On the Cordiality of Elongated Plys

Samina Boxwala and N. B. Limaye[1]

Abstract: An elongated ply $T(n; t^{(1)}, t^{(2)}, \cdots, t^{(n)})$ is a snake of n number of plys $P_{t^{(i)}}(u_i, u_{i+1})$ where any two adjacent plys $P_{t^{(i)}}$ and $P_{t^{(i+1)}}$ have only the vertex u_{i+1} in common. That means the block cut vertex graph of T_n is thus a path of length $n-1$. In this paper, the cordiality of the Elongated Ply T_n is investigated.

1: Preliminaries

Let $V(G)$ and $E(G)$ denote the vertex set and edge set of a finite simple graph G. A labeling $f : V(G) \longrightarrow \{0, 1\}$, called a binary vertex labeling of G, induces a labeling $f : E(G) \longrightarrow \{0, 1\}$ given by $f(uv) = \mid f(u) - f(v) \mid$. By $v_f(0), v_f(1)$ we denote the number of vertices in G having labels 0 and 1 respectively under f. The numbers $e_f(0), e_f(1)$ denote the number of edges having labels 0 and 1 respectively under f.

Definition: For a graph G, by the **index of cordiality** $i(G)$, we mean $\min\{\mid e_f(0) - e_f(1) \mid\}$ where the minimum is taken over all binary labelings of G with $\mid v_f(0) - v_f(1) \mid \leq 1$.

A graph G is called a **cordial graph** if $i(G) \leq 1$, and a binary labeling f of G is called a **cordial labeling** if $\mid v_f(0) - v_f(1) \mid \leq 1$ and $\mid e_f(0) - e_f(1) \mid \leq 1$. In [1], Cahit proved the following:

Theorem 1: If G is an Eulerian graph with e edges, where $e \equiv 2(\bmod 4)$, then G has no cordial labeling.

A t-ply $P_t(u, v)$ is a graph with t paths, each of length at least two and such that no two paths have a vertex in common except for the end vertices u and v. In [3], the cordiality of t-plys was investigated. **An elongated ply** $T(n; t^{(1)}, t^{(2)}, \cdots, t^{(n)})$, often denoted by T_n, is a snake of n number of plys $P_{t^{(i)}}(u_i, u_{i+1})$ where any two adjacent plys $P_{t^{(i)}}$ and $P_{t^{(i+1)}}$ have only the vertex u_{i+1} in common. Each $t^{(i)}$-ply $P_{t^{(i)}}$ is a block of T_n. The block cut vertex graph of T_n is thus a path of length $n-1$. In this sequel of [3], we investigate cordiality of elongated plys.

Consider a t-ply $P_t(u, v)$ and let $P \equiv \{u, v_1, \cdots, v_n, v\}$ be a typical path with the end points u and v in $P_t(u, v)$. The length $l(P)$of this path is $n+1$. We say that the path P is of type i if $l(P) \equiv i(\text{mod}4)), i = 1, 2, 3, 4$. Denote by t_i, the number of paths of the type $i, i = 1, 2, 3, 4$. Then

$$t = t_1 + t_2 + t_3 + t_4 \cdots\cdots (\text{I}).$$

[1]This work was partially supported by the Department of Science and Technology, Government of India.

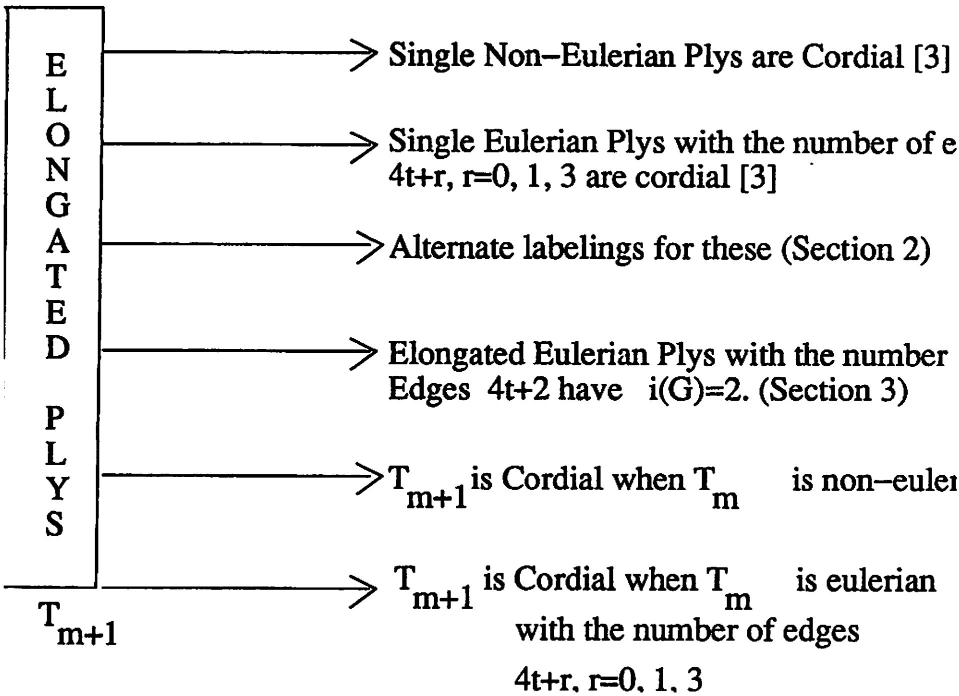
It follows clearly that if $e = | E(P_t(u,v)) |$, then

$$e \equiv (t_1 + 2\,t_2 + 3\,t_3)(\text{mod}4) \cdots\cdots (\text{II}).$$

Further, let $t_1 = 4\,s_1 + x_1, t_3 = 4\,s_3 + x_3, t_2 = 2\,s_2 + x_2, t_4 = 2\,s_4 + x_4$, $0 \leq x_1,\ x_3 \leq 3$ and $0 \leq x_2,\ x_4 \leq 1$. By (II), it follows that

$$e \equiv (x_1 + 2\,x_2 + 3\,x_3)(\text{mod}4) \cdots\cdots (\text{III}).$$

Following road map indicates the flow of the proof.



We now list the classification and all the labelings and the corresponding parameters found in the $t-ply$ paper [3], in a tabular form so that they can be referred to as and when required. This will be useful later, when we define some alternate labelings.

2: Classification:

The single t-ply graphs are classified into various types depending on whether they are Eulerian or Non-Eulerian and on whether $e \equiv 0, 1, 2$ or $3(\bmod 4)$. Accordingly, we have the following 8 types.

Type A_1. These are Eulerian with $e \equiv 0$ mod 4 and the quadruples (x_1, x_2, x_3, x_4) corresponding to them are $(0,0,0,0), (0,1,2,1), (1,0,1,0), (1,1,3,1), (2,0,2,0), (2,1,0,1), (3,0,3,0), (3,1,1,1)$.
Type A_2. These are Non-Eulerian with $e \equiv 1$ mod 4 and the quadruples (x_1, x_2, x_3, x_4) corresponding to them are $(0,0,3,0), (0,1,1,1), (1,0,0,0), (1,1,2,1), (2,0,1,0), (2,1,3,1), (3,0,2,0), (3,1,0,1)$.
Type A_3. These are Non-Eulerian with $e \equiv 3$ mod 4 and the quadruples (x_1, x_2, x_3, x_4) corresponding to them are $(0,0,1,0), (0,1,3,1), (1,0,2,0), (1,1,0,1), (2,0,3,0), (2,1,1,1), (3,0,0,0), (3,1,2,1)$.
Type A_4. These are Non-Eulerian with $e \equiv 2$ mod 4 and the quadruples (x_1, x_2, x_3, x_4) corresponding to them are $(0,0,2,1), (0,1,0,0), (1,0,3,0), (1,1,1,0), (2,0,0,1), (2,1,2,0), (3,0,1,1), (3,1,3,0)$.
Type A_5. These are Eulerian with $e \equiv 2$ mod 4 and the quadruples (x_1, x_2, x_3, x_4) corresponding to them are $(0,0,1,1), (0,1,3,0), (1,0,2,1), (1,1,0,0), (2,0,3,1), (2,1,1,0), (3,0,0,1), (3,1,2,0)$.
Type A_6. These are Eulerian with $e \equiv 2$ mod 4 and the quadruples (x_1, x_2, x_3, x_4) corresponding to them are $(0,0,3,1), (0,1,1,0), (1,0,0,1), (1,1,2,0), (2,0,1,1), (2,1,3,0), (3,0,2,1), (3,1,0,0)$.
Type B. These are Non-Eulerian with $e \equiv 2$ mod 4 and the quadruples (x_1, x_2, x_3, x_4) corresponding to them are $(0,0,0,1), (0,1,2,0), (1,0,1,1), (1,1,3,0), (2,0,2,1), (2,1,0,0), (3,0,3,1), (3,1,1,0)$.
Type C. These are Eulerian with $e \equiv 2$ mod 4 and the quadruples (x_1, x_2, x_3, x_4) corresponding to them are $(0,0,0,1), (0,1,2,0), (1,0,1,1), (1,1,3,0), (2,0,2,1), (2,1,0,0), (3,0,3,1), (3,1,1,0)$.

