Upper bounds on the domination number of a graph in terms of diameter and girth

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Abstract

A vertex set D of a graph G is a dominating set if every vertex not in D is adjacent to some vertex in D. The domination number γ of a graph G is the minimum cardinality of a dominating set in G. In 1989, Brigham and Dutton [1] proved

$$\gamma \leq \left\lceil \frac{3n-g}{6} \right\rceil$$

for each graph G of order n, minimum degree $\delta \geq 2$, and girth $g \geq 5$. If G is a graph of order n, minimum degree $\delta \geq 2$, girth $g \geq 5$ and neither a cycle nor one of two exceptional graphs, then we give in this paper the better bound

$$\gamma \leq \left\lceil \frac{3n-g}{6} \right\rceil - 1. \qquad (*)$$

For $\delta \geq 3$ and $g \geq 5$, we also prove $\gamma \leq \lceil (6n-g)/15 \rceil$, and this inequality is better than $(*)$ when $n > g + 10$. In addition, if $\delta \geq 3$, then we show that

$$2\gamma \leq n - (\delta - 2)(1 + \lfloor d/3 \rfloor),$$

where d is the diameter of the graph. Some related bounds in terms of the diameter, girth, order, and minimum degree are also presented.

Keywords: Domination number; Diameter of a graph; Girth of a graph

1. Terminology

We consider finite, undirected, and simple graphs G with the vertex set $V(G)$ and the edge set $E(G)$. The number of vertices $|V(G)|$ of a graph G is called the *order* of G and is denoted by $n = n(G)$. The *open neighborhood* $N(v) = N(v, G)$ of the vertex v consists of the vertices adjacent to v, and the *closed neighborhood* of v is $N[v] = N[v, G] = N(v) \cup \{v\}$. For a subset $S \subseteq V(G)$, we define $N(S) = N(S, G) = \bigcup_{v \in S} N(v)$ and $N[S] = N[S, G] = N(S) \cup S$. The vertex v is an *endvertex* if $d(v, G) = 1$, and an *isolated vertex* if $d(v, G) = 0$, where $d(v) = d(v, G) = |N(v)|$ is the degree of $v \in V(G)$. An edge incident with an endvertex is called a *pendant edge*. Let $\Omega(G)$ be the set of endvertices in a graph G. By $\delta = \delta(G)$ we denote the *minimum degree* of the graph G. Furthermore, the *diameter* $d = d(G)$ of a graph G is the maximum distance between two vertices of G, and the *girth* $g = g(G)$ is the length of a shortest cycle of G. We write C_n for a cycle of length n and K_n for the complete graph of order n. A cycle with length n is also called an n-cycle.

A set $D \subseteq V(G)$ is a *dominating set* of G if $N[D, G] = V(G)$. The *domination number* $\gamma = \gamma(G)$ of G is the cardinality of any smallest dominating set.

The *corona* $H \circ K_1$ of the graph H is the graph constructed from a copy of H, where for each vertex $v \in V(H)$, a new vertex v' and a pendant edge vv' are added.

For detailed information on domination and related topics see the comprehensive monograph [4] by Haynes, Hedetniemi, and Slater.

2. Preliminary results

The following well-known results play an important role in our investigations.

Proposition 2.1 (Ore [6] 1962). *If G is a graph without isolated vertices, then*

$$\gamma(G) \leq \left\lfloor \frac{n(G)}{2} \right\rfloor.$$

Theorem 2.2 (Payan, Xuong [8] 1982, Fink, Jacobson, Kinch, Roberts [2] 1985). For a graph G with even order n and no isolated vertices, $\gamma(G) = \lfloor n/2 \rfloor$ if and only if the components of G are the cycle C_4 or the corona $H \circ K_1$ for any connected graph H.

In 1998, Randerath and Volkmann [9] and independently, in 2000, Xu, Cockayne, Haynes, Hedetniemi, and Zhou [12] (cf. also [4], pp. 42-48) characterized the odd order graphs G for which $\gamma(G) = \lfloor n/2 \rfloor$. In the next theorem, we only note the part of this characterization which we will use in Section 4.

Theorem 2.3 (Randerath, Volkmann [9] 1998, Xu, Cockayne, Haynes, Hedetniemi, Zhou [12] 2000). Let G be a connected graph of odd order n with $\delta(G) \geq 2$. Then $\gamma(G) \leq (n-3)/2$, unless $G = C_5$, $G = C_7$, or G belongs to a family of 10 graphs of order at most 7 with girth less than or equal 4.

