Eternal Security in Graphs

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ABSTRACT. Consider placing a guard on each vertex of a dominating set S_0 of a graph. If for every vertex $v \notin S_0$, there is a corresponding guard at an adjacent vertex u for which the resulting set $S_1 = S_0 - \{u\} \cup \{v\}$ is dominating, then we say that S_0 is *1-secure*. It is *eternally 1-secure* if for any sequence $v_1, v_2, \dots, v_k$ of vertices, there exists a sequence of guards $u_1, u_2, \dots, u_k$ with $u_i \in S_{i-1}$ and u_i equal to or adjacent to v_i, such that each set $S_i = S_{i-1} - \{u_i\} \cup \{v_i\}$ is dominating. We investigate the minimum cardinality of an eternally secure set. In particular, we refute a conjecture of Burger et al. We also investigate *eternal m-security*, in which all guards can move simultaneously.

1 Introduction

A dominating set in a graph can be thought of as a "secure set": for example, surveillance cameras that monitor every room in a museum, or troops that guard every intersection. The cameras are fixed and permanent, but the guards might be required to respond to an attack by moving there. However, since this response could leave some location unmonitored, one might need extra guards to respond to a further attack. This is the idea behind several recent generalizations of domination, such as Roman domination [4, 12, 13, 14, 15], weak Roman domination [5, 9], and secure domination [5, 11].

A more general problem is to cope with an arbitrary sequence of attacks. This idea was first considered by Burger et al. [2, 3]. We informally define an ***eternally secure set*** of a graph as a placement of guards that can respond to any sequence of attacks. In this paper we assume that each attack is at a single vertex.

We focus on two versions of the eternal security problem. In the first version, which we call ***1-security***, only one guard moves in response to an attack; in the second, which we call ***m-security***, all guards can move in response to an attack. The first version was introduced by Burger et

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al. [2, 3], though being able to withstand two attacks with a single-guard movement was explored in [5, 6, 10, 11, 12]. On the other hand, the idea that all guards may move in response to an attack appears to have been considered only in [12].

We define an ***eternal 1-secure set*** of a graph $G = (V, E)$ as a set $S_0 \subseteq V$ that can defend against any sequence of single-vertex attacks by means of single-guard shifts along edges of G. That is, for any k and any sequence $v_1, v_2, \ldots, v_k$ of vertices, there exists a sequence of guards $u_1, u_2, \ldots, u_k$ with $u_i \in S_{i-1}$ and either $u_i = v_i$ or $u_i v_i \in E$, such that each set $S_i = S_{i-1} - \{u_i\} \cup \{v_i\}$ is dominating. It follows that each S_i can be chosen to be an eternal 1-secure set. We define the ***eternal 1-security number***, denoted $\sigma_1(G)$, as the minimum cardinality of an eternal 1-secure set. This parameter was introduced by Burger et al. [3] using the notation γ_∞.

In order to reduce the number of guards needed for eternal security, we consider allowing more guards to move. Suppose that in responding to each attack, every guard may shift along an incident edge. We define the ***eternal m-security number*** $\sigma_m(G)$ as the minimum number of guards to handle an arbitrary sequence of single attacks using multiple-guard shifts. A suitable placement of the guards is called an ***eternal m-secure set***.

One simple example is the star $K_{1,m}$ with m leaves. Here $\sigma_1(G) = m$ but $\sigma_m(G) = 2$. Obviously,

$$\sigma_m(G) \leq \sigma_1(G)$$

for all graphs G.

Several weaker variations of eternal security have been defined. Cockayne et al. [6] define the ***secure domination number***: this is like σ_1, except one only has to respond to every possible single attack and leave a dominating set. (A similar goal was suggested earlier by Ochmanek [11].) Burger et al. [2] extended this to ***smart k-secure domination number***, where one has to be able to respond to any sequence of k one-vertex attacks. See also the paper by Pagourtzis et al. [12].

In this paper we examine bounds on and values of the two parameters σ_1 and σ_m. For the first parameter, we provide some examples that refute a conjecture in an earlier version of [3]. For the second, we show that there are several lower and upper bounds, and explore when these are attained.

