Altitude of $K_{3,n}$

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Abstract

An *edge-ordering* of a graph $G = (V, E)$ is a one-to-one function f from E to the set of positive integers. A path of length k in G is called a (k, f)-*ascent* if f increases along the edge sequence of the path. The *altitude* $\alpha(G)$ of G is the greatest integer k such that for all edge-orderings f, G has a (k, f)-ascent.

We obtain a recursive lower bound for $\alpha(K_{m,n})$ and show that

$$\alpha(K_{3,n}) = \begin{cases} 4 & \text{if } 5 \le n \le 9 \\ 5 & \text{if } 10 \le n \le 12 \\ 6 & \text{if } n \ge 13. \end{cases}$$

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1 Introduction

A one-to-one function f from E to the set of positive integers is called an *edge-ordering* of the graph $G = (V, E)$. For $e \in E$, we call $f(e)$ the *label* of e, and use e and $f(e)$ interchangeably. Denote the set of all edge-orderings of G by $\mathcal{F}$. For $f \in \mathcal{F}$, a path of G for which f increases along the edge sequence is called an f-*ascent* of G, and a (k, f)-*ascent* if it has length k. The *height* $h(f)$ of f is the maximum length of an f-ascent. The *altitude* $\alpha(G)$ of G is defined by

$$\alpha(G) = \min_{f \in \mathcal{F}} h(f).$$

Observe that $\alpha(G)$ is the greatest integer k such that G has a (k, f)-ascent for each edge-ordering $f \in \mathcal{F}$.

Clearly, $\alpha(G) \geq 2$ for any graph G with a vertex of degree at least two. It is also evident that if H is a subgraph of G, then $\alpha(H) \leq \alpha(G)$. The altitude of some classes of graphs is easy to determine, for example, $\alpha(C_{2n}) = 2$, $\alpha(C_{2n+1}) = 3$ for all $n \geq 2$ and $\alpha(C_3) = 2$ (since C_3 has no path of length three).

Although the term "altitude" was first used in [3], the concept was introduced by Chvatál and Komlós [5] who posed the problem of determining $\alpha(K_n)$. The best general bounds for $\alpha(K_n)$ are

$$\alpha(K_n) > \tfrac{1}{2}(\sqrt{4n-3} - 1)$$

obtained by Graham and Kleitman [6], and

$$\alpha(K_n) \leq 8\left\lceil \tfrac{n-4}{12} \right\rceil + 2 \tag{1}$$

(see e.g. [3], [7]). For small even n, the upper bound

$$\alpha(K_{n-1}) \leq \alpha(K_n) \leq \begin{cases} \left\lceil \frac{11n}{16} \right\rceil & \text{if } n \equiv 10 \pmod{16} \\ \left\lfloor \frac{11n-1}{16} \right\rfloor & \text{otherwise,} \end{cases}$$

obtained in [3], is better than the bound in (1). (The largest integer for which this bound is smaller than that in (1) is $n = 270$.) The best asymptotic upper bound

$$\alpha(K_n) \leq (\tfrac{1}{2} + o(1))n$$

was obtained by Calderbank, Chung and Sturtevant [4]. Results on the altitude of other classes of graphs can be found in [1, 2, 8, 9].

This paper is mainly concerned with the altitude of complete bipartite graphs $K_{3,n}$. It was proved in [3] that for $m \leq n$,

$$\alpha(K_{m,n}) \leq \min\{2m, \left\lceil \tfrac{3}{2}\left\lceil \tfrac{n}{2} \right\rceil \right\rceil\}, \tag{2}$$

thus $\alpha(K_{3,n}) \leq 6$. Combining results in [2], [3] and [8], we have the following exact values for $\alpha(K_{m,n})$.

Proposition 1

(*i*) $\alpha(K_{1,2}) = \alpha(K_{2,2}) = 2$.

(*ii*) $\alpha(K_{m,n}) = 3$ *for* $2 \leq m \leq 4$, $3 \leq n \leq 4$.

(*iii*) $\alpha(K_{2,n}) = 4$ *for* $n \geq 5$.

(*iv*) $\alpha(K_{m,n}) = 4$ *for* $3 \leq m \leq 4$, $5 \leq n \leq 6$.

(*v*) $\alpha(K_{5,5}) = 4$.

Thus we see that $\alpha(K_{2,n})$ is known for all $n \geq 2$, while $\alpha(K_{3,n})$ is known for $3 \leq n \leq 6$. In this paper we obtain a recursive lower bound for the altitude of $K_{m,n}$ and determine $\alpha(K_{3,n})$ for all $n \geq 7$.

2 Lower bounds

We denote a (k, f)-ascent whose edges (in sequence) have the labels $u_1, ..., u_k$ by $(u_1...u_k)$. If $\lambda = (u_1...u_k)$ is a (k, f)-ascent and $(v_1...v_l)$ is an (l, f)-ascent with $v_l < u_1$, then $(v_1...v_l)\lambda$ denotes the $(k+l, f)$-ascent $(v_1...v_l u_1...u_k)$.