Theorem 2.4 (McCuaig, Shepherd [5] 1989). Let G be a connected graph of order n with $\delta(G) \geq 2$. Then $\gamma(G) \leq 2n/5$, unless $G = C_7$ or G belongs to a family of 6 graphs of order at most 7 and with girth less than or equal 4.

Theorem 2.5 (Flach, Volkmann [3] 1990). Let G be a graph of order n and minimum degree $\delta \geq 2$. If $A \subset V(G)$ is an arbitrary subset, then

$$2\gamma(G) \leq n + |A| - (\delta - 1)\frac{|N(A) - A|}{\delta}.$$

Proofs of the Theorems 2.5 and 2.2 can also be found in [11], pp. 217-219 and 223-224.

Theorem 2.6 (Reed [10] 1996). If G is a graph of order n with $\delta(G) \geq 3$, then $\gamma(G) \leq 3n/8$.

3. Upper bounds based on minimum degree, diameter and order

Theorem 3.1. If G is a connected graph of order n and minimum degree $\delta \geq 3$, then

$$2\gamma \leq n - (\delta - 2)(1 + \lfloor d/3 \rfloor).$$

Proof. Let $d = 3t + r$ with $0 \leq r \leq 2$ and let $x_0 x_1 \ldots x_d$ be a minimum length path between the vertices x_0 and x_d. If $A = \{x_0, x_3, \ldots, x_{3t}\}$, then $|A| = 1 + \lfloor d/3 \rfloor$ and $N(A) \cap A = \emptyset$. This implies

$$|N(A) - A| = |N(A)| = \left|\bigcup_{i=0}^{t} N(x_{3i})\right| = \sum_{i=0}^{t} |N(x_{3i})| \geq \delta|A|.$$

Applying Theorem 2.5, we obtain

$$\begin{aligned} 2\gamma &\le n + |A| - (\delta - 1)\frac{|N(A) - A|}{\delta} \\ &\le n + |A| - (\delta - 1)|A| \\ &= n - (\delta - 2)(1 + \lfloor d/3 \rfloor). \ \square \end{aligned}$$

Corollary 3.2 (Payan [7] 1975). If G is a graph of order n with $\delta \ge 3$, then

$$2\gamma \le n + 2 - \delta.$$

For the special family of graphs with no C_4 subgraphs, Brigham and Dutton [1] have presented the following better bound.

Theorem 3.3 (Brigham, Dutton [1] 1989). Let G be a connected graph of order n with $\delta \ge 3$. If G has no C_4 subgraphs, then

$$2\gamma \le n - 1 - (\delta - 1)(\lfloor d/3 \rfloor - 1 + \delta/2).$$

Inspired by Theorem 3.3, we will prove, similarly to the proof of Theorem 3.1, the following related bound.

Theorem 3.4. Let G be a connected graph of order n with $\delta \ge 4$. If G does not contain the 4-cycle and the diamond (a 4-cycle with a chord) as induced subgraphs, then

$$2\gamma \le n - 1 - (\delta - 3)(1 + \lfloor d/2 \rfloor) - \lfloor d/2 \rfloor/\delta.$$

Proof. Let $d = 2t + r$ with $0 \le r \le 1$ and let $x_0x_1 \dots x_d$ be a minimum length path between the vertices x_0 and x_d. If $A = \{x_0, x_2, \dots, x_{2t}\}$, then $|A| = 1 + \lfloor d/2 \rfloor = 1 + t$ and $N(A) \cap A = \emptyset$. Since G does not contain the 4-cycle and the diamond as induced subgraphs, we observe that

$$\begin{aligned} |N(A) - A| &= |N(A)| = \left| \bigcup_{i=0}^{t} N(x_{2i}) \right| \\ &= \sum_{i=0}^{t} |N(x_{2i})| - t \ge \delta|A| - |A| + 1. \end{aligned}$$

Thus, it follows from Theorem 2.5 that

$$\begin{aligned} 2\gamma &\le n + |A| - (\delta - 1)\frac{|N(A) - A|}{\delta} \\ &\le n + |A| - (\delta - 1)|A| + |A| - \frac{|A|}{\delta} - \frac{\delta - 1}{\delta} \end{aligned}$$

$$= n - 1 - (\delta - 3)|A| - \frac{|A| - 1}{\delta}$$
$$= n - 1 - (\delta - 3)(1 + \lfloor d/2 \rfloor) - \frac{\lfloor d/2 \rfloor}{\delta}. \square$$

Note that the family of graphs with no C_4 subgraphs is a subclass of the graphs which do not contain the 4-cycle and the diamond as induced subgraphs. In addition, for $\delta \geq 8$ and $d(G)$ great enough, for example $d(G) \geq 3\delta(\delta - 1)/(\delta - 7)$, the bound in Theorem 3.4 is better than this one in Theorem 3.3.