2 The Eternal 1-Security Number

2.1 Fundamental bounds

A fundamental lower bound for the eternal 1-security number is the independence number $\beta(G)$, and a fundamental upper bound is the clique cover number $\theta(G)$ (or equivalently the chromatic number of the complement of G). This was first observed in [3]. For completeness we include a proof.

Theorem 1 *[Burger et al.] For any graph G,*

$$\beta(G) \leq \sigma_1(G) \leq \theta(G).$$

Proof. Lower bound: consider a sequence of attacks at the vertices of a maximum independent set. Each attack requires a new guard.

Upper bound: partition the graph into a minimum number of cliques and assign one guard to each clique. Each guard can always respond to an attack on its clique. qed

The graphs for which $\beta(G) = \theta(G)$ include, by definition, the perfect graphs. The following graphs are all perfect and thus their eternal 1-security number is known:

Corollary 2 $\sigma_1(G) = \beta(G) = \theta(G)$ *for bipartite graphs, complete graphs, complete multipartite graphs, and the cartesian product of two complete graphs.*

Equality for trees was also observed by others. At one stage, Burger et al. conjectured equality in the upper bound: that is, that $\sigma_1(G) = \theta(G)$ for all graphs G. We provide below a counterexample. Indeed, $\theta(G)$ can be arbitrarily larger than $\sigma_1(G)$. Nevertheless, a much weaker implication of their conjecture seems true: namely, if $\sigma_1(G) = \beta(G)$ then $\beta(G) = \theta(G)$.

2.2 Small eternal 1-security numbers

Burger et al. [3] gave two examples of graphs that have $\beta(G) < \theta(G)$ but where $\sigma_1(G)$ is known (and equal to $\theta(G)$): the odd cycle and its complement.

Theorem 3 *[Burger et al.] For n odd,*
(a) $\sigma_1(C_n) = (n+1)/2$.
(b) $\sigma_1(\bar{C_n}) = 3$.

Using Theorem 3b, Burger et al. [3] showed that if $\theta(G) \leq 3$ then $\sigma_1(G) = \theta(G)$.

However, we show there exist many graphs G with $\sigma_1(G) < \theta(G) = 4$.

Theorem 4 *If* $\beta(G) = 2$, *then* $\sigma_1(G) \leq 3$.

Proof. Assume $\beta(G) = 2$. Define a set S of 3 vertices as *good* if the subgraph induced by S is not complete. We claim that any good set is an eternal 1-secure set.

Consider any good set S; say $S = \{x, y, z\}$ with vertices x and y non-adjacent. Suppose there is an attack at vertex a.

If z is adjacent to a, then one can move the guard at z to a and still have a good set. Otherwise, since $\{x, y\}$ is a maximum independent set, it is dominating and so one can move a guard from x or y to a, and the set remains good. qed

It is well-known that a triangle-free graph can have arbitrarily high chromatic number. Thus, by considering complements, a graph with independence number 2 can have arbitrarily high clique cover number. The Grötzsch graph M (shown in Figure 1) is the smallest triangle-free graph with chromatic number four: thus $\sigma_1(\bar{M}) = 3$ and $\theta(\bar{M}) = 4$. It seems likely that this is the smallest example of a graph with $\sigma_1(G) < \theta(G)$. It is unclear whether a similar result holds for graphs with larger independence number. For example, does there exist a constant s_3 such that $\beta(G) = 3$ implies $\sigma_1(G) \leq s_3$?

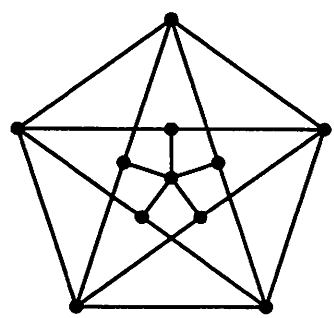

Figure 1: The Grötzsch graph M: $\sigma_1(\bar{M}) \neq \theta(\bar{M})$

There are other graphs where $\sigma_1(G) < \theta(G)$. One example is the (triangle-free) circulant $C_{18}[1, 3, 8]$, which has $\beta = 6$, $\sigma_1 = 8$ and $\theta = 9$. The graph with the biggest ratio σ_1/β that we know is the (triangle-free) circulant $C_{21}[1,3,8]$, which has $\beta = 6$ and $\sigma_1 = 10$. (All these calculations were performed by computer.)

