LIFTS OF AUTOMORPHISMS OF SYMMETRIC DIGRAPHS ASSOCIATED WITH CYCLIC ABELIAN COVERS III

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Abstract

Let D be a connected symmetric digraph, Γ a group of automorphisms of D, and A a finite abelian group. For a cyclic A-cover of D, we consider a lift of $\gamma \in \Gamma$, and the associated group automorphism of some subgroup of $Aut\ A$. Furthermore, we give a characterization for any $\gamma \in \Gamma$ to have a lift in terms of some matrix.

Key words: digraph automorphism; digraph covering; lift

1 Introduction

Graphs and digraphs treated here are finite and simple.

Let G be a graph and $D(G)$ the arc set of the symmetric digraph corresponding to G. For $e = (u, v) \in D(G)$, let $i(e) = u$ and $t(e) = v$. The inverse arc of e is denoted by e^{-1}. A *walk* P in G is a sequence $P = (e_1, \cdots, e_l)$ of arcs with $t(e_i) = i(e_{i+1})$ for $i = 1, \cdots, l-1$. Also, P is called an $(i(e_1), t(e_l))$-*walk*. Furthermore, set $P^{-1} = ({e_l}^{-1}, \cdots, {e_1}^{-1})$ A (v, w)-walk is called a *closed v-walk* if $v = w$. For two walks $P = (e_1, \cdots, e_m)$ and $Q = (f_1, \cdots, f_l)$ such that $t(e_m) = i(f_1)$, set $PQ = (e_1, \cdots, e_m, f_1, \cdots, f_l)$.

A graph H is called a *covering* of a graph G with projection $\pi : H \longrightarrow G$ if there is a surjection $\pi : V(H) \longrightarrow V(G)$ such that $\pi|_{N(v')} : N(v') \longrightarrow N(v)$ is a bijection for all vertices $v \in V(G)$ and $v' \in \pi^{-1}(v)$. The projection $\pi : H \longrightarrow G$ is an *n-fold covering* of G if π is n-to-one. A covering $\pi : H \longrightarrow G$ is said to be *regular* if there is a subgroup B of the automorphism group $Aut\ H$ of H acting freely on H such that the quotient graph H/B is isomorphic to G.

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Let G be a graph and A a finite group. Then a mapping $\alpha : D(G) \longrightarrow A$ is called an *ordinary voltage assignment* if $\alpha(v,u) = \alpha(u,v)^{-1}$ for each $(u,v) \in D(G)$. The (*ordinary*) *derived graph* G^α derived from an ordinary voltage assignment α is defined as follows:

$V(G^\alpha) = V(G) \times A$, and $((u,h),(v,k)) \in D(G^\alpha)$ if and only if $(u,v) \in D(G)$ and $k = h\alpha(u,v)$.

The graph G^α is called an *A-covering* of G . The A-covering G^α is an $|A|$-fold regular covering of G. Every regular covering of G is an A-covering of G for some group A (see [2]).

Cheng and Wells [1] discussed isomorphism classes of cyclic triple covers (1-cyclic Z_3-covers) of a complete symmetric digraph. For a connected symmetric digraph D and a finite group A, Mizuno and Sato [6] introduced a cyclic A-cover of D as a generalization of regular covering graphs and cyclic triple covers, and discussed the number of isomorphism classes of cyclic A-covers of D with respect to the group Γ of automorphisms of D.

Let D be a symmetric digraph and A a finite group. Let $A(D)$ be the set of arcs in D. A function $\alpha : A(D) \longrightarrow A$ is called *alternating* if $\alpha(y,x) = \alpha(x,y)^{-1}$ for each $(x,y) \in A(D)$. Let $C(D)$ denote the set of alternating functions from $A(D)$ to A. For $g \in A$, a *g-cyclic A-cover* $D_g(\alpha)$ of D is the digraph as follows:

$V(D_g(\alpha)) = V(D) \times A$, and $((u,h),(v,k)) \in A(D_g(\alpha))$ if and only if $(u,v) \in A(D)$ and $k^{-1}h\alpha(u,v) = g$.

The *natural projection* $\pi : D_g(\alpha) \longrightarrow D$ is a function from $V(D_g(\alpha))$ onto $V(D)$ which erases the second coordinates. A digraph D' is called a *cyclic A-cover* of D if D' is a g-cyclic A-cover of D for some $g \in A$. In the case that A is abelian, then $D_g(\alpha)$ is called simply a *cyclic abelian cover*. Furthermore the 1-cyclic A-cover $D_1(\alpha)$ of a symmetric digraph D can be considered as the A-covering G^α of the underlying graph G of D.