4. Upper bounds based on girth, order and minimum degree

In 1989, Brigham and Dutton [1] gave the following upper bound for the domination number based on the girth and the order (a proof of this theorem can also be found in [4], pp. 56-57).

Theorem 4.1 (Brigham, Dutton [1] 1989). If G is a graph of order n, minimum degree $\delta \geq 2$, and girth $g \geq 5$, then

$$\gamma \leq \left\lceil \frac{n - \lfloor g/3 \rfloor}{2} \right\rceil = \left\lceil \frac{3n - g}{6} \right\rceil. \tag{1}$$

The main theorem of this paper is the following improvement of Theorem 4.1, which shows in particular, that equality holds in (1) if and only if G is a cycle, the twin-C_7 (see the figure), or $G = 2C_7$.

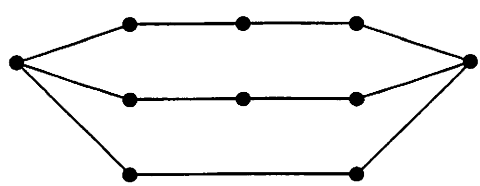

Figure: twin-C_7

Theorem 4.2 Let G be a graph of order n, minimum degree $\delta \geq 2$, and girth $g \geq 5$. If G is not a cycle and not isomorphic to $2C_7$ and to the twin-C_7, then

$$\gamma \leq \left\lceil \frac{3n - g - 6}{6} \right\rceil = \left\lceil \frac{3n - g}{6} \right\rceil - 1. \tag{2}$$

Proof. Observe that in general, a g-cycle can be dominated by $\lceil g/3 \rceil$ vertices. Assume that G is not a cycle, and remove a g-cycle C_g from G to a form a graph H. Since $g \geq 5$ and $\delta \geq 2$, the graph H has minimum degree at least $\delta - 1 \geq 1$.

Case 1. One of the components of H is a corona graph F.

Subcase 1.1. The corona graph $F = F' \circ K_1$ has at least four vertices. Let u be an endvertex of F. Because of $\delta \geq 2$, the vertex u is adjacent with a vertex $x \in V(C_g)$. If we choose, without loss of generality, a minimum dominating set D_g of C_g such that $x \in D_g$, then D_g dominates the vertex u. Since $F - u$ is a connected graph of odd order with at least three vertices, Proposition 2.1 implies $\gamma(F - u) \leq \lfloor (n(F) - 2)/2 \rfloor$. Applying once more Proposition 2.1 on all the remaining components of H, we obtain

$$\gamma \leq \left\lfloor \frac{n(H) - 2}{2} \right\rfloor + \left\lceil \frac{g}{3} \right\rceil = \left\lfloor \frac{n - g - 2}{2} \right\rfloor + \left\lceil \frac{g}{3} \right\rceil \leq \left\lceil \frac{3n - g - 6}{6} \right\rceil.$$

Subcase 1.2. The corona graph $F = F' \circ K_1$ consists of two vertices u and v. Since each of the two vertices u and v has a neighbor in C_g, we conclude that $g \leq 6$.

If $C_g = x_1x_2x_3x_4x_5x_1$, then let, without loss of generality, $ux_1, vx_3 \in E(G)$. Obviously, the vertices x_1 and x_3 dominate $V(C_g) \cup \{u, v\}$, and thus, it follows from Proposition 2.1 that

$$\gamma \leq 2 + \left\lfloor \frac{n - 7}{2} \right\rfloor = \left\lfloor \frac{3n - 9}{6} \right\rfloor \leq \left\lceil \frac{3n - g - 6}{6} \right\rceil.$$

