taking two copies of each of the two distinct quadruples from $[0, 2, 4, 6]_8$. Next take a NKTS(30) on the point set $B \cup C$ to obtain a PBD, say $\mathcal{D}$, having 140 triples arranged into 14 parallel classes of triples together with 15 pairs forming a single parallel class of pairs which we may take to be $\{0'0'', 1'1'', \ldots, 14'14''\}$. Then take two copies of $\mathcal{D}$ and form 40 quadruples of the solution by placing x on each triple of four parallel classes. Form a further 240 quadruples by placing each point from A on each triple of three of the remaining 24 parallel classes of triples obtained from the two copies of $\mathcal{D}$. By this stage, x appears with every other point four times, every pair from $B \cup C$ (apart from the pairs $0'0'', 1'1'', \ldots, 14'14''$) appears twice, the pairs $[0,1]_8, [0,2]_8, [0,3]_8, [0,4]_8$ (four distinct pairs) appear respectively 1, 3, 1, 3 times, and each pair ad with $a \in A$ and $d \in B \cup C$ appears three times. To complete a 4-covering of pairs ad with $a \in A$ and $d \in B \cup C$ and simultaneously a 4-covering of pairs from A, take four copies of each of the pairs $0'0'', 1'1'', \ldots, 14'14''$ and assign these to 15 one-factors on A which cover the missing pairs from A. One way of doing this is provided by the 60 quadruples

$0\ 1\ 0'\ 0''$	$2\ 3\ 0'\ 0''$	$4\ 5\ 0'\ 0''$	$6\ 7\ 0'\ 0''$
$1\ 2\ 1'\ 1''$	$3\ 4\ 1'\ 1''$	$5\ 6\ 1'\ 1''$	$7\ 0\ 1'\ 1''$
$0\ 1\ 2'\ 2''$	$2\ 3\ 2'\ 2''$	$4\ 5\ 2'\ 2''$	$6\ 7\ 2'\ 2''$
$1\ 2\ 3'\ 3''$	$3\ 4\ 3'\ 3''$	$5\ 6\ 3'\ 3''$	$7\ 0\ 3'\ 3''$
$0\ 1\ 4'\ 4''$	$2\ 3\ 4'\ 4''$	$4\ 5\ 4'\ 4''$	$6\ 7\ 4'\ 4''$
$1\ 2\ 5'\ 5''$	$3\ 4\ 5'\ 5''$	$5\ 6\ 5'\ 5''$	$7\ 0\ 5'\ 5''$
$0\ 2\ 6'\ 6''$	$1\ 3\ 6'\ 6''$	$4\ 6\ 6'\ 6''$	$5\ 7\ 6'\ 6''$
$2\ 4\ 7'\ 7''$	$3\ 5\ 7'\ 7''$	$6\ 0\ 7'\ 7''$	$7\ 1\ 7'\ 7''$
$0\ 3\ 8'\ 8''$	$2\ 5\ 8'\ 8''$	$4\ 7\ 8'\ 8''$	$6\ 1\ 8'\ 8''$
$1\ 4\ 9'\ 9''$	$3\ 6\ 9'\ 9''$	$5\ 0\ 9'\ 9''$	$7\ 2\ 9'\ 9''$
$0\ 3\ 10'\ 10''$	$2\ 5\ 10'\ 10''$	$4\ 7\ 10'\ 10''$	$6\ 1\ 10'\ 10''$
$1\ 4\ 11'\ 11''$	$3\ 6\ 11'\ 11''$	$5\ 0\ 11'\ 11''$	$7\ 2\ 11'\ 11''$
$0\ 3\ 12'\ 12''$	$2\ 5\ 12'\ 12''$	$4\ 7\ 12'\ 12''$	$6\ 1\ 12'\ 12''$
$1\ 4\ 13'\ 13''$	$3\ 6\ 13'\ 13''$	$5\ 0\ 13'\ 13''$	$7\ 2\ 13'\ 13''$
$0\ 4\ 14'\ 14''$	$1\ 5\ 14'\ 14''$	$2\ 6\ 14'\ 14''$	$3\ 7\ 14'\ 14''$.