Theorem 2 *Given arbitrary m and n, if, for any edge-ordering f, $K_{m,n}$ has a (k, f)-ascent starting at a vertex of degree m, then for any edge-ordering f', $K_{m+1,(m+1)n+1}$ has a $(k+2, f')$-ascent starting at a vertex of degree $m+1$.*

Proof. Let $p = (m+1)n+1$ and suppose $K_{m+1,p}$ has bipartition (U, X), where $X = \{X_1, ..., X_p\}$. Consider an arbitrary edge-ordering f' of $K_{m+1,p}$. For each $i = 1, ..., p$, let e_i be the edge incident with X_i with smallest label and let $E_1 = \{e_i : i = 1, ..., p\}$. By the pigeonhole principle, some vertex $A \in U$ is incident with at least $n+1$ edges in E_1. Let these edges be AX_i with labels $a_i = f'(AX_i)$, $i = 1, ..., n+1$. Assume without loss of generality that $a_1 = \min_{i=1}^{n+1}\{a_i\}$. Consider the subgraph $H \cong K_{m,n}$ of $K_{m+1,p}$ induced by $(U - \{A\}, \{X_2, ..., X_{n+1}\})$. For the edge-ordering $f = f'|E(H)$, let λ be a (k, f)-ascent of H starting at X_i (a vertex of degree m in H) for some $i = 2, ..., n+1$. Since a_i is the smallest of the labels of all edges incident with X_i, $(a_1 a_i)\lambda$ is a $(k+2, f')$-ascent of $K_{m+1,p}$ starting at vertex X_1, which has degree $m+1$. ■

Corollary 3 *Let m and n be integers such that for any edge-ordering f, $K_{m,n}$ has a (k, f)-ascent starting at a vertex of degree m.*

(*i*) $\alpha(K_{m+1,(m+1)n+1}) \geq k+2$.

(*ii*) *If* $p = [\frac{n}{m!} + \frac{1}{(m+1)!} + \cdots + \frac{1}{(m+q)!}](m+q)!$, *then* $\alpha(K_{m+q,p}) \geq k+2q$.

Proof. Statement (*i*) is a direct consequence of Theorem 2 and (*ii*) is obtained by applying Theorem 2 q times. ■

Corollary 4

(*i*) $\alpha(K_{2,3}) \geq 3$, $\alpha(K_{3,10}) \geq 5$, $\alpha(K_{4,41}) \geq 7$. *If* $n \geq \lfloor(e-1)m!\rfloor$, *then* $\alpha(K_{m,n}) \geq 2m-1$.

(*ii*) $\alpha(K_{2,5}) \geq 4$, $\alpha(K_{3,16}) \geq 6$, $\alpha(K_{4,65}) \geq 8$. *If* $n \geq \lfloor em!\rfloor$, *then* $\alpha(K_{m,n}) \geq 2m$.

Proof. (*i*) Clearly $K_{1,1}$ satisfies the hypothesis of Theorem 2 and $\alpha(K_{1,1}) = 1$, so if we substitute $m = n = k = 1$ and $q = 1$, 2 and 3 respectively in Corollary 3(*ii*), we obtain $\alpha(K_{2,3}) \geq 3$, $\alpha(K_{3,10}) \geq 5$ and $\alpha(K_{4,41}) \geq 7$.

Further, if we first substitute $m = n = k = 1$ and then $q + 1 = m$ in Corollary 3(*ii*) and use the expansion of e, we get

$$p = [1 + \frac{1}{2!} + \cdots + \frac{1}{(q+1)!}](q+1)! = [1 + \frac{1}{2!} + \cdots + \frac{1}{m!}]m! < (e-1)m!$$

and so $p \leq \lfloor (e-1)m! \rfloor$. Since any supergraph H of a graph G satisifes $\alpha(H) \geq \alpha(G)$, we obtain $\alpha(K_{m,n}) \geq 2m - 1$ for $n \geq \lfloor (e-1)m! \rfloor$.
(*ii*) Similarly, using $K_{1,2}$ as the initial case (thus $m = 1$, $n = k = 2$) and the fact that $2 + \frac{1}{2!} + \cdots + \frac{1}{r!} < e$, we obtain (*ii*). ■

In some cases the bounds given by Corollary 4 are best possible in two ways. For example, by Proposition 1, $\alpha(K_{2,5}) = 4$, thus the bound is exact, and also 5 is the smallest integer n such that $\alpha(K_{2,n}) = 4$, i.e. such that $\alpha(K_{2,n})$ attains the bound in (2).

For m fixed, we denote the smallest integer n such that $\alpha(K_{m,n}) = 2m$ by $\theta(m)$. Thus $\theta(2) = 5$ and by Corollary 4(*ii*), $\theta(m) \leq \lfloor em! \rfloor$. We do not expect this bound to be good in general. For example, the bound gives $\theta(3) \leq 16$, but more detailed analysis with the proof technique of Theorem 2 will establish $\theta(3) = 13$ (Corollary 12).

We now consider $K_{3,n}$ with bipartition $(\{A, B, C\}, \{X_1, ..., X_n\})$ and edge-orderings f with labels $f(AX_i) = a_i$, $f(BX_i) = b_i$ and $f(CX_i) = c_i$ for each $i = 1, ..., n$. We will be concerned with the ordered partition (E_1, E_2, E_3) of $E(K_{3,n})$ induced by f in the following way. For each $i = 1, ..., n$ and any edge $e_i \in \{a_i, b_i, c_i\}$,

$$e_i \in \begin{cases} E_1 & \text{if } e_i = \min\{a_i, b_i, c_i\}, \\ E_3 & \text{if } e_i = \max\{a_i, b_i, c_i\}, \\ E_2 & \text{otherwise.} \end{cases}$$

Our next result concerns ascents in $K_{2,3}$, and with notation as above, we consider the bipartition $(\{B, C\}, \{X_1, X_2, X_3\})$ of $K_{2,3}$, with its edges labelled b_i and c_i, $i = 1, 2, 3$.