Let α and β be two alternating functions from $A(D)$ into A, and let Γ be a subgroup of the automorphism group $Aut\ D$ of D, denoted $\Gamma \leq Aut\ D$. Let $g,h \in A$ and $\gamma \in \Gamma$. Then two cyclic A-covers $D_g(\alpha)$ and $D_h(\beta)$ are called *γ-isomorphic*, denoted $D_g(\alpha) \cong_\gamma D_h(\beta)$, if there exists an isomorphism $\Phi : D_g(\alpha) \longrightarrow D_h(\beta)$ such that $\pi\Phi = \gamma\pi$, i.e., the diagram

$$\begin{array}{ccc} D_g(\alpha) & \xrightarrow{\ \Phi\ } & D_h(\beta) \\ \pi \downarrow & & \downarrow \pi \\ D & \xrightarrow{\ \gamma\ } & D \end{array}$$

commutes. Furthermore, cyclic A-covers $D_g(\alpha)$ and $D_h(\beta)$ are called Γ-*isomorphic*, denoted $D_g(\alpha) \cong_\Gamma D_h(\beta)$, if there exist an isomorphism $\Phi : D_g(\alpha) \longrightarrow D_h(\beta)$ and $\gamma \in \Gamma$ such that $\pi\Phi = \gamma\pi$. Let $I = \{1\}$ be the trivial group of automorphisms.

Let D be a symmetric digraph, A a finite group, $\alpha \in C(D)$ and $\Gamma \leq Aut\ D$. Furthermore, let $g \in A$ and $\gamma \in \Gamma$. Then an automorphism Φ of the g-cyclic A-cover $D_g(\alpha)$ is called a *lift* of γ such that $\pi\Phi = \gamma\pi$. If each $\gamma \in \Gamma$ has a lift, then the set of all such lifts is called the *lifted group* or the *lift* of Γ.

Širáň [9] gave a necessary and sufficient condition for graph automorphisms to have a lift, and discussed the lift of a group of automorphisms of a graph which is a split extension of the voltage group by it. Associated with lifts of a group of automorphisms of a graph, Gvozdjak and Širáň [4] presented a construction of highly symmetric regular maps. A general approarch based on group actions and their morphisms to coverings of topological spaces related to graphs can be found in Malnič [5]. Sato [7] gave a necessary and sufficient condition for a group Γ of automorphisms of a symmetric digraph D to have a lift with respect to a g-cyclic A-cover of D for a finite group A, and discussed the lift of Γ which is isomorphic to a split extension of A by Γ for a finite abelian group A and a $g \in A$ of odd order. Furthermore, Sato [8] discussed the lift of Γ which is isomorphic to a split extension of A by Γ for a finite abelian group A. Širáň [10] presented an algebraic characterization for an automorphism of a graph to lift to an automorphism of its A-covering for a finite abelian group A by using certain matrices and orthogonality in $\mathbb{Z}$-modules.

In Sections 2 and 3 we consider a lift of $\gamma \in \Gamma$, and describe the compatiblity of alternating function and the associated group automorphism of some subgroup of $Aut\ A$. In Section 4 we give a characterization for any $\gamma \in \Gamma$ to have a lift in terms of some matrix. In Section 5 we discuss characterizations for any $\gamma \in \Gamma$ to have a lift in the special cases.

A general theory of graph coverings is developed in [3].

2 Compatible, proper voltage assignments

Let D be a connected symmetric digraph and A a finite group. For any $\alpha \in C(D), g \in A$ and walk W in G, the *net-g-voltage* of W, denoted $\alpha_g(W)$, is defined by

$$\alpha_g(W) = \alpha(e_1)g^{-1} \cdots \alpha(e_n)g^{-1}, W = (e_1, \cdots, e_n).$$

Note that the net-1-voltage of W is the net-voltage of W, where 1 is the unit of A(see [3]).

An alternating function $\alpha : A(D) \longrightarrow A$ is called *g-proper* if there exists some closed u-walk W of D such that $\alpha_g(W) = h$ for each $u \in V(D)$ and $h \in A$. If α is g-proper, then there exists a (u,v)-walk W such that $\alpha_g(W) = h$ for any $u, v \in V(D)$ and any $h \in A$. In the g-cyclic A-cover $D_g(\alpha)$, set $v_h = (v,h)$ and $e_h = (e,h)$, where $v \in V(D), e \in A(D), h \in A$. For $e = (u,v) \in A(D)$, the arc e_h emanates from u_h and terminates at $v_{h\alpha_g(e)}$.