If $C_g = x_1x_2x_3x_4x_5x_6x_1$, then let, without loss of generality, $ux_1, vx_4 \in E(G)$. Obviously, the vertices x_1 and x_4 dominate $V(C_g) \cup \{u, v\}$, and thus, it follows from Proposition 2.1 that

$$\gamma \leq 2 + \left\lfloor \frac{n - 8}{2} \right\rfloor = \left\lfloor \frac{3n - 12}{6} \right\rfloor \leq \left\lceil \frac{3n - g - 6}{6} \right\rceil.$$

Case 2. None of the components of H is a corona graph, and H contains a component F of even order. The hypothesis $g \geq 5$ implies $F \neq C_4$ and $n(F) \geq 4$. Hence, it follows from Theorem 2.2 that $\gamma(F) \leq (n(F) - 2)/2$. Therefore, Proposition 2.1 leads to

$$\gamma \leq \left\lfloor \frac{n(H) - 2}{2} \right\rfloor + \left\lceil \frac{g}{3} \right\rceil \leq \left\lceil \frac{3n - g - 6}{6} \right\rceil.$$

Case 3. The graph H contain two odd components H_1 and H_2. We conclude from Proposition 2.1 that $\gamma(H_i) \leq (n(H_i) - 1)/2$ for $i = 1, 2$ and hence, we arrive at

$$\gamma \leq \left\lfloor \frac{n(H) - 2}{2} \right\rfloor + \left\lceil \frac{g}{3} \right\rceil \leq \left\lceil \frac{3n - g - 6}{6} \right\rceil.$$

Case 4. The graph H is connected and of odd order.

Case 4.1. The graph H has an endvertex u and $H-u$ is not a corona graph. The vertex u is adjacent with a vertex $x \in V(C_g)$. If we choose, without loss of generality, a minimum dominating set D_g of C_g such that $x \in D_g$, then D_g dominates the vertex u. Since $H-u$ is connected, of even order, and not a corona graph, it follows from Theorem 2.2 that $\gamma(H-u) \leq \lfloor (n(H-u)-2)/2 \rfloor$. This leads to

$$\gamma \leq \left\lfloor \frac{n(H-u)-2}{2} \right\rfloor + \left\lceil \frac{g}{3} \right\rceil = \left\lfloor \frac{n(H)-3}{2} \right\rfloor + \left\lceil \frac{g}{3} \right\rceil \leq \left\lceil \frac{3n-g-9}{6} \right\rceil.$$

Case 4.2. The graph H has an endvertex u and $H-u$ is a corona graph.

Subcase 4.2.1. The vertex u is adjacent with an endvertex w of $H-u$ and $H-u$ consists of two vertices w and z. Since z is adjacent with a vertex of C_g, we observe that $g \leq 8$.

If $g=5$, then

$$\gamma = 3 = \left\lceil \frac{13}{6} \right\rceil = \left\lceil \frac{3n-g-6}{6} \right\rceil.$$

If $g=6$, then

$$\gamma = 3 = \left\lceil \frac{15}{6} \right\rceil = \left\lceil \frac{3n-g-6}{6} \right\rceil.$$

If $g=7$, then G is isomorphic to the forbidden graph twin-C_7, and we observe that $\gamma(G) = 4 = \lceil (3n-g)/6 \rceil$.

If $g=8$, then

$$\gamma = 4 = \left\lceil \frac{19}{6} \right\rceil = \left\lceil \frac{3n-g-6}{6} \right\rceil.$$

Subcase 4.2.2. The vertex u is adjacent with an endvertex w of $H-u$, and $H-u$ consists of at least 4 vertices. Let $x \in V(C_g)$ be adjacent with u.

If $g = 3s+1$ with $s \geq 2$, then observe that u dominates the vertices x and w and $\gamma(C_g - x) = (g-1)/3$. As $H - \{u,w\}$ is connected and of odd order with at least three vertices, Proposition 2.1 yields $\gamma(H-\{u,w\}) \leq (n(H)-3)/2$. Altogether, we obtain

$$\gamma \leq \left\lfloor \frac{n(H)-3}{2} \right\rfloor + \left\lceil \frac{g}{3} \right\rceil \leq \left\lceil \frac{3n-g-9}{6} \right\rceil.$$

Let $g = 3s+2$ with $s \geq 1$. If we choose a minimum dominating set D_g of C_g such that $x \in D_g$, then D_g dominates the vertex u. As $H - \{u\}$ is connected with at least four vertices, Proposition 2.1 yields $\gamma(H-\{u\}) \leq (n(H)-1)/2$. Thus, it follows that

$$\gamma \leq \left\lfloor \frac{n(H)-1}{2} \right\rfloor + \left\lceil \frac{g}{3} \right\rceil \leq \left\lceil \frac{3n-g-3}{6} \right\rceil.$$