Proposition 5

(*i*) *In any edge-ordering f of $K_{2,3}$ there is a $(3, f)$-ascent starting at a vertex of degree two and one starting at a vertex of degree three.*

(*ii*) *If f is an edge-ordering of $K_{2,3}$ with $b_i < c_i$ for $i = 1, 2, 3$, then there is a $(4, f)$-ascent starting at a vertex X_j, $j = 1, 2, 3$.*

Proof. (*i*) Consider an arbitrary edge-ordering f of $K_{2,3}$. Without loss of generality we may assume that $b_1 < c_1$ and $b_2 < c_2$, and so $(b_1 b_2 c_2)$ (if $b_1 < b_2$) or $(b_2 b_1 c_1)$ (if $b_2 < b_1$) is a $(3, f)$-ascent starting at a vertex of degree two. Similarly, $(b_2c_2c_1)$ (if $c_2 < c_1$) or $(b_1c_1c_2)$ (if $c_1 < c_2$) is a $(3, f)$-ascent starting at a vertex of degree three.

(ii) Assume without loss of generality that $b_1 = \min\{b_1, b_2, b_3\}$. Then $(b_1b_2c_2c_3)$ (if $c_2 < c_3$) or $(b_1b_3c_3c_2)$ (if $c_3 < c_2$) is a $(4, f)$-ascent starting at a vertex of degree two. ■

The following two propositions involve ascents in $K_{3,4}$ and will be used extensively in the proof of Theorem 8. In each of these propositions, $K_{3,4}$ has bipartition $(\{A, B, C\}, \{X_1, ..., X_4\})$ and we use the notation defined above.

Proposition 6

(a) *Let f be an edge-ordering of $K_{3,4}$ such that $a_i \in E_1$ for $i = 1, ..., 4$ and $a_1 = \min_{i=1}^{4}\{a_i\}$.*

  (*i*) *There is a $(5, f)$-ascent starting at vertex X_1.*
  (*ii*) *If $b_i \in E_2$ for $i = 2, 3, 4$, then there is a $(6, f)$-ascent starting at vertex X_1.*

(b) *Let f be an edge-ordering of $K_{3,4}$ such that $a_i \in E_3$ for $i = 1, ..., 4$ and $a_1 = \max_{i=1}^{4}\{a_i\}$.*

  (*i*) *There is a $(5, f)$-ascent terminating at vertex X_1.*
  (*ii*) *If $b_i \in E_2$ for $i = 2, 3, 4$, then there is a $(6, f)$-ascent terminating at vertex X_1.*

Proof. (a) Consider the $K_{2,3}$ induced by $\{B, C, X_2, X_3, X_4\}$.

(*i*) By Proposition 5(*i*) there is a $(3, f)$-ascent λ beginning at X_j for some $j = 2, 3, 4$. Then $(a_1a_j)\lambda$ is a $(5, f)$-ascent.

(*ii*) By Proposition 5(*ii*) there is a $(4, f)$-ascent λ beginning at X_j for some $j = 2, 3, 4$, and thus $(a_1a_j)\lambda$ is a $(6, f)$-ascent.

(b) These statements follow by applying (a) to the edge-ordering formed by reversing the order of f. ■

Proposition 7 *Let f be an edge-ordering of $K_{3,4}$ such that $a_i \in E_1$ for $i = 1, 2, 3$, $c_i \in E_3$ for $i = 2, 3, 4$,*

$$a_1 < a_s \quad \textit{where} \quad b_s = \min\{b_2, b_3\} \tag{3}$$

and

$$c_4 > c_t \quad \textit{where} \quad b_t = \max\{b_2, b_3\}. \tag{4}$$

Then $K_{3,4}$ has the $(6, f)$-ascent $(a_1a_sb_sb_tc_tc_4)$.

Proof. The hypothesis implies that $\{b_2, b_3\} \subseteq E_2$ and the result follows. ■

It is important to observe that $a_1 = \min\{a_1, a_2, a_3\}$ (respectively $c_4 = \max\{c_2, c_3, c_4\}$) is a sufficient condition for (3) (respectively (4)) to hold. This fact will also be used repeatedly.

We now prove the final theorem of this section. The argument includes frequent applications of Propositions 6 and 7 or their contrapositives to specific induced subgraphs isomorphic to $K_{3,4}$. Earlier applications are detailed but in later cases we merely use statements such as "By Proposition 6 (or 7) applied to $(\{B, A, C\}, \{X_i, X_j, X_k, X_l\})$, ..." We mean that the proposition is applied to the $K_{3,4}$ with this bipartition. Moreover, in the application, B (respectively A, C) has the role of A (respectively B, C) in Proposition 6 (or 7), while X_i (respectively X_j, X_k, X_l) has the role of X_1 (respectively X_2, X_3, X_4).

Theorem 8 $\alpha(K_{3,13}) \geq 6$.

Proof. Let $K_{3,13}$ have bipartition $(\{A, B, C\}, \{X_1, ..., X_{13}\})$ and an arbitrary edge-ordering f. We use the notation a_i, b_i, c_i $(i = 1, ..., 13)$ and E_1, E_2, E_3 as defined above. By Corollary 4(i), $\alpha(K_{3,10}) \geq 5$, hence any edge-ordering of $K_{3,13}$ has height at least five. Suppose contrary to the statement of the theorem that there exists an edge-ordering f of $K_{3,13}$ with $h(f) = 5$.