Let $\Gamma \leq Aut\ D$. For a $\gamma \in \Gamma$, α is called *$\gamma - g-$compatible* if for each vertex $u \in V(D)$,

$$\alpha_g(W) = 1 \ if\ and\ only\ if\ \alpha_g(\gamma(W)) = 1$$

for each closed u-walk W of D, and

$$\alpha_g(U) = \alpha_g(V) \ if\ and\ only\ if\ \alpha_g(\gamma(U)) = \alpha_g(\gamma(V))$$

for any two closed u-walks U, V of D. Note that the second condition is obtained from the first condition if $g = 1$. Furthermore, α is called *$\Gamma - g-$compatible* if α is $\gamma - g-$compatible for each $\gamma \in \Gamma$.

Sato [7] gave a condition for Γ to have a lift.

Theorem 1 **(Sato)** *Let D be a connected symmetric digraph, A a finite group, $\Gamma \leq Aut\ D$, $g \in A$ and $\alpha \in C(D)$. Then a $\gamma \in \Gamma$ has a lift if and only if α is $\gamma - g-$compatible. If, in addition, α is g-proper, then, for a fixed vertex u, all lifts of a $\gamma \in \Gamma$ are of the form $\gamma_{h,u}, h \in A$:*

$$\gamma_{h,u}((v,w)_{\alpha_g(U)}) = (\gamma(v), \gamma(w))_{h\alpha_g(\gamma U)} \ for\ any\ (u,v) - walk\ U.$$

3 Lifts of automorphisms

From now on, assume that A is abelian.

Let D be a connected symmetric digraph, G its underlying graph and A a finite abelian group. The set of ordinary voltage assignments of G with voltages in A is denoted by $C^1(G;A)$. Note that $C(D) = C^1(G;A)$. Furthermore, let $C^0(G;A)$ be the set of functions from $V(G)$ into A. We consider $C^0(G;A)$ and $C^1(G;A)$ as additive groups. The homomorphism $\delta : C^0(G;A) \longrightarrow C^1(G;A)$ is defined by $(\delta s)(x,y) = s(x) - s(y)$ for $s \in C^0(G;A)$ and $(x,y) \in A(D)$. For each $\alpha \in C^1(G;A)$, let $[\alpha]$ be the element of $C^1(G;A)/Im\delta$ which contains α.

The group Γ of automorphisms of D acts on $C^0(G;A)$ and $C(D) = C^1(G;A)$ by

$$s^\gamma(x) = s(\gamma(x)) \ for\ all\ x \in V(G)$$

and

$$\alpha^\gamma(x,y) = \alpha(\gamma(x), \gamma(y)) \ for\ all\ (x,y) \in A(D),$$

where $s \in C^0(G; A)$, $\alpha \in C(D)$ and $\gamma \in \Gamma$. The automorphism group $Aut\ A$ acts on $C^0(G; A)$ and $C^1(G; A)$ as follows:

$$(\sigma s)(x) = \sigma(s(x)) \ for\ x \in V(D),$$

$$(\sigma\alpha)(x, y) = \sigma(\alpha(x, y)) \ for\ (x, y) \in A(D),$$

where $s \in C^0(G; A)$, $\alpha \in C^1(G; A)$ and $\sigma \in Aut\ A$.

For $g \in A$ and a g-proper alternating function $\alpha : A(D) \longrightarrow A$, the subgroup $A_{\alpha-g}$ of A generated by all net g-voltages of the closed v-walks for any $v \in V(D)$ is equal to A, and so $D_g(\alpha)$ is connected.

For a function $f : A(D) \longrightarrow A$, the *net-$f$-value* of W, denoted $f(W)$, is defined by

$$f(W) = f(e_1) \cdots f(e_n), W = (e_1, \cdots, e_n).$$

Form [8, Theorem 3], the following result follows. Let $d_T(u, v)$ be the distance between u and v in T.

Theorem 2 (Sato) *Let D be a connected symmetric digraph, A a finite abelian group, $g \in A \setminus \{0\}$ and $\alpha \in C(D)$. Furthermore, let G be the underlying graph of D, T a spanning tree of G and $\Gamma \leq Aut\ D$. Assume that α is g-proper. Then, for $\gamma \in \Gamma$, the following are equivalent:*

1. *α is $\gamma - g-$compatible.*

2. *There exist an automorphism $\sigma_\gamma \in (AutA)_{2g}$ and $s \in C^0(G, A)$ such that*

   $$\alpha^\gamma_{\epsilon g} = \sigma_\gamma \alpha_{\epsilon g} + \delta s,$$

   i.e.,

   $$\alpha^\gamma(u, v) - \epsilon^\gamma(u, v)g = \sigma_\gamma(\alpha(u, v) - \epsilon(u, v)g) + \delta s(u, v),$$

   where

   $$\epsilon^\gamma(u, v) = \begin{cases} 1 & \textit{if the distance } d_{\gamma T}(\gamma u, \gamma v) \textit{ is even,} \\ 0 & \textit{otherwise,} \end{cases}$$

   and $\epsilon(u, v) = \epsilon^1(u, v)$.