However, in this situation, $3n$ and g have opposite parity so it is straightforward to verify that

$$\left\lceil \frac{3n-g-3}{6} \right\rceil = \left\lceil \frac{3n-g-6}{6} \right\rceil .$$

Let $g = 3s$ with $s \geq 2$. If we choose a minimum dominating set D_g of C_g such that $x \in D_g$, then D_g dominates the vertex u. As above, it follows that

$$\gamma \leq \left\lfloor \frac{n(H)-1}{2} \right\rfloor + \left\lceil \frac{g}{3} \right\rceil \leq \left\lceil \frac{3n-g-3}{6} \right\rceil = \left\lceil \frac{3n-g-6}{6} \right\rceil .$$

Subcase 4.2.3. The vertex u is adjacent with a vertex w of $H-u$ and w is not an endvertex of $H-u$. Let z be an endvertex of $H-u$ which is not adjacent with w, and let v be adjacent with z in $H-u$. Since z is an endvertex of H, there exists a neighbor $y \in V(C_g)$ of z. If we choose a minimum dominating set D_g of C_g such that $y \in D_g$, then D_g dominates the vertex z. Furthermore, we observe that $V(H) - (\Omega(H) \cup \{v\})$ is a dominating set of $H - z$. Combining these two dominating sets, we deduce that

$$\gamma \leq \left\lfloor \frac{n(H)-3}{2} \right\rfloor + \left\lceil \frac{g}{3} \right\rceil \leq \left\lceil \frac{3n-g-9}{6} \right\rceil .$$

Subcase 4.3. The graph H has no endvertex. Since H is of odd order, it follows from Theorem 2.3 that $H = C_5$, $H = C_7$, or $\gamma(H) \leq (n(H)-3)/2$. In the last case, we obtain

$$\gamma \leq \left\lfloor \frac{n(H)-3}{2} \right\rfloor + \left\lceil \frac{g}{3} \right\rceil \leq \left\lceil \frac{3n-g-9}{6} \right\rceil .$$

If $H = C_5$, then $C_g = C_5$ and $\gamma \leq 4 = \lceil (3n-g-6)/6 \rceil$. If $H = C_7$, then $C_g = C_5$, $C_g = C_6$, or $C_g = C_7$. In the cases $C_g = C_5$ and $C_g = C_6$, the desired inequality $\gamma \leq 5 = \lceil (3n-g-6)/6 \rceil$ is immediate. In the remaining case $C_g = C_7$, we arrive at the forbidden graph $G = 2C_7$ or the cycles of length 7 are connected by an edge. In the last case it is easy to see that $\gamma \leq 5 = \lceil (3n-g-6)/6 \rceil$. Finally, we observe that $\gamma(2C_7) = 6 = \lceil (3n-g)/6 \rceil$. □

The next result of Brigham and Dutton [1] is identical with Theorem 4.1 when $\delta = 2$, and an improvement of (1) when $\delta \geq 3$.

Theorem 4.3 (Brigham, Dutton [1] 1989). If G is a graph of order n, minimum degree $\delta \geq 2$, and girth $g \geq 5$, then

$$\gamma \leq \left\lceil \frac{n - \lfloor g/3 \rfloor - (g-4)\frac{(\delta-2)(\delta-3)}{2} - 2(\delta-2)}{2} \right\rceil .$$

If the order n of a graph G is great enough, then the following results are better than Theorem 4.3.

Theorem 4.4. If G is a graph of order n, minimum degree $\delta \geq 4$, and girth $g \geq 5$, then

$$\gamma \leq \left\lceil \frac{9n-g}{24} \right\rceil.$$

Proof. Remove a g-cycle C_g from G to form a graph H. Since $g \geq 5$ and $\delta \geq 4$, the graph H has minimum degree at least $\delta - 1 \geq 3$. Thus, Theorem 2.6 leads to

$$\gamma \leq \left\lfloor \frac{3(n-g)}{8} \right\rfloor + \left\lceil \frac{g}{3} \right\rceil \leq \left\lceil \frac{9n-g}{24} \right\rceil. \ \square$$

Theorem 4.5. If G is a graph of order n, minimum degree $\delta \geq 3$, and girth $g \geq 5$, then

$$\gamma \leq \left\lceil \frac{6n-g}{15} \right\rceil.$$

Proof. Remove a g-cycle C_g from G to form a graph H. Since $g \geq 5$ and $\delta \geq 3$, the graph H has minimum degree at least $\delta - 1 \geq 2$. If F is a component of H, then it follows from Theorem 2.4 that $\gamma(F) \leq 2n(F)/5$ or $F = C_7$.