By the pigeonhole principle, one of the vertices A, B and C, say A, is incident with $k \geq 5$ edges in E_1, say $a_1, \cdots, a_k$ with $a_1 = \min_{i=1}^{k}\{a_i\}$. By Proposition 6(a)($ii$), B is incident with at most two edges $b_i \in E_2$ where $i \in \{2, ..., k\}$, and similarly (interchanging the roles of B and C in Proposition 6(a)(ii)), C is incident with at most two edges $c_i \in E_2$, $i \in \{2, ..., k\}$. Since each X_i is incident with exactly one edge in each E_j, $j = 1, 2, 3$, it follows that each of B and C joins exactly two of the vertices $X_2, ..., X_k$ with edges in E_2, and in particular, $k = 5$. Note that similar arguments show that each of B and C is incident with at most five edges in E_1.

Assume without loss of generality that $b_2, b_3 \in E_2$ and $b_4, b_5 \in E_3$. Then $c_2, c_3 \in E_3$ and $c_4, c_5 \in E_2$. Note that we make no assumption about the edges b_1 and c_1, but it is clear that $b_1, c_1 \in E_2 \cup E_3$ and that $b_1 \in E_2$ if and only if $c_1 \in E_3$.

We now define additional notation. For $j = 1, 2, 3$,

$$A_j = \{a_i : a_i \in E_j,\ i = 1, ..., 13\}.$$

The edge-sets B_j, C_j, $j = 1, 2, 3$ are defined similarly. Further, for $j = 2, 3$, let $C'_j = C_j - \{c_1\}$.

Lemma 8.1 $|C'_3 - \{c_2, c_3\}| = 2$.

Proof. We note that $A_1 = \{a_1, ..., a_5\}$ and deduce

$$\{a_i \ : \ c_i \in C_2' - \{c_4, c_5\}\} \subseteq A_3, \tag{5}$$
$$\{a_i \ : \ c_i \in C_3' - \{c_2, c_3\}\} \subseteq A_2, \tag{6}$$

hence

$$\{b_i \ : \ c_i \in C_2' - \{c_4, c_5\}\} \subseteq B_1, \tag{7}$$
$$\{b_i \ : \ c_i \in C_3' - \{c_2, c_3\}\} \subseteq B_1. \tag{8}$$

Suppose that $\{c_p, c_q, c_r, c_s\} \subseteq C_3' - \{c_2, c_3\}$. By (6), $\{a_p, a_q, a_r, a_s\} \subseteq A_2 \subseteq E_2$ and by (8), $\{b_p, b_q, b_r, b_s\} \subseteq B_1 \subseteq E_1$. Thus, regardless of the order of labels b_p, b_q, b_r, b_s, the $K_{3,4}$ with bipartition $(\{B, A, C\}, \{b_p, b_q, b_r, b_s\})$ satisfies the hypothesis of Proposition 6(a)(ii) and hence has a $(6, f)$-ascent. This contradiction shows that

$$|C_3' - \{c_2, c_3\}| \leq 3. \tag{9}$$

A similar argument involving (5), (7) and Proposition 6(a)(ii) establishes

$$|C_2' - \{c_4, c_5\}| \leq 3. \tag{10}$$

Next suppose that there is equality in (9) and, without loss of generality, $C_3' - \{c_2, c_3\} = \{c_6, c_7, c_8\}$. By (6) and (8), $\{a_6, a_7, a_8\} \subseteq A_2$ and $\{b_6, b_7, b_8\} \subseteq B_1$. Let $p \in \{6, 7, 8\}$. In the $K_{3,4}$ with bipartition $(\{A, B, C\}, \{X_1, X_2, X_3, X_p\})$,

$$\{a_1, a_2, a_3\} \subseteq E_1, \quad \{c_2, c_3, c_p\} \subseteq E_3$$

and

$$a_1 = \min_{i=1}^{3}\{a_i\}.$$

Hence by the contrapositive of Proposition 7,

$$c_p < c_t \quad \text{where} \quad b_t = \max\{b_2, b_3\}.$$

Now without loss of generality, let $b_6 = \min\{b_6, b_7, b_6\}$. In the $K_{3,4}$ with bipartition $(\{B, A, C\}, \{X_6, X_7, X_8, X_t\})$ we have

$$\{b_6, b_7, b_8\} \subseteq E_1, \quad \{c_7, c_8, c_t\} \subseteq E_3,$$

$$b_6 = \min_{i=6}^{8}\{b_i\} \quad \text{and} \quad c_t = \max\{c_7, c_8, c_t\}.$$

By Proposition 7 there exists a $(6, f)$-ascent. This contradiction together with (9) shows that

$$|C_3' - \{c_2, c_3\}| \leq 2. \tag{11}$$

In order to establish the lemma, it remains to show that $|C'_3 - \{c_2, c_3\}| \geq 2$. Suppose to the contrary that (without loss of generality) $C'_3 \subseteq \{c_2, c_3, c_6\}$. By (10), $|C'_2| \leq 5$ and the argument used for A_1 shows that $|C_1| \leq 5$. Since $|C_1| + |C'_2| + |C'_3| = 12$, there are three possible triples for these quantities. These will be eliminated in the following two cases.