Furthermore, if either of the above conditions is satisfied, then we have

$$\sigma_\gamma \alpha_{\epsilon g}(W) = \alpha^\gamma_{\epsilon g}(W),$$

i.e.,

$$\alpha^\gamma(W) - \sum_{e \in W} \epsilon^\gamma(e)g = \sigma_\gamma(\alpha(W) - \sum_{e \in W} \epsilon(e)g),$$

where W is any closed walk in D based at a fixed vertex u.

For $(v, z) \in A(T)$, we have $d_T(v, z) = 1$, and so $\epsilon(v, z) = 0$.

4 Matrices

We begin with recalling a few basic concepts from [10]. Let D be a connected symmetric digraph, G the underlying graph of D and T a spanning tree of G. Let $E(G) \setminus E(T) = \{e_1, \cdots, e_r\}$ be the set of all cotree edges, and let $E(T) = \{e_{r+1}, \cdots, e_{r+t}\}$ be the set of all edges of T. For any edge e_i $(1 \leq i \leq r+t)$, we fix one of the two arcs corresponding to e_i. We denote this fixed arc by x_i. Set $X = \{x_i \mid 1 \leq i \leq r+t\}$.

Let $\Gamma \leq Aut\ D$ and $\gamma \in \Gamma$. For $1 \leq i \leq r$, Let C_i be the unique (directed) cycle of the subgraph $T + e_i$ containing x_i. All the remaining arcs of C_i come from edges in T. We consider the image $\gamma(C_i)$ of C_i. Each arc of $\gamma(C_i)$ is either of the form x_j or of the form x_j^{-1} for some $j(1 \leq i \leq r+t)$. Let

$$\gamma(C_i) = (x_{j_1}^{\epsilon_1}, \cdots, x_{j_s}^{\epsilon_s}),$$

where $\epsilon_m \in \{+1, -1\}$ $(1 \leq m \leq s)$ are suitable exponents. Then we define an $r \times (r+t)$ matrix $\mathbf{C}_T(\gamma) = (c_{ij})$ as follows:

$$c_{ij} = \begin{cases} \epsilon_m & \text{if } j = j_m \ (1 \leq m \leq s), \\ 0 & \text{otherwise}. \end{cases}$$

The matrix $\mathbf{C}_T(\gamma)$ is called the *cycle basis matrix* of D cooresponding to γ and T. Furthermore, the $r \times r$ matrix $\mathbf{L}_T(\gamma)$ formed by the first r columns of $\mathbf{C}_T(\gamma)$ is called the *T-reduced matrix* of γ(see [10]).

Let A a finite abelian group, $g \in A \setminus \{0\}$ and $\alpha \in C(D)$. Then the *T-voltage* α_T of α is defined as follows: $\alpha_T(u,v) = \alpha(P_u) + \alpha(u,v) - \alpha(P_v)$ for each $(u,v) \in A(D)$, where P_u is the unique path from a fixed vertex w to u in T etc. Note that $\alpha_T(x) = 0$ for each $x \in A(T)$. Furthermore, it is well known that $D_g(\alpha) \cong_I D_g(\alpha_T)$. A $\beta \in C(D)$ is called *T-reduced* if $\beta(x) = 0$ for each $x \in A(T)$. Suppose that α is T-reduced. Let $\alpha_i = \alpha(x_i)$, $1 \leq i \leq r$ and let

$$\bar{\alpha} = {}^t(\alpha_1, \cdots, \alpha_r),$$

where ${}^t(\alpha_1, \cdots, \alpha_r)$ is the transpose of $(\alpha_1, \cdots, \alpha_r)$. Then $\bar{\alpha}$ is called *T-reduced voltage vector*. Furthermore, let

$$\bar{\epsilon} = {}^t(\epsilon(x_1), \cdots, \epsilon(x_r)).$$

For any $\sigma \in AutA$, let

$$\sigma(\bar{\alpha}) = {}^t(\sigma(\alpha_1), \cdots, \sigma(\alpha_r)).$$

Now, we give a characterization for any $\gamma \in \Gamma$ to have a lift in terms of some matrix.

Theorem 3 *Let D be a connected symmetric digraph, A a finite abelian group, $g \in A \setminus \{0\}$ and $\alpha \in C(D)$. Furthermore, let G be the underlying graph of D, T a spanning tree of G and $\Gamma \leq Aut\ D$. For $\gamma \in \Gamma$, let $\mathbf{L} = \mathbf{L}_T(\gamma)$. Assume that α is g-proper and α is T-reduced. Then the following two statements are equivalent:*

1. *α is $\gamma - g-$compatible.*

2. *There exists an automorphism $\sigma \in (Aut\ A)_{2g}$ such that*

$$\mathbf{L}\bar{\alpha} = \sigma(\bar{\alpha}) - (\sigma(g) - g)\bar{\epsilon}.$$

Proof. Let u be a root of T.