It now remains to construct ten triples and 135 quadruples on $B \cup C$ which cover every pair from $B \cup C$ twice, apart from the pairs $[0', 0'']_{15}$ which already appear four times. This may be achieved using $[0', 5', 10']_{15}$ (five distinct triples), $[0'', 5'', 10'']_{15}$ (five distinct triples), $[0', 1', 3', 7']_{15}$, $[0'', 1'', 3'', 7'']_{15}$, $[0', 1', 2'', 3'']_{15}$, $[0', 2', 8'', 12'']_{15}$, $[0', 3', 1'', 7'']_{15}$, $[0', 4', 11'', 14'']_{15}$, $[0', 5', 9'', 11'']_{15}$, $[0', 6', 9'', 14'']_{15}$, $[0', 7', 5'', 12'']_{15}$. Note that the ten triples and first 30 quadruples just given cover all pure differences once and the remaining 105 quadruples cover all pure differences once and all mixed differences, apart from zero, twice.

Summarizing the results of this section, we have shown that $g_4^{(4)}(v) = l_4^{(4)}(v)$ for all $v \geq 4$.

4 The case $\lambda = 5$

Table 5 lists the values of $l_5^{(4)}(v)$ for $v = 12s + i$ with $0 \le i \le 11$ and $v \ge 4$. We will show that, depending on the value of i and apart from $v = 7$, either $g_5^{(4)}(v) = l_5^{(4)}(v)$ or $g_5^{(4)}(v) = l_5^{(4)}(v) + 1$. Furthermore, for $v \not\equiv 2, 11 \pmod{12}$ and with some small exceptions, the solution may be obtained by either combining solutions for $g_1^{(4)}(v)$ and $g_4^{(4)}(v)$, or solutions for $g_2^{(4)}(v)$ and $g_3^{(4)}(v)$. In the cases $v \equiv 2, 11 \pmod{12}$ we give direct constructions. In the cases $v \equiv 1$ or $4 \pmod{12}$ $g_5^{(4)}(v)$ is also given directly by BIBDs with all blocks of size 4. For $v = 7$, we show below that $g_5^{(4)}(7) = l_5^{(4)}(7) + 2 = 20 = g_2^{(4)}(7) + g_3^{(4)}(7)$.

v	$l_5^{(4)}(v)$	$g_5^{(4)}(v)$	Comments
$12s$	$60s^2 - s$	$l_5^{(4)}(v)$	$g_2^{(4)}(v) + g_3^{(4)}(v)$
$12s+1$	$60s^2 + 5s$	$l_5^{(4)}(v)$	$g_2^{(4)}(v) + g_3^{(4)}(v)$
$12s+2$	$60s^2 + 17s + 2$	$l_5^{(4)}(v) + 1$	See subsequent text
$12s+3$	$60s^2 + 29s + 4$	$l_5^{(4)}(v)$	$g_1^{(4)}(v) + g_4^{(4)}(v)$
$12s+4$	$60s^2 + 35s + 5$	$l_5^{(4)}(v)$	$g_2^{(4)}(v) + g_3^{(4)}(v)$
$12s+5$	$60s^2 + 47s + 10$	$l_5^{(4)}(v) + 1$	$g_2^{(4)}(v) + g_3^{(4)}(v)$
$12s+6$	$60s^2 + 59s + 15$	$l_5^{(4)}(v)$	$g_1^{(4)}(v) + g_4^{(4)}(v), s \ge 2$
$12s+7$	$60s^2 + 65s + 18$	$l_5^{(4)}(v) + 1$	$g_2^{(4)}(v) + g_3^{(4)}(v), s \ge 1, v \ne 7$
$12s+8$	$60s^2 + 77s + 25$	$l_5^{(4)}(v) + 1$	$g_2^{(4)}(v) + g_3^{(4)}(v), s \ge 1$
$12s+9$	$60s^2 + 89s + 33$	$l_5^{(4)}(v)$	$g_2^{(4)}(v) + g_3^{(4)}(v)$
$12s+10$	$60s^2 + 95s + 38$	$l_5^{(4)}(v) + 1$	$g_2^{(4)}(v) + g_3^{(4)}(v)$
$12s+11$	$60s^2 + 107s + 48$	$l_5^{(4)}(v) + 1$	See subsequent text

Table 5: $g_5^{(4)}(v)$.