Case 1. $(|C_1|, |C'_2|, |C'_3|) = (4,5,3)$ or $(5,5,2)$

Suppose

$$\{c_7, ..., c_{10}\} \subseteq C_1 \text{ with } c_7 = \min_{i=7}^{10}\{c_i\} \text{ and } C'_2 = \{c_4, c_5, c_{11}, c_{12}, c_{13}\}.$$

From (7) we deduce that $\{b_{11}, b_{12}, b_{13}\} \subseteq B_1$ and without loss of generality we may assume that $b_{11} = \min_{i=11}^{13}\{b_i\}$. By (5), $\{a_{11}, a_{12}, a_{13}\} \subseteq A_3$. Observe that for $i \in \{8, 9, 10\}$,

$$a_i \in A_2 \text{ (respectively } A_3\text{) if and only if } b_i \in B_3 \text{ (respectively } B_2\text{)}. \quad (12)$$

Applying Proposition 6(a)(ii) to

$$(\{C, A, B\}, \{X_7, ..., X_{10}\}) \text{ and } (\{C, B, A\}, \{X_7, ..., X_{10}\}),$$

we deduce that

$$\{a_8, a_9, a_{10}\} \cap A_3 \neq \phi \text{ and } \{b_8, b_9, b_{10}\} \cap B_3 \neq \phi. \quad (13)$$

By (12) and (13),

$$\{b_8, b_9, b_{10}\} \cap B_2 \neq \phi \text{ and } \{a_8, a_9, a_{10}\} \cap A_2 \neq \phi. \quad (14)$$

The relations (12), (13) and (14) imply that the following two subcases are sufficient to complete Case 1.

Subcase 1(a) $A_2 \cap \{a_8, a_9, a_{10}\} = \{a_8\}$, $A_3 \cap \{a_8, a_9, a_{10}\} = \{a_9, a_{10}\}$, $B_2 \cap \{b_8, b_9, b_{10}\} = \{b_9, b_{10}\}$, $B_3 \cap \{b_8, b_9, b_{10}\} = \{b_8\}$.

Let $c_t = \max\{c_{12}, c_{13}\}$ and for each $p \in \{9, 10\}$, apply Proposition 7 to $(\{B, C, A\}, \{X_{11}, X_{12}, X_{13}, X_p\})$. To avoid the existence of a $(6, f)$-ascent, we have $a_p < a_t$. Now apply Proposition 7 to $(\{C, B, A\}, \{X_7, X_9, X_{10}, X_t\})$. There exists a $(6, f)$-ascent and this contradiction completes Subcase 1(a).

Subcase 1(b) $A_2 \cap \{a_8, a_9, a_{10}\} = \{a_9, a_{10}\}$, $A_3 \cap \{a_8, a_9, a_{10}\} = \{a_8\}$, $B_2 \cap \{b_8, b_9, b_{10}\} = \{b_8\}$, $B_3 \cap \{b_8, b_9, b_{10}\} = \{b_9, b_{10}\}$.

Let $a_t = \max\{a_9, a_{10}\}$ and for each $p \in \{4, 5\}$, apply Proposition 7 to $(\{C, A, B\}, \{X_7, X_9, X_{10}, X_p\})$. To avoid the $(6, f)$-ascent, $b_p < b_t$. But then Proposition 7 applied to $(\{A, C, B\}, \{X_1, X_4, X_5, X_t\})$ ensures the existence of a $(6, f)$-ascent. This contradiction completes Case 1.

Case 2. $(|C_1|, |C_2'|, |C_3'|) = (5, 4, 3)$

Without losing generality, let $C_1 = \{c_8, ..., c_{12}\}$ where $c_8 = \min_{i=8}^{12}\{c_i\}$, $C_2' = \{c_4, ..., c_7\}$ and $C_3' = \{c_2, c_3, c_{13}\}$. Then by (5) - (8),

$$\{a_6, a_7\} \subseteq A_3, \ a_{13} \in A_2 \text{ and } \{b_6, b_7, b_{13}\} \subseteq B_1.$$

Applications of Proposition 6(a)(ii) to $(\{C, B, A\}, \{X_8, X_i, X_j, X_k\})$ and $(\{C, A, B\}, \{X_8, X_i, X_j, X_k\})$, where $\{i, j, k\} \subseteq \{9, ..., 12\}$, establish that

$$|A_2 \cap \{a_9, ..., a_{12}\}| = |B_2 \cap \{b_9, ..., b_{12}\}| = 2,$$

and without losing generality we may assume that

$$a_9, a_{10} \in A_2, \quad a_{11}, a_{12} \in A_3, \quad b_{11}, b_{12} \in B_2, \quad b_9, b_{10} \in B_3.$$

Let $c_t = \max\{c_4, c_5\}$. For each $p \in \{9, 10\}$, to avoid the 6-ascent implied by Proposition 7 applied to $(\{A, C, B\}, \{X_1, X_4, X_5, X_p\})$, we obtain $b_p < b_t$. Now Proposition 7 applied to $(\{C, A, B\}, \{X_8, X_9, X_{10}, X_t\})$ gives a 6-ascent and this contradiction completes the proof of Lemma 8.1. □

By symmetry we also have the following lemma, where for $j = 2, 3$, $B_j' = B_j - \{b_1\}$.

Lemma 8.2 $|B_3' - \{b_4, b_5\}| = 2$.