Suppose that there exists a component $F = C_7 = x_1x_2x_3x_4x_5x_6x_7x_1$. This yields $5 \leq g \leq 7$, and because of $\delta \geq 3$, we conclude that each vertex of F is adjacent with a vertex of C_g.

Let $g = 7$. Since x_1 and x_2 have a neighbor in C_g, it follows immediately that x_1 and x_2 are contained in p-cycle with $p \leq 6$. This is a contradiction to the hypothesis that $g = 7$.

Let $g = 6$ such that $C_g = y_1y_2y_3y_4y_5y_6y_1$. We assume, without loss of generality, that $x_1y_1 \in E(G)$. This implies $x_2y_4 \in E(G)$. Since x_3 is also adjacent with one vertex of C_g, we observe that x_3 is contained in a p-cycle with $p \leq 5$, a contradiction.

Analogously to the case $g = 6$, one can show that $g = 5$ is also not possible.

Consequently, we have $\gamma(F) \leq 2n(F)/5$ for all components F of H and thus,

$$\gamma \leq \left\lfloor \frac{2(n-g)}{5} \right\rfloor + \left\lceil \frac{g}{3} \right\rceil \leq \left\lceil \frac{6n-g}{15} \right\rceil. \ \square$$

Following the idea of the proof of Theorem 4.3 by Brigham and Dutton [1], which is an improvement of Theorem 4.1 for $\delta \geq 3$, we will present similar improvements of Theorems 4.4 and 4.5 for $\delta \geq 5$ and $\delta \geq 4$, respectively.

Theorem 4.6. If G is a graph of order n, minimum degree $\delta \geq 4$, and girth $g \geq 5$, then

$$\gamma \leq \left\lceil \frac{9n - g - 4(g-4)(\delta-4)(\delta-2) - 18(\delta-4)}{24} \right\rceil.$$

Proof. We proceed by induction on $\delta \geq 4$. According to Theorem 4.4, the bound is valid for $\delta = 4$. Now let $\delta \geq 5$, and let F be the induced subgraph of G consisting of $g-4$ consecutive vertices of a g-cycle and all the neighbors of these vertices, and let $H = G - V(F)$. This implies $\delta(H) \geq \delta - 1 \geq 4$, $|V(F)| \geq (g-4)(\delta-1)+2$, $\gamma(F) \leq g-4$, and $n(H) \leq n-(g-4)(\delta-1)-2$. Now the induction hypothesis leads to

$$\begin{aligned}
\gamma &\leq \gamma(F) + \gamma(H) \\
&\leq g - 4 + \left\lceil \frac{9n(H) - g - 4(g-4)(\delta-5)(\delta-3) - 18(\delta-5)}{24} \right\rceil \\
&\leq \left\lceil \frac{9n - g - 18(\delta-4) - (g-4)(9(\delta-1) + 4(\delta-5)(\delta-3) - 24)}{24} \right\rceil \\
&= \left\lceil \frac{9n - g - 18(\delta-4) - (g-4)(4\delta^2 - 23\delta + 27)}{24} \right\rceil \\
&\leq \left\lceil \frac{9n - g - 4(g-4)(\delta-4)(\delta-2) - 18(\delta-4)}{24} \right\rceil,
\end{aligned}$$

since $\delta \geq 5$. □.

Analogously, one can give the following improvements of Theorem 4.5, where the second one is better than the first one when $4 \leq \delta \leq 5$.

Theorem 4.7. If G is a graph of order n, minimum degree $\delta \geq 3$, and girth $g \geq 5$, then

$$\gamma \leq \left\lceil \frac{6n - g - 3(g-4)(\delta-3)(\delta-4) - 12(\delta-3)}{15} \right\rceil.$$

Theorem 4.8. If G is a graph of order n, minimum degree $\delta \geq 3$, and girth $g \geq 5$, then

$$\gamma \leq \left\lceil \frac{6n - g - \frac{3(g-4)(\delta-3)(\delta-2)}{2} - 12(\delta-3)}{15} \right\rceil.$$

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