For Type B labeling f of type B were used. for which $v_f(0) + 1 = v_f(1), e_f(0) = e_f(1), f(u) = 0 = f(v)$. By Theorem 1, plys of type C are not cordial. Hence, neither labelings of type A nor of type B were used for these graphs. The cordial labelings in the remaining cases are done in two stages. For a typical path $u, v_1, \cdots, v_n, v$ we write $n = 4q + 1, 1 \leq r \leq 4$. In the first stage for type A labels $f(u), f(v)$ are $1, 0$ respectively. On each path the middle vertices are labeled repeatedly $0, 0, 1, 1$, in all q times. If $v_f^{"}(i), e^{"}(i), i = 0, 1$, are the number of vertices and the number of edges being labeled i in the second stage, then $| v_f(0) - v_f(1) | = | v^{"}(0) - v_f^{"}(1) |$ and $| e_f(0) - e_f(1) | = | e_f^{"}(0) - e_f^{"}(1) |$. Similarly, in the first stage for type B labels $f(u), f(v)$ are both 0. On each path the middle vertices are labeled repeatedly $1, 1, 0, 0$, in all q times. Clearly for these labelings $| v_f(0) - v_f(1) | = 2 + | v_f^{"}(0) - v_f^{"}(1) |$ and $| e_f(0) - e_f(1) | = | e_f^{"}(0) - e_f^{"}(1) |$.
We now tabulate all the labelings according to the values of $x_1, \cdots, x_4$.

Compendium Of Labelings

The various labelings used in stage 2 earlier, are required to be supplemented by additional labelings which will be used in the sequel. We present the labelings of [2] together in the form of a compendium given below:

Labelings of type A

$f(u) = 1; f(v) = 0$

1. $x_1 = 1$

f	$f(v_{n-3})$	$f(v_{n-2})$	$f(v_{n-1})$	$f(v_n)$	$v_f''(0)$	$v_f''(1)$	$e_f''(0)$	$e_f''(1)$
a_{11}	0	0	1	1	2	2	2	3
a_{12}	1	1	0	0	2	2	4	1
a_{13}	1	0	0	0	3	1	4	1
a_{14}	1	1	1	0	1	3	4	1
a_{15}	0	0	0	1	3	1	2	3
a_{16}	0	1	1	1	1	3	2	3

2. $x_1 = 2$

f	Path	$f(v_{n-3})$	$f(v_{n-2})$	$f(v_{n-1})$	$f(v_n)$	$v_f''(0)$	$v_f''(1)$	$e_f''(0)$	$e_f''(1)$
a_{21}	1	0	0	1	1	4	4	4	6
	2	0	0	1	1				
a_{22}	1	0	0	1	1	4	4	6	4
	2	1	1	0	0				
a_{23}	1	0	0	1	1	5	3	6	4
	2	1	0	0	0				
a_{24}	1	0	1	0	1	5	3	4	6
	2	1	0	0	0				
a_{25}	1	0	1	1	1	3	5	4	6
	2	0	0	1	1				
a_{26}	1	0	1	1	1	3	5	6	4
	2	1	1	0	0				

3. $x_1 = 3$

f	Path	$f(v_{n-3})$	$f(v_{n-2})$	$f(v_{n-1})$	$f(v_n)$	$v_f''(0)$	$v_f''(1)$	$e_f''(0)$	$e_f''(1)$
a_{31}	1 2 3	0 0 0	0 0 0	1 1 1	1 1 1	6	6	6	9
a_{32}	1 2 3	0 0 1	0 0 1	1 1 0	1 1 0	6	6	8	7
a_{33}	1 2 3	0 0 1	0 0 1	1 1 1	1 1 0	5	7	8	7

4. $x_2 = 1$

f	$f(v_n)$	$v_f''(0)$	$v_f''(1)$	$e_f''(0)$	$e_f''(1)$
b_{11}	0	1	0	1	1
b_{12}	1	0	1	1	1

5. $x_3 = 1$

f	$f(v_{n-1})$	$f(v_n)$	$v_f''(0)$	$v_f''(1)$	$e_f''(0)$	$e_f''(1)$
c_{11}	1	0	1	1	2	1
c_{12}	0	1	1	1	0	3
c_{13}	1	1	0	2	2	1
c_{14}	0	0	2	0	2	1

6. $x_3 = 2$

f	Path	$f(v_{n-1})$	$f(v_n)$	$v_f''(0)$	$v_f''(1)$	$e_f''(0)$	$e_f''(1)$
c_{21}	1 2	1 1	0 0	2	2	4	2
c_{22}	1 2	1 0	0 1	2	2	2	4
c_{23}	1 2	1 1	1 0	1	3	4	2
c_{24}	1 2	1 0	0 0	3	1	4	2
c_{25}	1 2	0 0	1 0	3	1	2	4
c_{26}	1 2	1 0	1 1	1	3	2	4

7. $x_3 = 3$

f	Path	$f(v_{n-1})$	$f(v_n)$	$v_f''(0)$	$v_f''(1)$	$e_f''(0)$	$e_f''(1)$
c_{31}	1 2 3	1 1 0	0 0 1	3	3	4	5
c_{32}	1 2 3	1 1 1	0 0 0	3	3	6	3
c_{33}	1 2 3	1 1 1	1 0 0	2	4	4	5
c_{34}	1 2 3	1 1 1	0 0 1	2	4	6	3

8. $x_4 = 1$

f	$f(v_{n-2})$	$f(v_{n-1})$	$f(v_n)$	$v_f''(0)$	$v_f''(1)$	$e_f''(0)$	$e_f''(1)$
d_{11}	1	0	1	1	2	1	3
d_{12}	1	1	0	1	2	3	1
d_{13}	1	1	1	0	3	3	1
d_{14}	1	0	0	2	1	3	1
d_{15}	0	1	0	2	1	1	3

Labelings of type B

f(u)=0, f(v)=0

1. $x_1 = 1$

f	$f(v_{n-3})$	$f(v_{n-2})$	$f(v_{n-1})$	$f(v_n)$	$v_f''(0)$	$v_f''(1)$	$e_f''(0)$	$e_f''(1)$
j_{11}	1	1	0	0	2	2	3	2
j_{12}	1	1	1	0	1	3	3	2
j_{13}	1	0	0	0	3	1	3	2

2. $x_1 = 2$

	Path	$f(v_{n-3})$	$f(v_{n-2})$	$f(v_{n-1})$	$f(v_n)$	$v_f''(0)$	$v_f''(1)$	$e_f''(0)$	$e_f''(1)$
j_{21}	1	1	1	0	0	4	4	6	4
	2	1	1	0	0				
j_{22}	1	1	1	1	1	3	5	6	4
	2	0	0	0	1				
j_{23}	1	1	1	1	1	2	6	6	4
	2	1	1	0	0				
j_{24}	1	1	0	0	1	4	6	2	6
	2	1	1	1	1				
j_{25}	1	1	1	0	0	4	4	4	6
	2	1	0	0	1				

3. $x_1 = 3$

f	Path	$f(v_{n-3})$	$f(v_{n-2})$	$f(v_{n-1})$	$f(v_n)$	$v_f''(0)$	$v_f''(1)$	$e_f''(0)$	$e_f''(1)$
j_{31}	1	1	1	0	0	6	6	9	6
	2	1	1	0	0				
	3	1	1	0	0				
j_{32}	1	1	1	1	0	5	7	7	8
	2	1	1	0	0				
	3	1	0	0	1				
j_{33}	1	1	1	1	1	4	8	9	6
	2	1	1	0	0				
	3	1	1	0	0				
j_{34}	1	1	1	1	1	4	8	7	8
	2	1	1	0	0				
	3	1	0	0	1				
j_{35}	1	1	1	0	0	6	6	7	8
	2	1	1	0	0				
	3	1	0	0	1				
j_{36}	1	1	0	0	0	7	5	9	6
	2	1	1	0	0				
	3	1	0	0	0				