Using Lemmas 8.1 and 8.2 we assume without loss of generality that $C_3' = \{c_2, c_3, c_8, c_9\}$ and $B_3' = \{b_4, b_5, b_6, b_7\}$. There is exactly one edge of each of the sets E_1, E_2 and E_3 incident with each X_i, and $A_1 = \{a_1, ..., a_5\}$. Hence

$$A_2 = \{a_6, ..., a_9\}, \quad A_3 = \{a_{10}, ..., a_{13}\}, \quad \{b_8, b_9\} \subseteq B_1 \text{ and } \{c_6, c_7\} \subseteq C_1.$$

Without loss of generality assume that $a_{13} = \max_{i=10}^{13}\{a_i\}$. By Proposition 6(b)$(ii)$ applied to $(\{A, B, C\}, \{X_{13}, X_{12}, X_{11}, X_{10}\})$,

$$B_1 \cap \{b_{10}, b_{11}, b_{12}\} \neq \phi \text{ and } C_1 \cap \{c_{10}, c_{11}, c_{12}\} \neq \phi.$$

Hence

$$C_2 \cap \{c_{10}, c_{11}, c_{12}\} \neq \phi \text{ and } B_2 \cap \{b_{10}, b_{11}, b_{12}\} \neq \phi.$$

There are two remaining cases to consider.

Case 1. $b_{10} \in B_1$ and $\{b_{11}, b_{12}\} \subseteq B_2$

Then $c_{10} \in C_2$ and $\{c_{11}, c_{12}\} \subseteq C_1$, hence $C_1 = \{c_6, c_7, c_{11}, c_{12}\}$. Let $c_q = \min\{c_6, c_7, c_{11}, c_{12}\}$. If $q = 6$ or 7, then Proposition 7 applied to $(\{C, B, A\}, \{X_q, X_{11}, X_{12}, X_{13}\})$ forces a 6-ascent. Hence without loss of generality $q = 11$. Now for each $p \in \{4, 5\}$, we apply Proposition 7 to $(\{C, A, B\}, \{X_{11}, X_6, X_7, X_p\})$. To avoid 6-ascents, $\max_{i=4}^{7}\{b_i\} = b_t$, where $t \in \{6, 7\}$. However, Proposition 7 applied to

$$(\{A, B, C\}, \{X_1, X_4, X_5, X_t\})$$

ensures the existence of a 6-ascent, which contradiction concludes Case 1.

Case 2. $\{b_{10}, b_{11}\} \subseteq B_1$ and $b_{12} \in B_2$

Then $\{c_{10}, c_{11}\} \subseteq C_2$, $c_{12} \in C_1$ and $B_1 = \{b_8, ..., b_{11}\}$. Let $b_q = \min_{i=8}^{11}\{b_i\}$. If $q = 8$ or 9, then by Proposition 7 applied to

$$(\{B, C, A\}, \{X_q, X_{10}, X_{11}, X_{13}\})$$

there is a 6-ascent. Hence without loss of generality $q = 10$. Now for each $t \in \{2, 3\}$ we apply Proposition 7 to $(\{B, A, C\}, \{X_{10}, X_8, X_9, X_t\})$. To avoid 6-ascents, $\max_{C_3'}\{c_i\} = c_q$, where $q \in \{8, 9\}$. However, Proposition 7 applied to $(\{A, B, C\}, \{X_1, X_2, X_3, X_q\})$ gives a 6-ascent. This final contradiction completes Case 2 and the proof of the theorem. ■

Observe that any path of length six, and thus 6-ascent, in $K_{3,13}$ necessarily starts at a vertex of degree three. The final result in this section extends and improves results in Corollary 4(i) and (ii).

Corollary 9 (i) $\alpha(K_{4,9}) \geq 5$ *and* $\alpha(K_{4,53}) \geq 8$.
(ii) *If* $m \geq 3$, *then* $\theta(m) \leq \lfloor (e - \frac{1}{2})m! \rfloor$.

Proof. (i) By Proposition 5(i), $K_{3,2}$ has a $(3, f)$-ascent starting at a vertex of degree three for any edge-ordering f, hence by Corollary 3(i), $\alpha(K_{4,9}) \geq 5$. Similarly, using 6-ascents in $K_{3,13}$ (Theorem 8), we get $\alpha(K_{4,53}) \geq 8$.
(ii) Substituting $m = 3$, $n = 13$ and $k = 6$ in Corollary 3(ii), we get

$$\begin{aligned} p &= [\frac{13}{3!} + \frac{1}{(3+1)!} + \cdots + \frac{1}{(3+q)!}](3+q)! \\ &= [2 + \frac{1}{3!} + \frac{1}{(3+1)!} + \cdots + \frac{1}{(3+q)!}](3+q)!, \\ &< (e - \tfrac{1}{2})(3 + q!) \end{aligned}$$

and so $p \leq \lfloor (e - \frac{1}{2})(3+q)! \rfloor$. Therefore, if $n \geq \lfloor (e - \frac{1}{2})(3+q)! \rfloor$, then $\alpha(K_{3+q,n}) \geq 6 + 2q$. Substituting $q + 3 = m$, we obtain that if $n \geq \lfloor (e - \frac{1}{2})(m!) \rfloor$, then $\alpha(K_{m,n}) \geq 2m$ and the result follows. ■

3 Upper bounds

Altitude upper bounds will be established using the following methods, which were also exploited in [4, 6, 8, 9].