In order to establish the values of $g_5^{(4)}(v)$ given in the Table 5, we first show that $g_5^{(4)}(v) > l_5^{(4)}(v)$ for $v \equiv 2, 5, 7, 8, 10, 11 \pmod{12}$. To do this, assume that there is a solution having $l_5^{(4)}(v)$ blocks so that $g_2 + g_3 + g_4 = l_5^{(4)}(v)$. Counting pairs gives $g_2 + 3g_3 + 6g_4 = 5\binom{v}{2}$. Note that if $v \equiv 2 \pmod{3}$ this gives $g_2 \equiv 2 \pmod{3}$ and hence $g_2 \ge 2$. Furthermore, for all v, $5g_2 + 3g_3 = 6l_5^{(4)}(v) - 5\binom{v}{2} = X(v)$, say, and for every point x we have $r_2(x) + 2r_3(x) + 3r_4(x) = 5(v-1)$. Again if $v \equiv 2 \pmod{3}$, this latter equation gives $r_2(x) + 2r_3(x) \equiv 2 \pmod{3}$, and so $r_2(x) + 2r_3(x) \ge 2$, implying that every point x is in at least one small block.

By computation, $X(12s+2) = 12s+7$, giving $5g_2 + 3g_3 = 12s+7$ and hence $2g_2+3g_3 \le 12s+1$, implying that at least one point does not appear in a small block, a contradiction. Similarly, $X(12s+5) = 12s+10$ giving $2g_2 + 3g_3 \le 12s+4$, $X(12s+8) = 12s+10$ giving $2g_2+3g_3 \le 12s+4$, and $X(12s+11) = 12s+13$ giving $2g_2+3g_3 \le 12s+7$, all of which yield contradictions. In the case $v = 12s+7$, we get $r_2(x) + 2r_3(x) \equiv 0 \pmod 3$ for every x. However, $X(12s+7) = 3$ which implies that $(g_2, g_3) = (0, 1)$ and hence that there exists a point x^* for which $r_2(x^*) = 0, r_3(x^*) = 1$, so that $r_2(x^*)+2r_3(x^*) \equiv 2 \pmod 3$, a contradiction. In the case $v = 12s+10$ we again have $r_2(x)+2r_3(x) \equiv 0 \pmod 3$ for every x. But $X(12s+10) = 3$ and this again gives a contradiction.

Appropriate combinations of solutions for $g_\lambda^{(4)}(v)$ for $\lambda = 1, 2, 3, 4$ now establish the validity of the entries in Table 5 apart from the cases $v = 12s+2$ and $v = 12s+11$, and with the exceptions of $g_5^{(4)}(v)$ for $v = 6, 7, 8, 18$.

For $g_5^{(4)}(6)$, a solution in $l_5^{(4)}(6)$ blocks is given by the three distinct pairs $[0,3]_6$ and the 12 quadruples $[0,1,2,3]_6, [0,2,3,4]_6$.

A solution for $g_5^{(4)}(7)$ in $l_5^{(4)}(7) + 1 = 19$ blocks does not exist. To see this, suppose the contrary and apply the usual arguments to show that $5g_2 + 3g_3 = 9$. The only solution is $(g_2, g_3, g_4) = (0, 3, 16)$. However, we also get $r_2(x) + 2r_3(x) + 3r_4(x) = 30$ for every x, and $g_2 = 0$ implies that $r_2(x) = 0$, so we can deduce that $r_3(x) = 0$ or 3 for each x. So there is a thrice repeated triple abc and the points a, b, c each appear in eight quadruples. But it is easy to see that we cannot arrange this in 16 quadruples with each pair ab, ac, bc appearing twice amongst the quadruples. A solution for $g_5^{(4)}(7)$ in $l_5^{(4)}(7) + 2 = 20$ blocks is however easily constructed since $g_2^{(4)}(7) = 7$ and $g_3^{(4)}(7) = 13$.

For $g_5^{(4)}(8)$, a solution in $l_5^{(4)}(8) + 1 = 26$ blocks is given by

$$ab, ab, 024, 135, ab01, ab23, ab45, a014, a014, a035, a035, a123, a125,$$
$$a234, a245, b012, b023, b035, b045, b125, b134, b134, b245, 0125, 0234, 1345.$$

For $g_5^{(4)}(18)$, we give a solution in $l_5^{(4)}(18) = 134$ blocks having three pairs, eight triples and 123 quadruples. We take the point set to be $A \cup B$ where $A = \{a, b, c, d, e, f\}$ and $B = \{0, 1, \dots, 11\}$. Take five copies of a one-factorization of the complete graph K_6 on A, giving 25 one-factors each comprising three pairs. The three pairs of the solution are given directly by one of these one-factors. Then place each of the 24 distinct pairs $[0,1]_{12}, [0,5]_{12}$ onto the three pairs of each of one of the remaining 24 one-factors to give 72 quadruples of the solution. By this stage, every pair of points from A appears five times, every pair of points $\{x, y\}$ with