4. $x_2 = 1$

f	$f(v_n)$	$v_f''(0)$	$v_f''(1)$	$e_f''(0)$	$e_f''(1)$
k_{11}	1	0	1	0	2
k_{12}	0	1	0	2	0

5. $x_3 = 1$

f	$f(v_{n-1})$	$f(v_n)$	$v_f''(0)$	$v_f''(1)$	$e_f''(0)$	$e_f''(1)$
l_{11}	1	1	0	2	1	2
l_{12}	1	0	1	1	1	2
l_{13}	0	0	2	0	3	0

6. $x_3 = 2$

f	Path	$f(v_{n-1})$	$f(v_n)$	$v_f''(0)$	$v_f''(1)$	$e_f''(0)$	$e_f''(1)$
l_{21}	1	1	1	1	3	2	4
	2	1	0				
l_{22}	1	1	1	0	4	2	4
	2	1	1				
l_{23}	1	1	1	2	2	4	2
	2	0	0				
l_{24}	1	1	0	2	2	2	4
	2	1	0				

7. $x_3 = 3$

f	Path	$f(v_{n-1})$	$f(v_n)$	$v_f''(0)$	$v_f''(1)$	$e_f''(0)$	$e_f''(1)$
l_{31}	1	0	0	3	3	5	4
	2	1	1				
	3	1	0				
l_{32}	1	1	1	2	4	3	6
	2	1	0				
	3	1	0				
l_{33}	1	1	1	1	5	3	6
	2	1	1				
	3	1	0				
l_{34}	1	1	1	2	4	5	4
	2	1	1				
	3	0	0				

8. $x_4 = 1$

f	$f(v_{n-2})$	$f(v_{n-1})$	$f(v_n)$	$v_f''(0)$	$v_f''(1)$	$e_f''(0)$	$e_f''(1)$
m_{11}	1	1	0	1	2	2	2
m_{12}	1	1	1	0	3	2	2
m_{13}	1	0	0	2	1	2	2

Let f be a binary labeling of a graph G. A labeling $\hat{f}$ such that $\hat{f}(v) = 1$, if $f(v) = 0$ and $\hat{f}(v) = 0$, if $f(v) = 1$ for each $v \in V(G)$, is called the dual labeling of f. Note that $v_{\bar{f}}(0) = v_f(1), v_{\bar{f}}(1) = v_f(0), e_{\bar{f}}(0) = e_f(0)$ and $e_{\bar{f}}(1) = e_f(1)$. Let f be a cordial labeling of a single t-ply graph $P_t(u, v,)$. Let $\tilde{f}$ be a binary labeling of $P_t(u, v)$ such that $\tilde{f}(u) = f(v), \tilde{f}(v) = f(u)$. Also, on any path, $u, v_1, v_2, \cdots, v_n, v$ in $P_t(u, v)$ let $\tilde{f}(v_i) = f(v_{n-i+1})$, that is, $\tilde{f}(v_1) = f(v_n)\tilde{f}(v_2) = f(v_{n-1})$ and so on. We call $\tilde{f}$ the **inversion labeling** of f. Then, $v_{\tilde{f}}(0) = v_f(0), v_{\tilde{f}}(1) = v_f(1)$; $e_{\tilde{f}}(0) = e_f(0), e_{\tilde{f}}(1) = e_f(1)$. Literally speaking, $\tilde{f}$ is a lateral inversion of f. This inversion labeling f will be used in those cases where we need to interchange the labels of the vertices u,v without disturbing the vertex and edge label conditions.

Using the dual labeling and inversion labeling as defined above, we obtain some more labelings for the t-ply graphs classified in the earlier paper[3].

Recall the labelings used in [3] for various types of $t-$ply graphs. For ease of reference, let $\alpha_1, \beta_1, \gamma_1, \delta_1, \theta_1, \phi_1, \mu_1$, denote the binary labelings given for $t-$ply graphs of type $A_1, A_2, A_3, A_4, A_5, A_6, B$ respectively in [3]. We use these notations consistently for them, throughout this paper.

The labelings mentioned above prove inadequate for the task ahead. It is therefore necessary to introduce some alternate labelings for each type of graph which we now set out to do. These labelings are not necessarily cordial. In most cases, unless otherwise mentioned, the labeling in Stage 1 remains as in the previous paper[3]. As before, we mention only the choice of the labelings used in Stage 2 in the following tables. Also the type of labeling used is mentioned alongside.

Alternate labeling for $t-$ply graph of Type A_1:
Here we use the labeling of type A.

x_1	x_2	x_3	x_4	$v''_f(0)$	$v''_f(1)$	$e''_f(0)$	$e''_f(1)$
0	0	0	0	0	0	0	0
0	$1(b_{11})$	$2(c_{23})$	$1(d_{11})$	3	5	6	6
$1(a_{16})$	0	$1(c_{11})$	0	2	4	4	4
$1(a_{16})$	$1(b_{11})$	$3(c_{31})$	$1(d_{12})$	6	8	10	10
$2(a_{25})$	0	$2(c_{21})$	0	5	7	8	8
$2(a_{25})$	$1(b_{11})$	0	$1(d_{12})$	5	7	8	8
$3(a_{33})$	0	$3(c_{31})$	0	8	10	12	12
$3(a_{32})$	$1(b_{11})$	$1(c_{12})$	$1(d_{12})$	8	10	12	12

In all cases, except the first, the labelings are described in the bracket and $v''_f(0) + 2 = v''_f(1)$ and $e''_f(0) = e''_f(1)$.

In the first case, since $x_1 = x_2 = x_3 = x_4 = 0$, hence $t_1 = 4s_1, t_2 = 2s_2$, $t_3 = 4s_3, t_4 = 2s_4$ and at least one of s_1, s_2, s_3, s_4 is non-zero. In this case, we disturb the labeling in Stage 1 to a small extent.

If $s_1 \neq 0$, then there is at least one path of type 1 on which the last 4 intermediate vertices are labeled $1, 1, 0, 0$ in that order. On precisely one

such path, we relabel these vertices as $1, 1, 1, 0$.
If $s_1 = 0$, then one of s_2, s_3, s_4 is non-zero. Suppose $s_2 \neq 0$. Then there is at least one path of type 2 on which the last intermediate vertex has the label 0. On exactly one such path, we replace this 0 by a 1.

If $s_1 = 0, s_2 = 0$, then one of s_3, s_4 is non-zero. Suppose $s_3 \neq 0$, then there is at least one path of type 3 on which the last 2 intermediate vertices are labeled $1, 0$ in that order. On precisely one such path, we relabel these vertices as $1, 1$.

If $s_1 = 0, s_2 = 0, s_3 = 0$, then $s_4 \neq 0$ and hence there is at least one path of type 4 on which the last 3 intermediate vertices are labeled $1, 1, 0$ in that order. On precisely one such path, we relabel these vertices as $1, 1, 1$.

Thus, we have obtained a labeling f of A_1 such that $v_f(0) + 2 = v_f(1), e_f(0) = e_f(1), f(u) = 1, f(v) = 0$. We denote this labeling henceforth by α_2.

Alternate labeling 1 for plys of Type A_2 : Here we use labeling of type A.

x_1	x_2	x_3	x_4	$v''_f(0)$	$v''_f(1)$	$e''_f(0)$	$e''_f(1)$
0	0	$3(c_{33})$	0	2	4	4	5
0	$1(b_{11})$	$1(c_{13})$	$1(d_{11})$	2	4	4	5
$1(a_{16})$	0	0	0	1	3	2	3
$1(a_{16})$	$1(b_{11})$	$2(c_{21})$	$1(d_{11})$	5	7	8	9
$2(a_{25})$	0	$1(c_{11})$	0	4	6	6	7
$2(a_{25})$	$1(b_{11})$	$3(c_{31})$	$1(d_{12})$	8	10	12	13
$3(a_{31})$	0	$2(c_{23})$	0	7	9	10	11
$3(a_{31})$	$1(b_{11})$	0	$1(d_{13})$	7	9	10	11

In each of these cases, $v_f(0) + 2 = v_f(1), e_f(0) + 1 = e_f(1), f(u) = 1, f(v) = 0$.
We denote this labeling henceforth by β_2
Alternate labeling 2 for plys of Type A_2: In this case we use the labeling of type B.

x_1	x_2	x_3	x_4	$v''_f(0)$	$v''_f(1)$	$e''_f(0)$	$e''_f(1)$
0	0	$3(l_{31})$	0	3	3	5	4
0	$1(k_{11})$	$1(l_{13})$	$1(m_{11})$	3	3	5	4
$1(j_{11})$	0	0	0	2	2	3	2
$1(j_{11})$	$1(k_{11})$	$2(l_{23})$	$1(m_{13})$	6	6	9	8
$2(j_{21})$	0	$1(l_{12})$	0	5	5	7	6
$2(j_{21})$	$1(k_{11})$	$3(l_{31})$	$1(m_{13})$	9	9	13	12
$3(j_{35})$	0	$2(l_{23})$	0	8	8	11	10
$3(j_{35})$	$1(k_{12})$	0	$1(m_{11})$	8	8	11	10

We observe that in all these cases $v_f''(0) = v_f''(1)$ and $e_f''(0) = e_f''(1) + 1$. Since $f(u) = f(v) = 0$; and $v_f(0) = v_f(1) + 2, e_f(0) = e_f(1) + 1$.
We denote this labeling by β_3.