Let $\mathbf{P} = (E_1, ..., E_t)$ be an ordered partition of the edge set E of G and let f be any edge-ordering of G satisfying

$$e_i \in E_i \text{ and } e_j \in E_j, \text{ where } i < j, \text{ implies } f(e_i) < f(e_j).$$

Such an edge-ordering is called **P**-*consistent.*

For $i = 1, ..., t$ we use the abbreviations $f_i = f|E_i$ and $G_i = G[E_i]$ (the subgraph of G induced by E_i); note that f_i is an edge-ordering of G_i.

Let f be **P**-consistent. In the edge-sequence X of any f-ascent of G, for each $i < j$, edges in E_i precede edges in E_j. Hence $X = X_1, ..., X_t$, where X_i (possibly empty) is an f_i-ascent of G_i.

Proposition 10 [3] *For any graph G, $\alpha(G) \leq \sum_{i=1}^{t} \alpha(G_i)$.*

Judicious choice of the ordered partition **P** and the **P**-consistent edge-ordering f often enables us to improve the upper bound of Proposition 10. Such a choice may allow us to find consecutive sets $E_j, ..., E_k$ so that the maximum length of an ascent in $f|(E_j \cup ... \cup E_k)$ is equal to $\Sigma_{i=j}^{k}\alpha(G_i) - c$ for some $c > 0$, in which case it is easily seen that the bound may be improved to $\Sigma_{i=1}^{t}\alpha(G_i) - c$. Situations of this type involving just two consecutive sets E_i, E_{i+1} of the partition include:

(*i*) G_i and G_{i+1} are vertex disjoint. In this case no edge of E_{i+1} may follow an edge of E_i in an f-ascent λ. Hence λ (considered as an edge set) satisfies $\lambda \cap E_i = \phi$ or $\lambda \cap E_{i+1} = \phi$, and the upper bound may be decreased by $\min\{\alpha_i, \alpha_{i+1}\}$.

(*ii*) Property (*i*) does not hold, but there is no vertex which is both the terminal vertex of an (α_i, f_i)-ascent in G_i and the initial vertex of an (α_{i+1}, f_{i+1})-ascent in G_{i+1}.

(*iii*) Properties (*i*) and (*ii*) do not hold. However, paths which negate Property (*ii*) have more than one common vertex.

Theorem 11 (*i*) $\alpha(K_{3,9}) \leq 4$
(*ii*) $\alpha(K_{3,12}) \leq 5$.

Proof. (*i*) Consider the ordered partition $\mathbf{P} = (E_1, E_2, E_3)$ of $E(K_{3,9})$ with $G_i = K_{3,9}[E_i] \cong 3K_{1,3}$ for each i, as shown in Figure 1, where we use the same labelling as defined in Section 2 and where the thinnest edges are in E_1 and the thickest in E_3. Note that any edge-ordering of $K_{1,3}$ has height two and so any edge-ordering of G_i, $i = 1, 2, 3$, has height two.

Let f with labels as in Figure 1 be an edge-ordering of $K_{3,9}$ such that

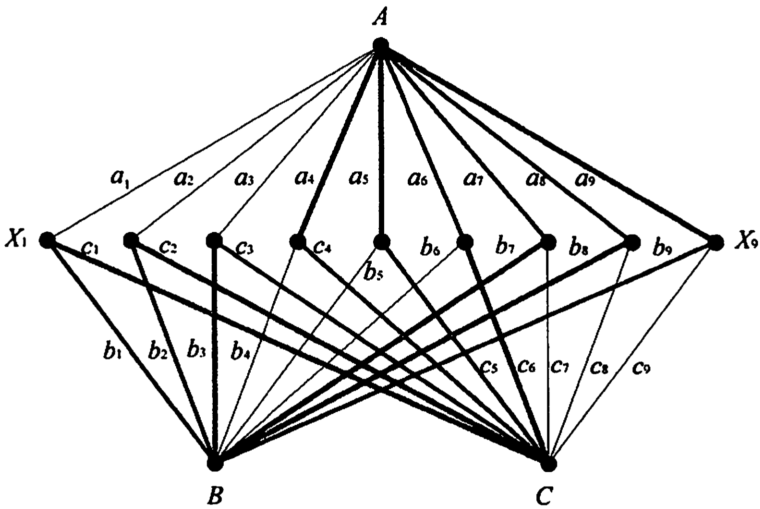

Figure 1: An edge-ordering of $K_{3,9}$ of height four

- f is **P**-consistent,
- in G_1, $a_1 < a_2 < a_3$, $b_4 < b_5 < b_6$ and $c_7 < c_8 < c_9$,
- in G_2, $a_7 < a_8$, $b_1 < b_2$ and $c_4 < c_5$, and
- in G_3, $a_9 < a_5$, $b_3 < b_8$ and $c_6 < c_2$.

Suppose that $\lambda = (uvxyz)$ is a $(5, f)$-ascent. Then λ contains at least one and at most two edges in each E_i, hence $u \in E_1$, $x \in E_2$ and $z \in E_3$. We consider edges in E_1 incident with A; the proofs in the cases where λ starts with edges in E_1 incident with B or C follow by symmetry.