$1 \Rightarrow 2$: Let $\mathbf{L}_i$ $(1 \leq i \leq r)$ denote the i th row of the matrix $\mathbf{L}$. Then we have

$$\mathbf{L}_i\bar{\alpha} = \alpha^{\gamma}(C_i),$$

where C_i is the unique cycle contained in $T + x_i$.

Let W_i be any closed u-walk contained in $T + x_i$. Then we have

$$\alpha(W_i) = \alpha(C_i) = \alpha(x_i).$$

By the hypothesis and Theorem 2, there exists $\sigma \in (AutA)_{2g}$ such that

$$\alpha^{\gamma}(W_i) - \sum_{e \in W_i} \epsilon^{\gamma}(e)g = \sigma(\alpha(W_i) - \sum_{e \in W_i} \epsilon(e)g).$$

Furthermore,

$$\alpha^{\gamma}(W_i) = \alpha^{\gamma}(C_i) \text{ and } \epsilon^{\gamma}(e) = \epsilon(e) = 0, \ \forall e \in A(T).$$

Thus,

$$\begin{aligned} \mathbf{L}_i\bar{\alpha} - \epsilon^{\gamma}(x_i)g &= \alpha^{\gamma}(C_i) - \epsilon^{\gamma}(x_i)g \\ &= \sigma(\alpha(C_i) - \epsilon(x_i)g) \\ &= \sigma(\alpha(x_i) - \epsilon(x_i)g) \\ &= \sigma(\alpha_i) - \epsilon(x_i)\sigma(g), \end{aligned}$$

i.e.,

$$\mathbf{L}_i\bar{\alpha} = \sigma(\alpha_i) - (\sigma(g) - g)\epsilon(x_i).$$

Therefore, it follows that

$$\mathbf{L}\bar{\alpha} = \sigma(\bar{\alpha}) - (\sigma(g) - g)\bar{\epsilon}.$$

$2 \Rightarrow 1$: Let $W = (\cdots, x_{i_1}^{\epsilon_1}, \cdots, x_{i_2}^{\epsilon_2}, \cdots, x_{i_m}^{\epsilon_m}, \cdots)$ be any closed u-walk in D, where x_i is the (not necessarily distinct) cotree arcs, $\epsilon_l = \pm 1$ and the dotted spaces correspond to arcs in T. Then we have

$$\alpha(W) = \epsilon_1 \alpha_{i_1} + \cdots + \epsilon_m \alpha_{i_m} = y_1 \alpha_1 + \cdots + y_r \alpha_r = \bar{y}\bar{\alpha},$$

where $\bar{y} = (y_1, \cdots, y_r)$ and each y_i is the integer determined by the number of times the walk W traverses the arc x_i minus the number of times W traverses the inverse x_i^{-1}.

Now, let W_i be the unique closed u-walk in $T + x_i$. Then the walk W is "homotopic" to the walk

$$W' = W_{i_1}^{\epsilon_1} \cdots W_{i_m}^{\epsilon_m}.$$

Thus,

$$\alpha(W) = \alpha(W') = \alpha(W_{i_1}^{\epsilon_1}) + \cdots + \alpha(W_{i_m}^{\epsilon_m}) = \epsilon_1 \alpha_{i_1} + \cdots + \epsilon_m \alpha_{i_m}.$$

But, the walk $\gamma(W)$ is homotopic to the walk $\gamma(W')$. Furthermore, we have

$$\alpha^{\gamma}(W) = \alpha^{\gamma}(W') = \epsilon_1 \alpha^{\gamma}(W_{i_1}) + \cdots + \epsilon_m \alpha^{\gamma}(W_{i_m}).$$

Since $\mathbf{L}_i \bar{\alpha} = \alpha^{\gamma}(W_i)$, we have

$$\alpha^{\gamma}(W) = \epsilon_1(\mathbf{L}_{i_1}\bar{\alpha}) + \cdots + \epsilon_m(\mathbf{L}_{i_m}\bar{\alpha}) = y_1(\mathbf{L}_1\bar{\alpha}) + \cdots + y_r(\mathbf{L}_r\bar{\alpha}) = \bar{y}(\mathbf{L}\bar{\alpha}).$$

Now, we have

$$\alpha_g(W) = \alpha(W) - \mid W \mid g = \bar{y}\bar{\alpha} - \mid W \mid g. \tag{1}$$

and

$$\alpha_g^{\gamma}(W) = \alpha^{\gamma}(W) - \mid W \mid g = \bar{y}(\mathbf{L}\bar{\alpha}) - \mid W \mid g. \tag{2}$$