Alternate labeling 3 for plys of Type A_2 : In this case we use the labeling of type B.

x_1	x_2	x_3	x_4	$v_f''(0)$	$v_f''(1)$	$e_f''(0)$	$e_f''(1)$
0	0	$3(l_{34})$	0	2	4	5	4
0	$1(k_{12})$	$1(l_{11})$	$1(m_{11})$	2	4	5	4
$1(j_{12})$	0	0	0	1	3	3	2
$1(j_{11})$	$1(k_{12})$	$2(l_{21})$	$1(m_{11})$	5	7	9	8
$2(j_{21})$	0	$1(l_{11})$	0	4	6	7	6
$2(j_{21})$	$1(k_{11})$	$3(l_{31})$	$1(m_{11})$	8	10	13	12
$3(j_{31})$	0	$2(l_{21})$	0	7	9	11	10
$3(j_{31})$	$1(k_{11})$	0	$1(m_{11})$	7	9	11	10

We observe that in all cases $v_f''(0) + 2 = v_f''(1)$ and $e_f''(0) = e_f''(1) + 1$. Since $f(u) = f(v) = 0$; hence $v_f(0) = v_f(1), e_f(0) = e_f(1) + 1$.
We denote this labeling by β_4.
Alternate labeling 1 for plys of Type A_3 : Here we use the labeling of type A.

x_1	x_2	x_3	x_4	$v_f''(0)$	$v_f''(1)$	$e_f''(0)$	$e_f''(1)$
0	0	$1(c_{13})$	0	0	2	2	1
0	$1(b_{11})$	$3(c_{33})$	$1(d_{12})$	4	6	8	7
$1(a_{16})$	0	$2(c_{21})$	0	3	5	6	5
$1(a_{16})$	$1(b_{11})$	0	$1(d_{12})$	3	5	6	5
$2(a_{26})$	0	$3(c_{31})$	0	6	8	10	9
$2(a_{25})$	$1(b_{11})$	$1(c_{11})$	$1(d_{12})$	6	8	10	9
$3(a_{33})$	0	0	0	5	7	8	7
$3(a_{31})$	$1(b_{11})$	$2(c_{23})$	$1(d_{12})$	9	11	14	13

In each of these cases, $v_f(0)+2 = v_f(1), e_f(0) = e_f(1)+1, f(u) = 1, f(v) = 0$.
We denote this labeling by γ_2.

Alternate labeling 2 for plys of Type A_3: Here we use the labeling of type B.

x_1	x_2	x_3	x_4	$v''_f(0)$	$v''_f(1)$	$e''_f(0)$	$e''_f(1)$
0	0	$1(l_{12})$	0	1	1	1	2
0	$1(k_{11})$	$3(l_{31})$	$1(m_{13})$	5	5	7	8
$1(j_{11})$	0	$2(l_{24})$	0	4	4	5	6
$1(j_{11})$	$1(k_{11})$	0	$1(m_{13})$	4	4	5	6
$2(j_{25})$	0	$3(l_{31})$	0	7	7	9	10
$2(j_{25})$	$1(k_{12})$	$1(l_{12})$	$1(m_{11})$	7	7	9	10
$3(j_{35})$	0	0	0	6	6	7	8
$3(j_{35})$	$1(k_{12})$	$2(l_{21})$	$1(m_{13})$	10	10	13	14

We observe that in all cases $v''_f(0) = v''_f(1)$ and $e''_f(0) + 1 = e''_f(1)$. Since $f(u) = f(v) = 0$; hence $v_f(0) = v_f(1) + 2, e_f(0) + 1 = e_f(1)$.
This labeling will be denoted by γ_3.

Alternate labeling 3 for plys of Type A_3:
Here we use the labeling of type B.

x_1	x_2	x_3	x_4	$v''_f(0)$	$v''_f(1)$	$e''_f(0)$	$e''_f(1)$
0	0	$1(l_{11})$	0	0	2	1	2
0	$1(k_{11})$	$3(l_{31})$	$1(m_{11})$	4	6	7	8
$1(j_{11})$	0	$2(l_{21})$	0	3	5	5	6
$1(j_{11})$	$1(k_{11})$	0	$1(m_{11})$	3	5	5	6
$2(j_{21})$	0	$3(l_{32})$	0	6	8	9	10
$2(j_{21})$	$1(k_{11})$	$1(l_{11})$	$1(m_{13})$	6	8	9	10
$3(j_{32})$	0	0	0	5	7	7	8
$3(j_{31})$	$1(k_{11})$	$2(l_{21})$	$1(m_{13})$	9	11	13	14

We observe that in all cases $v''_f(0) + 2 = v''_f(1)$ and $e''_f(0) + 1 = e''_f(1)$. Since $f(u) = f(v) = 0$; hence $v_f(0) = v_f(1), e_f(0) + 1 = e_f(1)$. This labeling will be denoted by γ_4.

Alternate labeling for plys of Type A_4:
Here we use the labeling of type B.

x_1	x_2	x_3	x_4	$v''_f(0)$	$v''_f(1)$	$e''_f(0)$	$e''_f(1)$
0	0	$2(l_{24})$	$1(m_{11})$	3	4	4	6
0	$1(k_{11})$	0	0	0	1	0	2
$1(j_{11})$	0	$3(l_{32})$	$1(m_{13})$	6	7	8	10
$1(j_{11})$	$1(k_{11})$	$1(l_{12})$	0	3	4	4	6
$2(j_{25})$	0	0	$1(m_{11})$	5	6	6	8
$2(j_{25})$	$1(k_{11})$	$2(l_{23})$	0	6	7	8	10
$3(j_{35})$	0	$1(l_{12})$	$1(m_{11})$	8	9	10	12
$3(j_{35})$	$1(k_{11})$	$3(l_{31})$	0	9	10	12	14

We observe that in all cases $v_f''(0)+1=v_f''(1)$ and $e_f''(0)+2=e_f''(1)$. Since $f(u)=f(v)=0$; hence $v_f(0)=v_f(1)+1, e_f(0)+2=e_f(1)$. We denote this labeling by δ_2.

Alternate labeling for plys of Type A_5:

Here we use labeling of type A.

x_1	x_2	x_3	x_4	$v_f''(0)$	$v_f''(1)$	$e_f''(0))$	$e_f''(1)$
0	0	$1(c_{11})$	$1(d_{12})$	2	3	5	2
0	$1(b_{12})$	$3(c_{32})$	0	3	4	7	4
$1(a_{11})$	0	$2(c_{21})$	$1(d_{12})$	5	6	9	6
$1(a_{12})$	$1(b_{12})$	0	0	2	3	5	2
$2(a_{22})$	0	$3(c_{31})$	$1(d_{12})$	8	9	13	10
$2(a_{22})$	$1(b_{12})$	$1(c_{11})$	0	5	6	9	6
$3(a_{32})$	0	0	$1(d_{12})$	7	8	11	8
$3(a_{32})$	$1(b_{12})$	$2(c_{21})$	0	8	9	13	10

With the above labeling for A_5, we have $v_f(0)+1=v_f(1), e_f(0)=e_f(1)+3; f(u)=1, f(v)=0$. We denote this labeling by θ_2.