Suppose $v \in E_1$. Then any $(3, f)$-ascent (uvx) terminates at vertex B or C and so can be extended to no more than a $(4, f)$-ascent $(uvxy)$ if $y \in E_3$. Hence $y \in E_2$. The only possible $(4, f)$-ascents $(uvxy)$ with $u, v \in E_1$ incident with A and $x, y \in E_2$ are $(a_1a_2b_2b_9)$ and (ua_3c_3y) where $u \in \{a_1, a_2\}$ and $y \in \{c_4, c_5\}$. In each case the addition of the unique edge $z \in E_3$ adjacent to y forms a 4-cycle $vxyz$ and thus $(uvxy)$ does not extend to a $(5, f)$-ascent.

We therefore conclude that $v \in E_2$ and so $y \in E_3$. Note that any $(4, f)$-ascent $(uvxy)$ with $u \in E_1$ incident with A and $v, x \in E_2$ starts at vertex A. With the properties of f mentioned above, the only such $(4, f)$-ascent is $(a_1b_1b_2c_2)$, which does not extend to a $(5, f)$-ascent because $c_6 < c_2$ and c_1 is adjacent to b_1.

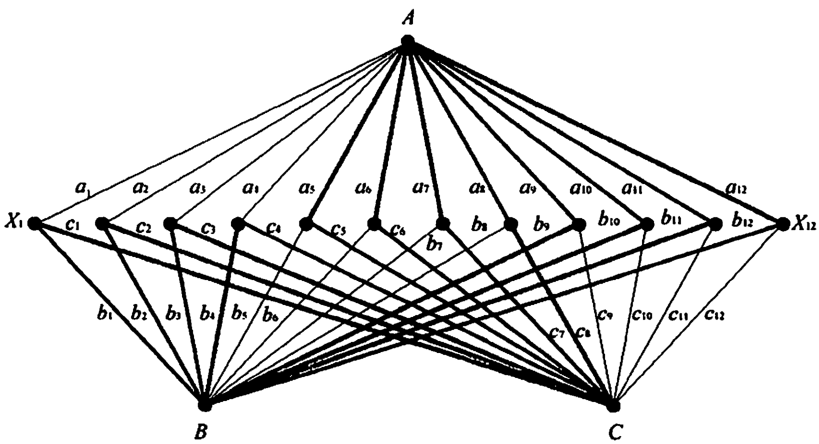

Figure 2: An edge-ordering of $K_{3,12}$ of height five

We have thus shown that there is no $(5, f)$-ascent in $K_{3,9}$, hence $h(f) \leq 4$ and the result follows.

(ii) Consider the ordered partition $\mathbf{P} = (E_1, E_2, E_3)$ of $E(K_{3,12})$ with $G_i = K_{3,12}[E_i] \cong 3K_{1,4}$ for each i, as shown in Figure 2. Note that any edge-ordering of $K_{1,4}$ has height two and so any edge-ordering of G_i, $i = 1, 2, 3$, has height two.

Let f with labels as in Figure 2 be an edge-ordering of $K_{3,12}$ such that

- f is $\mathbf{P}$-consistent,
- in G_1, $a_1 < \cdots < a_4$, $b_5 < \cdots < b_8$ and $c_9 < \cdots < c_{12}$,
- in G_2, $a_{11} < a_{10} < a_9$, $b_3 < b_2 < b_1$ and $c_7 < c_6 < c_5$, and
- in G_3, $a_{12} < \min\{a_5, a_6\}$, $b_4 < \min\{b_9, b_{10}\}$ and $c_8 < \min\{c_1, c_2\}$.

Suppose that $\lambda = (uvwxyz)$ is a $(6, f)$-ascent. Then λ contains exactly two edges in each E_i, hence λ starts at a vertex X_i (only X_1 and X_{12} are labelled), and $u, v \in E_1$, $w, x \in E_2$ and $y, z \in E_3$.

First consider the edges of E_1 incident with A. The only $(4, f)$-ascents $(uvxy)$ beginning with two of these edges are

$$(a_1a_3b_3b_2),\ (a_1a_3b_3b_{12}),\ (a_2a_3b_3b_1),\ (a_2a_3b_3b_{12})$$

and

$$(a_ia_4c_4c_j),\ \text{where } i = 1, 2, 3 \text{ and } j = 5, 6, 7.$$

Since $c_8 < \min\{c_1, c_2\}$, $(a_1a_3b_3b_2)$ and $(a_2a_3b_3b_1)$ cannot be extended to $(6, f)$-ascents, since a_{12} is incident with A, $(a_1a_3b_3b_{12})$ and $(a_2a_3b_3b_{12})$ cannot be extended to $(5, f)$-ascents, and since a_j, $j = 5, 6, 7$ is incident with A, $(a_ia_4c_4c_j)$ also cannot be extended to a $(5, f)$-ascent. By symmetry there are no $(6, f)$-ascents beginning with edges in E_1 incident with B or C and the result follows. ■

Combining the upper bound (2) in the introduction, Proposition 1, Corollary 4(i) and Theorems 8 and 11, and using the simple observation that if H is a subgraph of G, then $\alpha(H) \leq \alpha(G)$, we have therefore completed the evaluation of $\alpha(K_{3,n})$ for all $n \geq 3$, summarised as

$$\alpha(K_{3,n}) = \begin{cases} 3 & \text{if } 3 \leq n \leq 4 \\ 4 & \text{if } 5 \leq n \leq 9 \\ 5 & \text{if } 10 \leq n \leq 12 \\ 6 & \text{if } n \geq 13. \end{cases}$$

We also have

Corollary 12 $\theta(3) = 13$.

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