2.3 Other examples

Burger et al. [3] also consider the torus: the cartesian product of two cycles. If one of the cycles is C_3, then $\beta(C_3 \square C_n) = \theta(C_3 \square C_n) = n$, so that $\sigma_1(C_3 \square C_n) = n$. If both cycles are even, then the torus is bipartite and hence perfect. They also provided general bounds for $\sigma_1(C_m \square C_n)$, but these are no better than their fundamental bounds (Theorem 1).

Burger et al. [3] conjectured that the eternal 1-security number of a torus is always its clique cover number. The evidence we have supports this conjecture. We observe equality in the clique-cover bound for two small examples.

Theorem 5 *$\sigma_1(C_m \square C_n) = \theta(C_m \square C_n) = \lceil mn/2 \rceil$ for $\{m,n\} = \{4,5\}$ and $\{5,5\}$.*

Proof. The proof is by computer search. We sketch the idea of the search.

Fix a graph G and integer k. Determine the set of all $\binom{|V|}{k}$ possible placements of k guards on V. Start by marking a placement as good if and only if it is dominating. Then repeat the following process:

> *Consider each good placement in turn. If there exists an attack such that for every response the resulting placement is bad, then mark the placement as bad.*

So a placement gets marked bad if the adversary can force the guards into a nondominating set. On the other hand, when this process stabilizes, any placement that is still marked good is an eternal 1-secure set: for every attack there exists a response such that the resulting placement is good.

Now, the computer program was run with $G = C_4 \square C_5$ and $k = 9$, and with $G = C_5 \square C_5$ and $k = 12$. In both cases, all placements were eventually marked as bad. This shows that at least $\lceil mn/2 \rceil$ guards are required. qed

A related graph is the ladder $C_n \square K_2$. When n is even, the ladder is bipartite and thus covered by Corollary 2. When n is odd, the ladder has $\beta = n - 1$ and $\theta = n$. It can be shown that the latter value is the eternal 1-security number. We omit the proof.

Theorem 6 *For all $n \geq 3$, $\sigma_1(C_n \square K_2) = \theta(C_n \square K_2) = n$.*

3 The Eternal m-Security Number

3.1 Fundamental bounds

A fundamental lower bound for the eternal m-security number is the domination number $\gamma(G)$. For an upper bound, we consider a variation of the clique cover number. Define the ***clique-star cover number*** as follows. Define a ***colonization*** as a partition of the vertex set into subgraphs each with a dominator (a vertex adjacent to all other nodes in the subgraph). The ***weight*** of a colonization counts 1 for each clique and 2 for each non-clique. Then $\theta_S(G)$ is the minimum weight of a colonization. For example, if the graph has maximum degree 2, then $\theta_S(G) = \theta(G)$; in general, $\theta_S(G) \leq \theta(G)$.

Theorem 7 *For any graph* G*,*

$$\gamma(G) \leq \sigma_m(G) \leq \theta_S(G).$$

Proof. Lower bound: An eternal m-secure set must be dominating.

Upper bound: Assign one guard to each clique in the colonization and two guards to each non-clique. A single guard patrols its clique, while in a non-clique one guard is always on the dominator. qed

The upper bound is related to Roman domination. Cockayne et al. [4] defined the ***Roman domination number*** as the minimum total weight of a function $f: V \to \{0,1,2\}$ such that every vertex u of weight 0 has a neighbor of weight 2. It is denoted $\gamma_R(G)$. Later, the third author and Henning [9] defined weak Roman domination. The ***weak Roman domination number*** $\gamma_r(G)$ is the minimum total weight of a function $f: V \to \{0,1,2\}$ such that for any vertex u of weight 0 that has no neighbor of weight 2, there exists a neighbor v of weight 1 such that $(f^{-1}(1) \cup f^{-1}(2) \cup \{u\}) \setminus \{v\}$ dominates. Essentially, one is allowed to station double guards at some vertices, and must be able to respond to a single attack (and still dominate).

It is clear that

$$\gamma_r \leq \theta_S \leq \gamma_R.$$

Surprisingly perhaps, there is no relationship between the weak Roman domination number and the eternal m-security number:

$$\sigma_m \text{ and } \gamma_r \text{ are incomparable}$$

See the results on cycles and odd paths below.