Alternate labeling for plys of Type A_6:

Here we use labeling of type A.

x_1	x_2	x_3	x_4	$v_f''(0)$	$v_f''(1)$	$e_f''(0))$	$e_f''(1)$
0	0	$3(c_{31})$	$1(d_{11})$	4	5	5	8
0	$1(b_{12})$	$1(c_{12})$	0	1	2	1	4
$1(a_{11})$	0	0	$1(d_{11})$	3	4	3	6
$1(a_{11})$	$1(b_{12})$	$2(c_{22})$	0	4	5	5	8
$2(a_{21})$	0	$1(c_{12})$	$1(d_{12})$	6	7	7	10
$2(a_{21})$	$1(b_{12})$	$3(c_{31})$	0	7	8	9	12
$3(a_{31})$	0	$2(c_{21})$	$1(d_{11})$	9	10	11	14
$3(a_{31})$	$1(b_{12})$	0	0	6	7	7	10

With the above labeling for A_6, we have $v_f(0)+1=v_f(1), e_f(0)+3=e_f(1), f(u)=1, f(v)=0$. We denote this labeling by ϕ_2.

Alternate labeling 1 for plys of Type B:

Here we use labeling of type B.

x_1	x_2	x_3	x_4	$v_f''(0)$	$v_f''(1)$	$e_f''(0)$	$e_f''(1)$
0	0	0	$1(m_{11})$	1	2	2	2
0	$1(k_{11})$	$2(l_{23})$	0	2	3	4	4
$1(j_{11})$	0	$1(l_{11})$	$1(m_{13})$	4	5	6	6
$1(j_{11})$	$1(k_{12})$	$3(l_{32})$	0	5	6	8	8
$2(j_{21})$	0	$2(l_{21})$	$1(m_{13})$	7	8	10	10
$2(j_{21})$	$1(k_{11})$	0	0	4	5	6	6
$3(j_{31})$	0	$3(l_{32})$	$1(m_{13})$	10	11	14	14
$3(j_{31})$	$1(k_{11})$	$1(l_{12})$	0	7	8	10	10

For each of these graphs we observe that $v_f''(0) + 1 = v_f''(1)$ and $e_f''(0) = e_f''(1)$. But $f(u) = f(v) = 0$, hence $v_f(0) = v_f(1) + 1, e_f(0) = e_f(1)$. We denote this labeling by μ_2.

Alternate labeling 2 for graphs of Type B :

Here we use the labeling of type A.

x_1	x_2	x_3	x_4	$v_f''(0)$	$v_f''(1)$	$e_f''(0)$	$e_f''(1)$
0	0	0	$1(d_{12})$	1	2	3	1
0	$1(b_{12})$	$2(c_{21})$	0	2	3	5	3
$1(a_{12})$	0	$1(c_{11})$	$1(d_{11})$	4	5	7	5
$1(a_{12})$	$1(b_{12})$	$3(c_{31})$	0	5	6	9	7
$2(a_{22})$	0	$2(c_{21})$	$1(d_{11})$	7	8	11	9
$2(a_{22})$	$1(b_{12})$	0	0	4	5	7	5
$3(a_{32})$	0	$3(c_{32})$	$1(d_{11})$	10	11	15	13
$3(a_{32})$	$1(b_{12})$	$1(c_{11})$	0	7	8	11	9

In each of these cases, $v_f(0)+1 = v_f(1), e_f(0) = e_f(1)+2, f(u) = 1, f(v) = 0$. We denote this labeling by μ_3.

Recall that in [3], no labeling was given for graphs of Type C. This was specifically because graphs of Type C are not cordial. However, now it becomes necessary to give certain binary labelings for graphs of Type C.

Labeling 1 for plys of Type C :

Here we use the labeling of type A.

x_1	x_2	x_3	x_4	$v_f''(0)$	$v_f''(1)$	$e_f''(0))$	$e_f''(1)$
0	0	$2(c_{21})$	0	2	2	4	2
0	$1(b_{11})$	0	$1(d_{12})$	2	2	4	2
$1(a_{11})$	0	$3(c_{32})$	0	5	5	8	6
$1(a_{11})$	$1(b_{11})$	$1(c_{11})$	$1(d_{12})$	5	5	8	6
$2(a_{22})$	0	0	0	4	4	6	4
$2(a_{21})$	$1(b_{11})$	$2(c_{21})$	$1(d_{12})$	8	8	12	10
$3(a_{32})$	0	$1(c_{11})$	0	7	7	10	8
$3(a_{32})$	$1(b_{11})$	$3(c_{31})$	$1(d_{12})$	11	11	16	14

In each of these cases, $v_f(0) = v_f(1), e_f(0) = e_f(1) + 2, f(u) = 1, f(v) = 0$. We denote this labeling by ξ_1.

Labeling 2 for plys of Type C :

Here we use the labeling of type A.

x_1	x_2	x_3	x_4	$v''_f(0)$	$v''_f(1)$	$e''_f(0)$	$e''_f(1)$
0	0	$2(c_{22})$	0	2	2	2	4
0	$1(b_{11})$	0	$1(d_{11})$	2	2	2	4
$1(a_{11})$	0	$3(c_{31})$	0	5	5	6	8
$1(a_{11})$	$1(b_{11})$	$1(c_{11})$	$1(d_{11})$	5	5	6	8
$2(a_{21})$	0	0	0	4	4	4	6
$2(a_{21})$	$1(b_{11})$	$2(c_{21})$	$1(d_{11})$	8	8	10	12
$3(a_{31})$	0	$1(c_{11})$	0	7	7	8	10
$3(a_{31})$	$1(b_{11})$	$3(c_{31})$	$1(d_{12})$	11	11	14	16

With the above labeling for graphs of Type C, we have $v_f(0) = v_f(1), e_f(0)+2 = e_f(1), f(u) = 1, f(v) = 0$. We denote this labeling by ξ_2.
Labeling 3 for plys of Type C :
Here we use labeling of type A.

x_1	x_2	x_3	x_4	$v''_f(0)$	$v''_f(1)$	$e''_f(0)$	$e''_f(1)$
0	0	$2(c_{23})$	0	1	3	4	2
0	$1(b_{12})$	0	$1(d_{12})$	1	3	4	2
$1(a_{16})$	0	$3(c_{32})$	0	4	6	8	6
$1(a_{11})$	$1(b_{12})$	$1(c_{11})$	$1(d_{12})$	4	6	8	6
$2(a_{26})$	0	0	0	3	5	6	4
$2(a_{22})$	$1(b_{12})$	$2(c_{22})$	$1(d_{12})$	7	9	12	10
$3(a_{32})$	0	$1(c_{13})$	0	6	8	10	8
$3(a_{32})$	$1(b_{12})$	$3(c_{31})$	$1(d_{12})$	10	12	16	14

In each of these cases, $v_f(0)+2 = v_f(1), e_f(0) = e_f(1)+2, f(u) = 1, f(v) = 0$. We denote this labeling by ξ_3.
Labeling 4 for graphs of Type C :
Here we use the labeling of type A.

x_1	x_2	x_3	x_4	$v''_f(0)$	$v''_f(1)$	$e''_f(0)$	$e''_f(1)$
0	0	$2(c_{26})$	0	1	3	2	4
0	$1(b_{12})$	0	$1(d_{11})$	1	3	2	4
$1(a_{16})$	0	$3(c_{31})$	0	4	6	6	8
$1(a_{16})$	$1(b_{12})$	$1(c_{11})$	$1(d_{15})$	4	6	6	8
$2(a_{25})$	0	0	0	3	5	4	6
$2(a_{21})$	$1(b_{12})$	$2(c_{21})$	$1(d_{11})$	7	9	10	12
$3(a_{31})$	0	$1(c_{13})$	0	6	8	8	10
$3(a_{31})$	$1(b_{12})$	$3(c_{32})$	$1(d_{11})$	10	12	14	16

As we see, in each case above, $v_f(0) + 2 = v_f(1), e_f(0) + 2 = e_f(1), f(u) = 1, f(v) = 0$. We denote this labeling by ξ_4.

We summarize the classification of the single $t-$ply graphs below. Alongside, we also list the labelings that are available for each type. In the table below, let 'E' and 'NE' denote an Eulerian, non-Eulerian graph respectively. Further let 'e' be the number of edges in the corresponding $t-$ply graph and let $e \equiv r(\text{mod} 4), 0 \leq r \leq 3$.