We see that $\alpha_g(W) = 0$ if and only if $\bar{y}\bar{\alpha} = \mid W \mid g$. But,

$$\bar{y}\mathbf{L}\bar{\alpha} = \bar{y}\sigma(\bar{\alpha}) - (\sigma(g) - g)\bar{y}\bar{\epsilon} = \sigma(\bar{y}\bar{\alpha}) - (\sigma(g) - g)\bar{y}\bar{\epsilon},$$

and so $\alpha_g(W) = 0$ if and only if

$$\bar{y}\mathbf{L}\bar{\alpha} = \sigma(\mid W \mid g) - (\sigma(g) - g)\bar{y}\bar{\epsilon}.$$

Since $\sigma(2g) = 2g$, we have

$$\sigma(\mid W \mid g) = \begin{cases} \mid W \mid g & \text{if } \mid W \mid \text{ is even,} \\ (\mid W \mid -1)g + \sigma(g) & \text{if } \mid W \mid \text{ is odd.} \end{cases} \tag{3}$$

By (2), $\alpha_g(W) = 0$ if and only if

$$\alpha_g^{\gamma}(W) = \begin{cases} -(\sigma(g) - g)\bar{y}\bar{\epsilon} & \text{if } \mid W \mid \text{ is even,} \\ -(\sigma(g) - g)(\bar{y}\bar{\epsilon} - 1) & \text{otherwise.} \end{cases}$$

By the fact that $\epsilon(x_i) \equiv | C_i | \equiv | W_i | \pmod 2$ $(1 \leq i \leq r)$, we have

$$\bar{y}\bar{\epsilon} \equiv \sum_{k=1}^{m} | W_{i_k} | \equiv \sum_{k=1}^{m} | C_{i_k} | \equiv | W | \pmod 2. \tag{4}$$

Therefore, it follows that

$$\alpha_g(W) = 0 \text{ if and only if } \alpha_g(\gamma(W)) = 0.$$

Next, let U, V be any two closed u-walks of D. Set $\alpha(U) = \bar{u}\bar{\alpha}$ and $\alpha(V) = \bar{v}\bar{\alpha}$. By (1), $\alpha_g(U) = \alpha_g(V)$ if and only if $\alpha(U) - \alpha(V) = (| U | - | V |)g$, i.e.,

$$\bar{u}\bar{\alpha} - \bar{v}\bar{\alpha} = (\bar{u} - \bar{v})\bar{\alpha} = (| U | - | V |)g.$$

Then we have

$$\begin{aligned} (\bar{u} - \bar{v})\mathbf{L}\bar{\alpha} &= (\bar{u} - \bar{v})(\sigma(\bar{\alpha}) - (\sigma(g) - g)\bar{\epsilon}) \\ &= \sigma((\bar{u} - \bar{v})\bar{\alpha}) - (\sigma(g) - g)(\bar{u} - \bar{v})\bar{\epsilon} \\ &= \sigma((| U | - | V |)g) - (\sigma(g) - g)(\bar{u} - \bar{v})\bar{\epsilon} \end{aligned}$$

As in (3), we have

$$\sigma((| U | - | V |)g) = \begin{cases} (| U | - | V |)g & \text{if } | U | - | V | \text{ is even,} \\ (| U | - | V | - 1)g + \sigma(g) & \text{otherwise.} \end{cases}$$

By (2), $\alpha_g(U) = \alpha_g(V)$ if and only if

$$\alpha_g^{\gamma}(U) - \alpha_g^{\gamma}(V) = \begin{cases} -(\sigma(g) - g)(\bar{u} - \bar{v})\bar{\epsilon} & \text{if } | U | - | V | \text{ is even,} \\ -(\sigma(g) - g)((\bar{u} - \bar{v})\bar{\epsilon} - 1) & \text{otherwise.} \end{cases}$$

(4) implies that

$$\alpha_g(U) = \alpha_g(V) \text{ if and only if } \alpha_g(\gamma(U)) = \alpha_g(\gamma(V)).$$

Hence, α is $\gamma - g$–compatible. Q.E.D.

5 Special cases

Now, we present a characterization for any $\gamma \in \Gamma$ to have a lift in some special cases. Let $ord(g)$ be the order of $g \in A$.

Corollary 1 *Let D be a connected symmetric digraph, A a finite abelian group, $g \in A \setminus \{0\}$ and $\alpha \in C(D)$. Furthermore, let G be the underlying graph of D, T a spanning tree of G and $\Gamma \leq Aut\ D$. For $\gamma \in \Gamma$, let $\mathbf{L} = \mathbf{L}_T(\gamma)$. Assume that α is g-proper and α is T-reduced. If $ord(g)$ is odd or G is a bipartite graph, then the following two statements are equivalent:*

1. α is $\gamma - g-$compatible.