$x \in A$ and $y \in B$ appears four times, and the pairs $[0,1]_{12}, [0,5]_{12}$ appear three times. Next take $[0,3,6,9]_{12}$ (three distinct quadruples), two copies of $[0,2,3,6]_{12}$ (24 quadruples), and two copies of $[0,4,8]_{12}$ (eight triples). These cover the pairs $[0,1]_{12}$ twice, $[0,2]_{12}$ twice, $[0,3]_{12}$ five times, $[0,4]_{12}$ four times, and $[0,6]_{12}$ five times. It remains to cover the pairs $[0,2]_{12}$ three times, $[0,4]_{12}$ once and $[0,5]_{12}$ twice, as well as one copy of each of the pairs $\{x,y\}$ with $x \in A$ and $y \in B$. This final step can be achieved in 24 quadruples by starting with the 24 triples $[0,2,4]_{12}$ and $[0,2,7]_{12}$, partitioning these into six parallel classes and placing each element of A onto one of these classes. An explicit example of how this may be done is as follows.

a	0	2	4	b	1	3	5	c	2	4	6
a	3	5	7	b	4	6	8	c	5	7	9
a	6	8	10	b	7	9	11	c	8	10	0
a	9	11	1	b	10	0	2	c	11	1	3
d	0	2	7	e	1	3	8	f	2	4	9
d	3	5	10	e	4	6	11	f	5	7	0
d	6	8	1	e	7	9	2	f	8	10	3
d	9	11	4	e	10	0	5	f	11	1	6.

To treat the general case $v = 12s + 11$, we first give a solution for $v = 11$ in $l_5^{(4)}(11) + 1 = 49$ blocks. There are two pairs, three triples and 44 quadruples. We take the point set to be $\{a,b\} \cup A$ where $A = \{0,1,\ldots,8\}$. The pairs are ab and ab. Next take an STS(7) on $\{0,1,\ldots,6\}$ and delete the point 6 from the triples in which it lies to leave three pairs, P_1, P_2 and P_3, together with four triples T_1, T_2, T_3 and T_4. The three triples of the solution are obtained by placing the points 6, 7 and 8 respectively onto the pairs P_1, P_2 and P_3. We form 20 quadruples of the solution by placing the pair ab onto each of the pairs P_1, P_2 and P_3, the pairs 67, 68 and 78 respectively onto the pairs P_3, P_2 and P_1 (note the order), the points 6, 7 and 8 each onto all four triples T_1, T_2, T_3 and T_4, and then taking two additional quadruples $a678$ and $b678$. The remaining 24 quadruples are formed by adjoining a to the twelve blocks of any STS(9) on the base set A, and doing likewise for b.

Using this solution we may obtain a general solution for $v = 12s + 11$ with $s \geq 3$. There exists a $Q(s, 13^4 4^1)$ for $s \geq 3$ having 56 parallel classes of triples on $12s$ points. Take eleven new points, place each such point on the triples of five parallel classes, and add a solution for $g_5^{(4)}(11)$ on the eleven new points. The resulting design, which has two pairs, $4s + 3$ triples and $60s^2 + 103s + 44$ quadruples, establishes that $g_5^{(4)}(12s+11) = l_5^{(4)}(12s+11) + 1$ for $s \geq 3$. The values of $v \equiv 11 \pmod{12}$ omitted by this argument are $v = 23$ and 35; we return to these later.

To treat the general case $v = 12s+2$, we first give a solution for $v = 14$ in $l_5^{(4)}(14)+1 = 80$ blocks. There are two pairs, five triples and 73 quadruples. We take the point set to be $\{a,b,c,d,e,f\} \cup A$ where $A = \{0,1,\ldots,7\}$. The pairs are ab and ab. The triples are $c01, c23, c45, c67$ and def. The first four quadruples are $ac01, ac23, bc45, bc67$ (note that both here and in the triples, the pairs $01, 23, 45$ and 67 form a one-factor of K_8 on the set A, call it F_1). Add 48 further quadruples formed by adjoining each of $ad, ae, af, bd, be, bf, cd, ce, cf, de, df, ef$ to a one-factor of K_8 where the twelve one-factors together with the two occurrences of F_1 in the triples and first four quadruples form two one-factorizations of K_8 on the set A. Finally add the 21 quadruples $abcd, abce, abcf, a045, a156, a267, a347, b017, b036,$ $b124, b235, 0123, 0146, 0245, 0267, 0357, 1257, 1347, 1356, 2346, 4567$.