Type	E/NE	r	f	Label for u	Label for v	Relation of vertex labels	Relation of edge labels
A_1	E	0	α_1	1	0	$v_f(0) = v_f(1)$	$e_f(0) = e_f(1)$
			$\hat{\alpha_1}$	0	1	$v_f(0) = v_f(1)$	$e_f(0) = e_f(1)$
			α_2	1	0	$v_f(0) + 2 = v_f(1)$	$e_f(0) = e_f(1)$
			$\hat{\alpha_2}$	0	1	$v_f(0) = v_f(1) + 2$	$e_f(0) = e_f(1)$
A_2	NE	1	β_1	1	0	$v_f(0) = v_f(1)$	$e_f(0) + 1 = e_f(1)$
			$\hat{\beta_1}$	0	1	$v_f(0) = v_f(1)$	$e_f(0) + 1 = e_f(1)$
			β_2	1	0	$v_f(0) + 2 = v_f(1)$	$e_f(0) + 1 = e_f(1)$
			$\hat{\beta_2}$	0	1	$v_f(0) = v_f(1) + 2$	$e_f(0) + 1 = e_f(1)$
			β_3	0	0	$v_f(0) = v_f(1) + 2$	$e_f(0) = e_f(1) + 1$
			$\hat{\beta_3}$	1	1	$v_f(0) + 2 = v_f(1)$	$e_f(0) = e_f(1) + 1$
			β_4	0	0	$v_f(0) = v_f(1)$	$e_f(0) = e_f(1) + 1$
			$\hat{\beta_4}$	1	1	$v_f(0) = v_f(1)$	$e_f(0) = e_f(1) + 1$
A_3	NE	3	γ_1	1	0	$v_f(0) = v_f(1)$	$e_f(0) = e_f(1) + 1$
			$\hat{\gamma_1}$	0	1	$v_f(0) = v_f(1)$	$e_f(0) = e_f(1) + 1$
			γ_2	1	0	$v_f(0) + 2 = v_f(1)$	$e_f(0) = e_f(1) + 1$
			$\hat{\gamma_2}$	0	1	$v_f(0) = v_f(1) + 2$	$e_f(0) = e_f(1) + 1$
			γ_3	0	0	$v_f(0) = v_f(1) + 2$	$e_f(0) + 1 = e_f(1)$
			$\hat{\gamma_3}$	1	1	$v_f(0) + 2 = v_f(1)$	$e_f(0) + 1 = e_f(1)$
			γ_4	0	0	$v_f(0) = v_f(1)$	$e_f(0) + 1 = e_f(1)$
			$\hat{\gamma_4}$	1	1	$v_f(0) = v_f(1)$	$e_f(0) + 1 = e_f(1)$
A_4	NE	2	δ_1	1	0	$v_f(0) + 1 = v_f(1)$	$e_f(0) = e_f(1)$
			$\hat{\delta_1}$	0	1	$v_f(0) = v_f(1) + 1$	$e_f(0) = e_f(1)$
			δ_2	0	0	$v_f(0) = v_f(1) + 1$	$e_f(0) + 2 = e_f(1)$
			$\hat{\delta_2}$	1	1	$v_f(0) + 1 = v_f(1)$	$e_f(0) + 2 = e_f(1)$
A_5	E	3	θ_1	1	0	$v_f(0) + 1 = v_f(1)$	$e_f(0) + 1 = e_f(1)$
			$\hat{\theta_1}$	0	1	$v_f(0) = v_f(1) + 1$	$e_f(0) + 1 = e_f(1)$
			θ_2	1	0	$v_f(0) + 1 = v_f(1)$	$e_f(0) = e_f(1) + 3$
			$\hat{\theta_2}$	0	1	$v_f(0) = v_f(1) + 1$	$e_f(0) = e_f(1) + 3$
A_6	E	1	ϕ_1	1	0	$v_f(0) + 1 = v_f(1)$	$e_f(0) = e_f(1) + 1$
			$\hat{\phi_1}$	0	1	$v_f(0) = v_f(1) + 1$	$e_f(0) = e_f(1) + 1$
			ϕ_2	1	0	$v_f(0) + 1 = v_f(1)$	$e_f(0) + 3 = e_f(1)$
			$\hat{\phi_2}$	0	1	$v_f(0) = v_f(1) + 1$	$e_f(0) + 3 = e_f(1)$

B	NE	0	μ_1	0	0	$v_f(0)+1=v_f(1)$	$e_f(0)=e_f(1)$
			$\hat{\mu_1}$	1	1	$v_f(0)=v_f(1)+1$	$e_f(0)=e_f(1)$
			μ_2	0	0	$v_f(0)=v_f(1)+1$	$e_f(0)=e_f(1)$
			$\hat{\mu_2}$	1	1	$v_f(0)+1=v_f(1)$	$e_f(0)=e_f(1)$
			μ_3	1	0	$v_f(0)+1=v_f(1)$	$e_f(0)=e_f(1)+2$
			$\hat{\mu_3}$	0	1	$v_f(0)=v_f(1)+1$	$e_f(0)=e_f(1)+2$
C	E	2	ξ_1	1	0	$v_f(0)=v_f(1)$	$e_f(0)=e_f(1)+2$
			$\hat{\xi_1}$	0	1	$v_f(0)=v_f(1)$	$e_f(0)=e_f(1)+2$
			ξ_2	1	0	$v_f(0)=v_f(1)$	$e_f(0)+2=e_f(1)$
			$\hat{\xi_2}$	0	1	$v_f(0)=v_f(1)$	$e_f(0)+2=e_f(1)$
			ξ_3	1	0	$v_f(0)+2=v_f(1)$	$e_f(0)=e_f(1)+2$
			$\hat{\xi_3}$	0	1	$v_f(0)=v_f(1)+2$	$e_f(0)=e_f(1)+2$
			ξ_4	1	0	$v_f(0)+2=v_f(1)$	$e_f(0)+2=e_f(1)$
			$\hat{\xi_4}$	0	1	$v_f(0)+2=v_f(1)$	$e_f(0)+2=e_f(1)$

Consider the elongated ply T_n which is a snake of n number of $t^{(i)}$ plys $P_{t^{(i)}}(u_i, u_{i+1}), 1 \le i \le n$. These are blocks of the graph T_n. Clearly the number of edges incident on u_i in $P_{t^{(i-1)}}$ is $t^{(i-1)}$ and the number of edges incident on u_i in $P_t^{(i)}$ is $t^{(i)}$. Thus $d(u_i) = t^{(i-1)} + t^{(i)}$.

3: Non-cordial Elongated Plys.

Before we deal with the problem of cordiality of T_n in general, we digress a little to consider the index of cordiality of T_n, when T_n is Eulerian with the number of edges $e \equiv 2(\text{mod}4)$. Clearly, by Theorem 1, T_n is not cordial. However, we can prove that $i(T_n) = 2$. The result obtained in this Section will be used later in proving the main result.

Theorem 2: If an elongated ply T_n is Eulerian with $|E(T_n)| \equiv 2(\text{mod}4)$, then $i(T_n) = 2$.

Proof: As T_n is Eulerian, each block $P_t^{(i)}$ is Eulerian for all $i = 1, 2, \cdots, n$. Hence, each block $P_t^{(i)}$ is of the type A_1, A_5, A_6 or C.

Let the number of blocks of Type C be $2p_1 + r_1; r_1 = 0$ or 1. Let the number of blocks of type A_5 be $4p_2 + r_2; 0 \le r_2 \le 3$ and the number of blocks of type A_6 be $4p_3 + r_3; 0 \le r_3 \le 3$. We do not need to know the number of blocks of Type A_1 as in each such block, the number of edges is congruent to 0(mod 4).

The labeling g_1 for T_n is done in two steps as follows: We first label the end vertices $u_1, u_2, \cdots, u_m, u_{m+1}$, as

$g_1(u_i) = 1, \cdots i$ odd $\quad g_1(u_i) = 0, \cdots i$ even.

Secondly, to label the intermediate vertices, we label each block by a different labeling as described below. The restriction of the labeling g_1 then, to each of these blocks is precisely the labeling so chosen. Due to the labelings chosen for the end vertices, we use the labelings mentioned below for the odd numbered blocks and their inversions for the even numbered blocks.

In each of the blocks of Type A_1, use the corresponding labeling α_1. Out of the $2p_1 + r_1$ blocks of Type C, for the first p_1 blocks use ξ_1 and for the next p_1 use ξ_2. In $2p_2$ blocks of Type A_5, use the labeling $\hat{\theta_1}$, in p_2 blocks of Type A_5, use θ_1 and in the next p_2, use the alternate labeling θ_2. In $2p_3$ blocks of Type A_6, use the labeling $\hat{\phi_1}$, in p_3 blocks of Type A_6, use ϕ_1 and in the next p_3, use the alternate labeling ϕ_2.