2. There exists an automorphism $\sigma \in (Aut\ A)_{2g}$ such that

$$\mathbf{L}\bar{\alpha} = \sigma(\bar{\alpha}).$$

Proof. In the case that *ord*(g) is odd, we have $\sigma(g) = g$ if $\sigma(2g) = 2g$. Furthermore, if G is a bipartite graph, then all cycles are of even length, and so we have $\bar{\epsilon} = \mathbf{0}$. By Theorem 3, the result follows. Q.E.D.

Corollary 2 *Let D be a connected symmetric digraph, A a finite abelian group, $g \in A \setminus \{0\}$ and $\alpha \in C(D)$. Furthermore, let G be the underlying graph of D, T a spanning tree of G and $\Gamma \leq Aut\ D$. For $\gamma \in \Gamma$, let $\mathbf{L} = \mathbf{L}_T(\gamma)$. Assume that α is g-proper and α is T-reduced. If ord(g) is odd, then the following two statements are equivalent:*

1. α is $\gamma - g-$compatible.

2. There exists an automorphism $\sigma \in (Aut\ A)_g$ such that

$$\mathbf{L}\bar{\alpha} = \sigma(\bar{\alpha}).$$

3. $\bar{y}\bar{\alpha} = kg$ if and only if $\bar{y}\mathbf{L}\bar{\alpha} = kg$ for $\bar{y} \in \mathbf{Z}^r$, where $k \in \mathbf{Z}$ and $r = \mid E(G) \setminus E(T) \mid$.

Proof. $1 \Leftrightarrow 2$: Clear.

$2 \Rightarrow 3$: Let $\bar{y}$ be an r-dimensional integer row vector. Then $\bar{y}\bar{\alpha} = kg$ if and only if

$$\bar{y}(\mathbf{L}\bar{\alpha}) = \bar{y}\sigma(\bar{\alpha}) = \sigma(\bar{y}\bar{\alpha}) = \sigma(kg) = kg.$$

$3 \Rightarrow 1$: Let u be a root of T. Let $W = (\cdots, x_{i_1}^{\epsilon_1}, \cdots, x_{i_2}^{\epsilon_2}, \cdots, x_{i_m}^{\epsilon_m}, \cdots)$ be any closed u-walk in D, where x_i is the (not necessarily distinct) cotree arcs, $\epsilon_l = \pm 1$ and the dotted spaces correspond to arcs in T. Then we have

$$\alpha(W) = \epsilon_1\alpha_{i_1} + \cdots + \epsilon_m\alpha_{i_m} = y_1\alpha_1 + \cdots + y_r\alpha_r = \bar{y}\bar{\alpha},$$

where $\bar{y} = (y_1, \cdots, y_r)$ and each y_i is the integer determined by the number of times the walk W traverses the arc x_i minus the number of times W traverses the inverse x_i^{-1}.

Similarly to the proof of Theorem 3, we have

$$\alpha^{\gamma}(W) = \bar{y}(\mathbf{L}\bar{\alpha}).$$

By (1) and the hypothesis, $\alpha_g(W) = 0$ if and only if $\bar{y}\bar{\alpha} = \mid W \mid g$, i.e., $\bar{y}\mathbf{L}\bar{\alpha} = \mid W \mid g$. By (2), $\alpha_g(W) = 0$ if and only if $\alpha_g(\gamma(W)) = 0$.

Next, let U, V be any two closed u-walks of D. Set $\alpha(U) = \bar{u}\bar{\alpha}$ and $\alpha(V) = \bar{v}\bar{\alpha}$. By (1), $\alpha_g(U) = \alpha_g(V)$ if and only if $\alpha(U) - \alpha(V) = (|U| - |V|)g$, i.e., $\bar{u}\bar{\alpha} - \bar{v}\bar{\alpha} = (\bar{u} - \bar{v})\bar{\alpha} = (|U| - |V|)g$. Then we have

$$(\bar{u} - \bar{v})\mathbf{L}\bar{\alpha} = (|U| - |V|)g.$$

By (2), $\alpha_g(U) = \alpha_g(V)$ if and only if $\alpha_g(\gamma(U)) = \alpha_g(\gamma(V))$. Hence, α is $\gamma - g$–compatible. Q.E.D.