3.2 Calculations and lower bounds

Theorem 8 *a)* $\sigma_m(K_n) = 1$.
b) $\sigma_m(K_{r,s}) = 2$ *for* $r, s \geq 1$, $r + s \geq 3$.
c) $\sigma_m(P_n) = \theta_S(P_n) = \lceil n/2 \rceil$.
d) $\sigma_m(C_n) = \gamma(C_n) = \lceil n/3 \rceil$.

Proof. a) Immediate.

b) Place guards, one on each side of the bipartition. Respond to an attack by moving a guard to that vertex, and moving the other guard to any vertex on the opposite side.

c) Assume the vertices are numbered consecutively $0, 1, 2 \ldots$. Then consider a sequence of attacks at vertices $0, 2, 4, \ldots$. Each time a new guard is needed. Thus $\sigma_m(P_n) \geq \lceil n/2 \rceil$.

d) Place guards on a minimum dominating set. Then by shifting all guards either clockwise or counter-clockwise along one edge, one can eternally respond to any attack. qed

The cycles provide examples where $\sigma_m < \gamma_r$. The odd paths provide examples where $\sigma_m > \gamma_r$.

Theorem 8c can be generalized to the following bound.

Corollary 9 *For any graph* G, $\sigma_m(G) \geq (\text{diam}(G) + 1)/2$.

Theorem 8d can be generalized to other symmetric graphs such as the torus.

Theorem 10 *For any Cayley graph* G, $\sigma_m(G) = \gamma(G)$.

Proof. Recall that a Cayley graph G is defined by a group Γ and a subset D of the elements of Γ: the vertex set of G is the group elements, and two vertices u and v are adjacent if and only if $u = hv$ for some $h \in D$.

We claim that any dominating set S is an eternal m-secure set. For, suppose there is an attack at vertex u. Then there is a vertex $v \in S$ adjacent to u; that is, $u = hv$ for some $h \in D$. But then $hS = \{hs : s \in S\}$ is another dominating set (and reachable by a guard shift). qed

The result is probably true for any vertex-transitive graph.

3.3 More upper bounds

Recall that the ***2-domination number*** $\gamma_2(G)$ of a graph [7, 8] is the minimum cardinality of a set S such that every vertex not in S is adjacent to at least two members of S.

Theorem 11 *For any graph G, $\sigma_m(G) \leq \gamma_2(G)$.*

Proof. Start by placing guards on the vertices in a minimum 2-dominating set S—call the vertex a guard starts on its ***home***. For every attack on a vertex in $V - S$, send any adjacent guard to that vertex, and recall the guard used last time to its home. Since S is 2-dominating, $S - \{v\}$ dominates everything except possibly v, and so one can forever respond to attacks. qed

For example, consider the subdivision $S(G)$ of a graph $G = (V, E)$: since V is a 2-dominating set of $S(G)$, it follows that $\gamma_2(S(G)) \leq |V|$ (it can be shown that actually $\gamma_2(S(G)) = |V|$). Thus $\sigma_m(S(G)) \leq |V|$.

Corollary 12 *For $n \geq 4$ and $H_n = S(K_n)$, $\gamma(H_n) = n - 1$, $\sigma_m(H_n) = n$ and $\theta_S(H_n) = 2n - 3$.*

Proof. The domination number of H_n is given in [1]. We give only the proof of the lower bound on the eternal m-security number.

It can readily be shown that every minimum dominating set of H_n has the following form: $n - 2$ original vertices and the one subdivision vertex adjacent to the remaining two original vertices. So, suppose one tries $n - 1$ guards; then an attack on a subdivision vertex between two guards does not allow a dominating set to be maintained. qed

Though it is probably a weak bound, we next observe that the independence number is an upper bound on the eternal m-security number. And thus there is a clean separation between the eternal 1- and m-security numbers.

Theorem 13 *For any graph G, $\sigma_m(G) \leq \beta(G)$.*

Proof. By induction on the independence number. Clearly if $\beta(G) = 1$ then $\sigma_m(G) = 1$.

Consider a graph with independence number $k \geq 2$. If there is a vertex v such that $G' = G - N[v]$ has independence number at most $k - 2$ (where the null graph has independence number 0), then since $\sigma_m(G[N[v]]) \leq \theta_S(G[N[v]]) = 2$, and by induction $\sigma_m(G') \leq k - 2$, it follows that $\sigma_m(G) \leq \beta(G)$.