Let $v'_g(0), v'_g(1), e'_g(0), e'_g(1)$ be the number of vertices and edges which have received the labels 0 and 1 respectively so far. Let $v''_g(0), v''_g(1), e''_g(0), e''_g(1$ be the number of vertices and edges which are to receive the labels 0 and 1 respectively at the next stage. There now remain r_1 blocks of Type C, r_2 blocks of Type A_5 and r_3 blocks of Type A_6 to be labeled. As $e \equiv 2(\bmod 4)$, only the following cases will arise.
In each case, we give two types of labelings as follows:
Case 1: $r_1 = 0, r_2 = 0, r_3 = 2$.
In this case, two blocks of Type A_6 remain to be labeled.
(a) In one of the blocks, use the labeling ϕ_1 and in the other use $\hat{\phi_1}$. This gives $v''_g(0) = v''_g(1)$ and $e''_g(0) = e''_g(1) + 2$.
(b) In one of the blocks, use the labeling $\hat{\phi_1}$ and in the other use ϕ_2. This gives $v''_g(0) = v''_g(1)$ and $e''_g(0) + 2 = e''_g(1)$.
Case 2: $r_1 = 0, r_2 = 1, r_3 = 3$.
There now remain one block of Type A_5 and three of Type A_6 to be labeled.
(a)For the block of Type A_5, use the labeling θ_1. In one of the blocks of Type A_6, use ϕ_1 and in each of the remaining two blocks of Type A_6 use $\hat{\phi_1}$. Then $v''_g(0) = v''_g(1)$ and $e''_g(0) = e''_g(1) + 2$.
(b)For the block of Type A_5, use the labeling θ_1. In one of the blocks of Type A_6, use the labeling ϕ_2; and in the other two use the labeling $\hat{\phi_1}$. Then, $v''_g(0) = v''_g(1)$ and $e''_g(0) + 2 = e''_g(1)$.
Case 3: $r_1 = 0, r_2 = 2, r_3 = 0$.
There now remain two blocks of Type A_5 to be labeled.
(a)For one of the blocks of Type A_5, use the labeling θ_1 and for the other block use $\hat{\theta_2}$. We get: $v''_g(0) = v''_g(1)$ and $e''_g(0) = e''_g(1) + 2$.
(b)For one of the block of Type A_5, use the labeling θ_1 and for the other block use $\hat{\theta_1}$. We get: $v''_g(0) = v''_g(1)$ and $e''_g(0) + 2 = e''_g(1)$.
Case 4: $r_1 = 0, r_2 = 3, r_3 = 1$.
There now remain three blocks of Type A_5 and one block of Type A_6 to be labeled.
(a)For each of two blocks of Type A_5, use the labeling $\hat{\theta_1}$ and for the third block of Type A_5 we use θ_2. For the block of type A_6 use ϕ_1. We get: $v''_g(0) = v''_g(1)$ and $e''_g(0) = e''_g(1) + 2$.

(b)For each of two blocks of Type A_5, use the labeling $\hat{\theta_1}$ and for the other block of Type A_5 we use θ_2. For the block of type A_6 use ϕ_2. We get: $v_g''(0) = v_g''(1)$ and $e_g''(0) + 2 = e_g''(1)$.

Case 5: $r_1 = 1, r_2 = 0, r_3 = 0$.

There is one block of Type C remaining to be labeled.

(a)For this block, use the labeling ξ_1. Then, $v_g''(0) = v_g''(1)$ and $e_g''(0) = e_g''(1) + 2$.

(b)For this block, use the labeling ξ_2. Then, $v_g''(0) = v_g''(1)$ and $e_g''(0) + 2 = e_g''(1)$.

Case 6: $r_1 = 1, r_2 = 1, r_3 = 1$.

In this case, there is one block of Type C, one of Type A_5 and one of Type A_6 remaining to be labeled.

(a)For the block of Type C, use the labeling ξ_1. For the block of Type A_5, use either the labeling $\hat{\theta_1}$; and for the block of Type A_6, use the labeling ϕ_1. Then, $v_g''(0) = v_g''(1)$ and $e_g''(0) = e_g''(1) + 2$.

(b)For the block of Type C, use the labeling ξ_2. For the block of Type A_5, use the labeling $\hat{\theta_1}$ and for the block of Type A_6 use the labeling ϕ_1. Then, $v_g''(0) = v_g''(1)$ and $e_g''(0) + 2 = e_g''(1)$.

Case 7: $r_1 = 1, r_2 = 2, r_3 = 2$.

There now remain one block of Type C, two of Type A_5 and two of Type A_6 to be labeled.

(a)For the block of Type C, use ξ_1. For each of the blocks of Type A_5 use the labeling $\hat{\theta_1}$ and for each of the blocks of Type A_6, use ϕ_1. We get: $v_g''(0) = v_g''(1)$ and $e_g''(0) = e_g''(1) + 2$.

(b)For the block of Type C, use either the labeling ξ_2. For each of the blocks of Type A_5, use the labeling $\hat{\theta_1}$ and in each of the blocks of Type A_6 use ϕ_1. Then, $v_g''(0) = v_g''(1)$ and $e_g''(0) + 2 = e_g''(1)$.

Case 8: $r_1 = 1, r_2 = 3, r_3 = 3$.

There now remain one block of Type C, three of Type A_5 and three of Type A_6 to be labeled.

(a)For the block of Type C, use ξ_1. For each of the blocks of Type A_5 use the labeling $\hat{\theta_1}$ and for each of the blocks of Type A_6, use ϕ_1. We get: $v_g''(0) = v_g''(1)$ and $e_g''(0) = e_g''(1) + 2$.

(b)For the block of Type C, use the labeling ξ_2. For each of the blocks of Type A_5, use the labeling $\hat{\theta_1}$ and in each of the blocks of Type A_6 use ϕ_1. Then, $v_g''(0) = v_g''(1)$ and $e_g''(0) + 2 = e_g''(1)$.

From the above, it is clear that we have obtained two types of labelings for T_n, one in which $e_g(0) = e_g(1) + 2$ (as given in (a)), and the other in which $e_g(0) + 2 = e_g(1)$(as given in (b)). Further, we have $v_g'(0) = v_g'(1)$ and $v_g''(0) = v_g''(1)$. The relation between $v_g(0)$ and $v_g(1)$ now entirely depends upon the labels of the shared vertices. Clearly, $v_g(0) = v_g'(0) +$

$v_g''(0) - \lceil (n-1)/2 \rceil$
$v_g(1) = v_g'(1) + v_g''(1) - \lfloor (n-1)/2 \rfloor$. From this, it is clear that, if n is odd, then $v_g(0) = v_g(1)$ and if n is even then $v_g(0)+1 = v_g(1)$. Hence $i(T_n) = 2$. This completes the proof. ⊗

4: Cordiality of T_n:

Theorem 3: T_n is cordial if and only if it is not Euler with $| E(T_n) | \equiv 2 (\text{mod} 4)$.

Proof: Suppose T_n is Eulerian and it does not satisfy Cahit's condition of Theorem 1. In this case we prove that T_n is cordial. The proof is by induction on n. For $n = 1$, the graph T_n is simply a single $t-$ply, for which cordiality has already been established in [3].

Now, assume that each elongated ply $\tilde{T_m}$ which is not Eulerian with $| E(\tilde{T_m}) | \equiv 2(\text{mod}4)$ is cordial. We now establish the result for $m+1$. Firstly, note that T_{m+1} is a one point union of T_m with a single $t-$ply graph $P_{t(m+1)}(u_{m+1}, u_{m+2})$. The graph T_m can be of one of the following types:

(I) T_m is non-Eulerian. (II) T_m is Eulerian and $| E(T_m) | \not\equiv 2(\text{mod}4)$.
(III) T_m is Eulerian and $| E(T_m) | \equiv 2(\text{mod}4)$.

We give the proof in four parts, three parts for the three cases listed here and the fourth part to deal with certain problematic cases which arise in the first part, Part I. In each of these parts, in certain cases, the labelings used are inadequate to give a cordial labeling f of T_{m+1}. Each such case will be indicated by a 'o' if the condition $| e_f(0) - e_f(1) | \leq 1$ is not satisfied and by a '⊗' if both the vertex as well as the edge label conditions are not satisfied. These problematic cases will be dealt with separately in part **IV**.

Part I: T_m is non-Eulerian We give a binary labeling g for T_{m+1} as follows:

Since T_m is not Euler, T_m is cordial, hence there exists a cordial labeling g_1 of T_m. Let g_2 be a binary labeling of $P_{t(m+1)}(u_{m+1}, u_{m+2})$. The choice of g_2 will be indicated in the tables that follow.