Using this solution we may obtain a general solution for $v = 12s + 2$ with $s \geq 4$. There exists a $Q(s, 16^4 7^1)$ for $s \geq 3$ having 71 parallel classes of triples on $12s$ points. Take 14 new points, place each such point on the triples of five parallel classes, and add a solution for $g_5^{(4)}(14)$ on the 14 new points. The resulting design, which has two pairs, $4s + 5$ triples and $60s^2+133s+73$ quadruples, establishes that $g_5^{(4)}(12s+14) = l_5^{(4)}(12s+14)+1$ for $s \geq 3$. The values of $v \equiv 2 \pmod{12}$ omitted by this argument are $v = 26$ and 38.

All that remains in this section is to discuss the outstanding values $v = 23, 26, 35$ and 38.

For $g_5^{(4)}(23)$ we give a solution in $l_5^{(4)}(23) + 1 = 216$ blocks having two pairs, seven triples and 207 quadruples. Take the point set to be $\{a,b\} \cup A \cup B$ where $A = \{0,1,\ldots,5\}$ and $B = \{0',1',\ldots,14'\}$. The pairs are ab and ab. The triples are $012, 345$ and $[0', 5', 10']_{15}$. Twelve of the quadruples are $ab03, ab14, ab25, a015, a024, a123, b045, b135, b234, 0134, 0235, 1245$. By this stage, the pair ab appears five times, every pair of points from A appears three times, the pairs $a0, a1, a2, b3, b4, b5$ appear three times and the pairs $a3, a4, a5, b0, b1, b2$ appear twice. Next consider the following 15 one-factors of the complete graph K_8 on $\{a,b\} \cup A$.

$F_0 = \{a0, b4, 12, 35\}$ $F_1 = \{a0, b5, 12, 34\}$ $F_2 = \{a1, b3, 02, 45\}$
$F_3 = \{a1, b5, 02, 34\}$ $F_4 = \{a2, b3, 01, 45\}$ $F_5 = \{a2, b4, 01, 35\}$
$F_6 = \{a3, b0, 14, 25\}$ $F_7 = \{a3, b1, 05, 24\}$ $F_8 = \{a3, b2, 04, 15\}$
$F_9 = \{a4, b0, 15, 23\}$ $F_{10} = \{a4, b1, 03, 25\}$ $F_{11} = \{a4, b2, 05, 13\}$
$F_{12} = \{a5, b0, 13, 24\}$ $F_{13} = \{a5, b1, 04, 23\}$ $F_{14} = \{a5, b2, 03, 14\}$

These 15 one-factors cover all remaining pairs from the set $\{a,b\} \cup A$. Now place the points i' and $(i+4)'$, $0 \le i \le 14$ on each of the four pairs in F_i, thus forming 60 further quadruples. Here each point of $\{a,b\} \cup A$ appears with each point of B precisely twice and, together with the triples $[0', 5', 10']_{15}$ already included, the pairs $[0', 4']_{15}$ are covered four times and the pairs $[0', 5']_{15}$ are covered once. Next include the 15 quadruples $[0', 2', 5', 8']_{15}$ and consider the eight sets of triples (each of 15 triples) $[0', 1', 3']_{15}, [0', 1', 3']_{15}, [0', 1', 7']_{15}, [0', 1', 7']_{15}, [0', 1', 7']_{15}, [0', 2', 5']_{15}, [0', 2', 7']_{15}, [0', 4', 9']_{15}$. Note that these cover the pairs $[0', 4']_{15}$ once, the pairs $[0', 5']_{15}$ four times and all remaining pairs from B five times. Finally adjoin each point of $\{a,b\} \cup A$ to one of these sets of 15 triples to form 120 additional quadruples, thereby completing the design.

For $g_5^{(4)}(26)$ we give a solution in $l_5^{(4)}(26) + 1 = 277$ blocks having two pairs, nine triples and 266 quadruples. Take the point set to be $\{a,b,c,d,e\} \cup A$ where $A = \{0,1,\ldots,20\}$. A solution for $g_5^{(4)}(5)$ on $\{a,b,c,d,e\}$ gives the two pairs, two of the triples and seven of the quadruples. Take the seven additional triples as $[0,7,14]_{21}$. Next consider two copies of a KTS(21) on the base set A. This gives 140 triples arranged into 20 parallel classes. Put each of a, b, c and d onto five of these classes to give a further 140 quadruples. By this stage every pair of points from $\{a,b,c,d\}$ appears five times, every pair ax, bx, cx and dx for $x \in A$ appears five times, and every pair from A appears twice apart from the pairs $[0,7]_{21}$ which appear three times. The design is completed by adding the quadruples $[e,0,7,14]_{21}$ twice (14 blocks), $[e,0,1,4]_{21}, [0,1,4,9]_{21}, [0,1,10,16]_{21}, [0,2,4,13]_{21}$ and $[0,2,8,18]_{21}$.