In the case of $g = 0$, the 0-cyclic A-cover $D_0(\alpha)$ of D is the A-covering G^α of the underlying graph G of D. Note that the second condition in the definition of the $\gamma - 0$–compatibility is obtained from the first condition if $g = 0$. If α is $\gamma - 0$–compatible, then it is called *γ-compatible*(see [9,10]). By Theorem 3, we obtain a part of [10, Theorem 5]. Furthermore, α is called *proper* if it is 0-proper(see [9,10]).

Corollary 3 **(Širáň)** *Let G be a connected graph, A a finite abelian group, $\Gamma \leq Aut\ G$ and $\alpha : D(G) \longrightarrow A$ an ordinary voltage assignment. Furthermore, let T be a spanning tree of G and $\mathbf{L} = \mathbf{L}_T(\gamma)$ for $\gamma \in \Gamma$. Assume that α is proper and T-reduced. Then the following four statements are equivalent:*

1. *α is γ-compatible.*

2. *There exists an automorphism $\sigma \in Aut\ A$ such that*

$$\mathbf{L}\bar{\alpha} = \sigma(\bar{\alpha}).$$

6 Example

We conclude with an example. Let D be the symmetric digraph corresponding to the complete graph K_4 with four vertices 1,2,3 and 4, and $A = Z_4 = \{0, 1, 2, -1\}$ (the additive group). Then we have

$$Aut\ D = S_4.$$

Furthermore, let T be the spanning tree with edges $\{1,3\}$,$\{3,4\}$ and $\{2,3\}$.

Let $\gamma = (123) \in S_4$ and $g = 1$. We shall give a T-reduced alternating function $\alpha : A(D) \longrightarrow \mathbf{Z}_4$ which γ lifts to an automorphism of $D_1(\alpha)$. Let α be such a T-reduced alternating function which is g-proper. Furthermore, let

$$x_1 = (1,2), x_2 = (2,4), x_3 = (4,1), x_4 = (3,1), x_5 = (3,4), x_6 = (3,2).$$

Then we have

$$\mathbf{L} = \mathbf{L}_T(\gamma) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \end{bmatrix}.$$

Set

$$\alpha_i = \alpha(x_i)\ (i = 1,2,3);\ \bar{\alpha} = (\alpha_1, \alpha_2, \alpha_3).$$

By Theorem 3, there exists $\lambda \in (Aut\ \mathbf{Z}_4)_2$ such that

$$\mathbf{L}\bar{\alpha} = \lambda\bar{\alpha} - (\lambda g - g)\bar{\epsilon}.$$

But we have

$$(Aut\ \mathbf{Z}_4)_2 = Aut\ \mathbf{Z}_4 = \{1, -1\}.$$

Furthermore,

$$\bar{\epsilon} = {}^t(1,1,1).$$

Thus,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \lambda \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} - (\lambda - 1)g \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

i.e.,

$$\begin{cases} \alpha_1 = \lambda\alpha_1 - (\lambda - 1), \\ \alpha_3 = \lambda\alpha_2 - (\lambda - 1), \\ -\alpha_1 - \alpha_2 - \alpha_3 = \lambda\alpha_3 - (\lambda - 1). \end{cases}$$

In the case of $\lambda = 1$, we have

$$\alpha_1 = \alpha_2 = \alpha_3,\ i.e.,\ \bar{\alpha} = a\,{}^t(1,1,1), a = 0,1,2,-1.$$

Next, we determine the functions α which are 1-proper. For $1 = 1,2,3$, let C_i be the unique (directed) cycle of the subgraph $T + e_i$ containing x_i, where each e_i is the undirected edge deleting the direction from x_i. Then we have

$$\alpha_1(C_i) = a - 3$$

for each $i = 1,2,3$. Thus,

$$\alpha_1(C_i) = \begin{cases} 1 & \text{if } a = 0, \\ 2 & \text{if } a = 1, \\ -1 & \text{if } a = 2, \\ 0 & \text{if } a = -1. \end{cases}$$

If $\alpha_1 = 0,2$, then $\alpha_1(C_i^n) = \pm n$ for $n \in \mathbf{Z}$. Otherwise, $\alpha_1(C_i^n) = 0, 2n,\ n \in \mathbf{Z}$. If $\alpha_1 = 0,2$, then α is 1-proper.

If $\lambda = -1$, then we have

$$\bar{\alpha} = {}^t(1,1,3), {}^t(3,3,1)$$

as in the case of $\lambda = 1$. For each $i = 1, 2, 3$, we have

$$\alpha_1(C_i) = 0, 2,$$

and so

$$\alpha_1(C_i^n) = 0, 2n, \ n \in \mathbf{Z}.$$

In this case, no function α is 1-proper.

Therefore, $\gamma = (123)$ lifts to an automophism of $D_1(\alpha)$ if $\bar{\alpha} = {}^t(1, 1, 1)$, ${}^t(3, 3, 3)$.

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