So assume there is no such vertex. That is, every vertex is in a maximum independent set. For each vertex v, pick a maximum independent set S_v. Place guards on a maximum independent set, and to respond to an attack at a vertex w, we will move guards to S_w.

This is possible since given two maximum independent sets S and T in a graph, there is a matching between $S-T$ and $T-S$. This follows from Hall's marriage theorem or the well-known result that a bipartite graph has independence number $n/2$ if and only if it has a perfect matching.

Hence one can always respond to an attack using $\beta(G)$ guards. qed

Equality holds for example for graphs with $\gamma(G) = \beta(G)$ (the coronas $H \circ K_1$).

Finally in this section, we note one can improve the fundamental upper bound. Recall that the ***connected domination number*** $\gamma_c(G)$ of a graph G is the minimum cardinality of a connected dominating set. Define a ***neocolonization*** as a partition of the vertex set $\{V_1, \ldots, V_t\}$ such that each V_i induces a connected subgraph. The weight of V_i is 1 if $G[V_i]$ is a clique, and $1 + \gamma_c(G[V_i])$ otherwise. Then define the ***clique-connected cover number*** $\theta_C(G)$ as the minimum weight of a neocolonization.

Theorem 14 *For any graph G, $\sigma_m(G) \leq \theta_C(G) \leq \gamma_c(G) + 1$.*

Proof. For each subgraph $G[V_i]$ of the neocolonization, we place the appropriate number of guards as follows. A clique receives one guard. For a non-clique, we choose a minimum connected dominating set D_i, and place guards on all of D_i and on any one other vertex (called the rover). We will maintain the property that there are always guards on D_i.

The guards on each subgraph are only responsible for attacks on that subgraph. To respond to an attack in a non-clique, consider a path P from the rover to the attack: this can be chosen such that all internal vertices are in D_i, since D_i is connected and dominating. Then shift each guard found on P one vertex along P. The net result is that the rover is now on the attack. qed

For example, it follows that $\sigma_m(T)$ for a tree T is at most one more than the number of non-leaf vertices. We do not know of a tree for which $\sigma_m(T) < \theta_C(T)$.

4 Summary of Parameters

Figure 2 gives a Hasse diagram with all the relationships between the various parameters discussed.

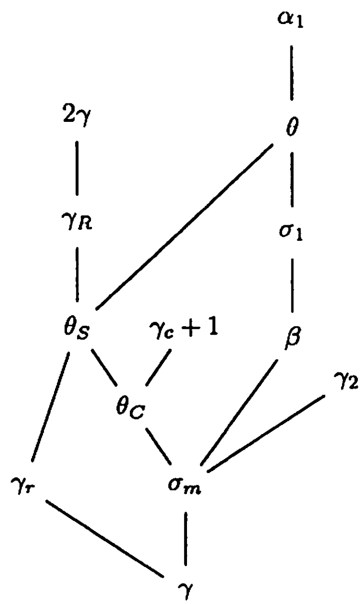

Figure 2: How the parameters compare

5 Open Questions

We conclude with some open questions, some of which have already been mentioned.

1. Is there a constant upper bound on the eternal 1-security number of graphs with independence number 3?

2. Is $\sigma_1(C_m \square C_n) = \lceil mn/2 \rceil$ for all $m, n \geq 4$?

3. Is $\sigma_m(G) = \gamma(G)$ for every vertex-transitive graph?

4. What is the complexity of the associated recognition problems? For example, how hard is it to tell whether a set is an eternal 1-secure or m-secure set? We expect such questions to lie within the first few levels of the polynomial hierarchy. And what is the complexity of the associated decision problems testing whether $\sigma_1(G) \leq k$ or $\sigma_m(G) \leq k$?

5. What about an algorithm for trees for $\sigma_m(T)$? Is $\sigma_m(T) = \theta_C(T)$ for every tree T?

6. Burger et al. [3] observed that there is no point in allowing multiple guards in the definition of eternal 1-security. For, if the double guards always remain together, their double-ness is of no use; but if they separate then the result must still be an eternal 1-secure set (since the adversary can ensure guards never rejoin). Is it the same story with eternal m-security? We conjecture so.

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