For $g_5^{(4)}(35)$ we give a solution in $l_5^{(4)}(35) + 1 = 503$ blocks having two pairs, eleven triples and 490 quadruples. Take the point set to be $A \cup B$ where $A = \{a,b,c,d,e,f,g,h,i,j,k\}$ and $B = \{0,1\ldots,23\}$. A solution for $g_5^{(4)}(11)$ on A gives the two pairs, three of the triples and 44 of the quadruples. Next consider five non-identical NKTS(24)s on the base set B. Each of these five designs has 100 blocks comprising twelve pairs which form a one-factor of B, and 88 triples arranged into eleven parallel classes of triples. The five systems are chosen so that the five one-factors are as listed in the columns below.

F_1	F_2	F_3	F_4	F_5
0,1	0,5	0,8	0,16	0,4
2,3	1,4	1,9	1,17	1,5
4,5	2,7	2,10	2,18	2,6
6,7	3,6	3,11	3,19	3,7
8,9	8,13	4,12	4,20	8,16
10,11	9,12	5,13	5,21	9,17
12,13	10,15	6,14	6,22	10,18
14,15	11,14	7,15	7,23	11,19
16,17	16,21	16,20	8,12	12,20
18,19	17,20	17,21	9,13	13,21
20,21	18,23	18,22	10,14	14,22
22,23	19,22	19,23	11,15	15,23

Place each point of A onto five of the 55 parallel classes of triples obtained from the NKTS(24)s, giving a further 440 quadruples. By this stage every pair of points from A appears five times, every pair αx for $\alpha \in A$ and $x \in B$ appears five times, and every pair from B appears five times apart from the pairs which appear in F_1, F_2, F_3, F_4 and F_5. The pairs from these one-factors may be incorporated into the following eight triples and six quadruples: $[0,8,16]_{24}$ (eight blocks), $\{0,1,4,5\}, \{2,3,6,7\}, \{8,9,12,13\}, \{10,11,14,15\}, \{16,17,20,21\}, \{18,19,22,23\}$. These complete the design.

For $g_5^{(4)}(38)$ we give a solution in $l_5^{(4)}(38)+1 = 594$ blocks having two pairs, 13 triples and 579 quadruples. Take the point set to be $\{a,b,c,d,e\} \cup A$ where $A = \{0,1,\ldots,32\}$. A solution for $g_5^{(4)}(5)$ on $\{a,b,c,d,e\}$ gives the two pairs, two of the triples and seven of the quadruples. Next consider a KTS(33) on the base set A. This has 176 triples arranged into 16 parallel classes. Put each of a, b and c onto five of these classes to give a further 165 quadruples, leaving one parallel class of eleven triples which we may take to be $[0,11,22]_{33}$ and which we also include in the solution. Next take $[d,0,11,22]_{33}$ twice (22 blocks) and $[e,0,11,22]_{33}$ twice (22 blocks) to give a further 44 quadruples. By this stage every pair of points from $\{a,b,c,d,e\}$ appears five times, every pair ax, bx and cx for $x \in A$ appears five times, every pair dx and ex for $x \in A$ appears twice, and every pair from A appears once apart from the pairs $[0,11]_{33}$ which appear five times. The design is completed with the following 363 quadruples: $[d,0,2,7]_{33}, [e,0,6,14]_{33}, [0,1,3,15]_{33}, [0,1,4,9]_{33}, [0,1,4,16]_{33}$, $[0,1,10,29]_{33}, [0,2,8,18]_{33}, [0,2,10,20]_{33}, [0,3,9,16]_{33}, [0,4,16,28]_{33}$, $[0,6,13,20]_{33}$.

This completes the case $\lambda = 5$. Summarizing the results of this section, we have shown that $g_5^{(4)}(v) = l_5^{(4)}(v)$ for $v \equiv 0, 1, 3, 4, 6, 9 \pmod{12}$ and that, with the exception of $v = 7$, $g_5^{(4)}(v) = l_5^{(4)}(v) + 1$ for $v \equiv 2, 5, 7, 8, 10, 11 \pmod{12}$. The single exceptional value is $g_5^{(4)}(7) = 20$.

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