Let
$$g(v) = g_1(v) \text{ for } v \in V(T_m)$$
$$= g_2(v) \text{ for } v \in V(P_{t(m+1)}).$$

While choosing g_2, we ensure that $g_2(u_{m+1}) = g_1(u_{m+1})$.

Since g_1 is a cordial labeling of T_m, $| v_{g_1}(0) - v_{g_1}(1) | \leq 1$ and $| e_{g_1}(0) - e_{g_1}(1) | \leq 1$. Moreover, $e_g(i) = e_{g_1}(i) + e_{g_2}(i), i = 0, 1$. On the other hand,

$$v_g(0) = v_{g_1}(0) + v_{g_2}(0) - 1, \qquad v_g(1) = v_{g_1}(0) + v_{g_2}(1)$$

when $g_1(u_{m+1}) = 0$, whereas

$$v_g(0) = v_{g_1}(0) + v_{g_2}(0), \qquad v_g(1) = v_{g_1}(1) + v_{g_2}(1) - 1$$

when $g_1(u_{m+1}) = 1$.

Since T_{m+1} is a one point union of T_m with a $t-$ply $P_{t(m+1)}(u_{m+1}, u_{m+2})$, in this case T_{m+1} is also not Eulerian. We therefore have to show that T_{m+1} is cordial for all plys $P_{t(m+1)}$.

Depending on the existing vertex and edge label conditions for g_1, we consider the following choices, in which $g_2(u_{m+1}) = g_1(u_{m+1})$.

Case 1: $v_{g_1}(0) = v_{g_1}(1), e_{g_1}(0) = e_{g_1}(1)$. Case $P_{t(m+1)} = C$ has problem (o).

	$g_1(u_{m+1}) = 0$		$g_1(u_{m+1}) = 1$	
$P_{t(m+1)}$	g_2	g	g_2	g
A_1	$\hat{\alpha}_1$	$v_g(0) + 1 = v_g(1), e_g(0) = e_g(1)$	α_1	$v_g(0) = v_g(1) + 1, e_g(0) = e_g(1)$
A_2	$\hat{\beta}_1$	$v_g(0) + 1 = v_g(1), e_g(0) + 1 = e_g(1)$	β_1	$v_g(0) = v_g(1) + 1, e_g(0) + 1 = e_g(1)$
A_3	$\hat{\gamma}_1$	$v_g(0) + 1 = v_g(1), e_g(0) = e_g(1) + 1$	γ_1	$v_g(0) = v_g(1) + 1, e_g(0) = e_g(1) + 1$
A_4	$\bar{\hat{\delta}}_1$	$v_g(0) = v_g(1), e_g(0) = e_g(1)$	δ_1	$v_g(0) = v_g(1), e_g(0) = e_g(1)$
A_5	$\bar{\hat{\theta}}_1$	$v_g(0) = v_g(1), e_g(0) + 1 = e_g(1)$	θ_1	$v_g(0) = v_g(1), e_g(0) + 1 = e_g(1)$
A_6	$\bar{\hat{\phi}}_1$	$v_g(0) = v_g(1), e_g(0) = e_g(1) + 1$	ϕ_1	$v_g(0) = v_g(1), e_g(0) = e_g(1) + 1$
B	μ_2	$v_g(0) = v_g(1), e_g(0) = e_g(1)$	$\hat{\mu}_2$	$v_g(0) = v_g(1), e_g(0) = e(1)$

Case 2: $v_{g_1}(0) = v_{g_1}(1), e_{g_1}(0)+1 = e_{g_1}(1)$. Case $p_{t(m+1)} = A_5$ has problem (o).

	$g_1(u_{m+1}) = 0$		$g_1(u_{m+1}) = 1$	
$P_{t(m+1)}$	g_2	g	g_2	g
A_1	$\hat{\alpha}_1$	$v_g(0) + 1 = v_g(1), e_g(0) + 1 = e_g(1)$	α_1	$v_g(0) = v_g(1) + 1, e_g(0) + 1 = e_g(1)$
A_2	β_4	$v_g(0) + 1 = v_g(1), e_g(0) = e_g(1)$	$\hat{\beta}_4$	$v_g(0) = v_g(1) + 1, e_g(0) = e_g(1)$
A_3	$\hat{\gamma}_1$	$v_g(0) + 1 = v_g(1), e_g(0) = e_g(1)$	γ_1	$v_g(0) = v_g(1) + 1, e_g(0) = e_g(1)$
A_4	$\hat{\delta}_1$	$v_g(0) = v_g(1), e_g(0) + 1 = e_g(1)$	δ_1	$v_g(0) = v_g(1), e_g(0) + 1 = e_g(1)$
A_6	$\hat{\phi}_1$	$v_g(0) = v_g(1), e_g(0) = e_g(1)$	ϕ_1	$v_g(0) = v_g(1), e_g(0) = e_g(1)$
B	μ_2	$v_g(0) = v_g(1), e_g(0) + 1 = e_g(1)$	$\hat{\mu}_2$	$v_g(0) = v_g(1), e_g(0) + 1 = e_g(1)$
C	$\hat{\xi}_1$	$v_g(0) + 1 = v_g(1), e_g(0) = e_g(1) + 1$	ξ_1	$v_g(0) = v_g(1) + 1, e_g(0) = e_g(1) + 1$

Case 3: $v_{g_1}(0) = v_{g_1}(1), e_{g_1}(0) = e_{g_1}(1) + 1$. Case $P_{t((m+1))} = A_6$ has problem (o).

	$g_1(u_{m+1}) = 0$		$g_1(u_{m+1}) = 1$	
$P_{t(m+1)}$	g_2	g	g_2	g
A_1	$\hat{\alpha}_1$	$v_g(0) + 1 = v_g(1), e_g(0) = e_g(1) + 1$	α_1	$v_g(0) = v_g(1) + 1, e_g(0) = e_g(1) + 1$
A_2	$\hat{\beta}_1$	$v_g(0) + 1 = v_g(1), e_g(0) = e_g(1)$	β_1	$v_g(0) = v_g(1) + 1, e_g(0) = e_g(1)$
A_3	γ_4	$v_g(0) + 1 = v_g(1), e_g(0) = e_g(1)$	$\hat{\gamma}_4$	$v_g(0) = v_g(1) + 1, e_g(0) = e_g(1)$
A_4	$\hat{\delta}_1$	$v_g(0) = v_g(1), e_g(0) = e_g(1) + 1$	δ_1	$v_g(0) = v_g(1), e_g(0) = e_g(1) + 1$
A_5	$\hat{\theta}_1$	$v_g(0) = v_g(1), e_g(0) = e_g(1)$	θ_1	$v_g(0) = v_g(1), e_g(0) = e_g(1)$
B	μ_2	$v_g(0) = v_g(1), e_g(0) = e_g(1) + 1$	$\hat{\mu}_2$	$v_g(0) = v_g(1), e_g(0) = e_g(1) + 1$
C	$\hat{\xi}_2$	$v_g(0) + 1 = v_g(1), e_g(0) + 1 = e_g(1)$	ξ_2	$v_g(0) = v_g(1) + 1, e_g(0) + 1 = e_g(1)$

Case 4: $v_{g_1}(0) + 1 = v_{g_1}(1), e_{g_1}(0) = e_{g_1}(1)$. Case $P_{t(m+1)} = C$ has problem $\circ$ or $\otimes$.

	$g_1(u_{m+1}) = 0$		$g_1(u_{m+1}) = 1$	
$P_{t(m+1)}$	g_2	g	g_2	g
A_1	$\hat{\alpha}_2$	$v_g(0) = v_g(1), e_g(0) = e_g(1)$	α_1	$v_g(0) = v_g(1), e_g(0) = e_g(1)$
A_2	$\hat{\beta}_2$	$v_g(0) = v_g(1), e_g(0) + 1 = e_g(1)$	β_1	$v_g(0) = v_g(1), e_g(0) + 1 = e_g(1)$
A_3	$\hat{\gamma}_2$	$v_g(0) = v_g(1), e_g(0) = e_g(1) + 1$	γ_1	$v_g(0) = v_g(1), e_g(0) = e_g(1) + 1$
A_4	$\hat{\delta}_1$	$v_g(0) + 1 = v_g(1), e_g(0) = e_g(1)$	$\bar{\hat{\delta}}_1$	$v_g(0) = v_g(1) + 1, e_g(0) = e_g(1)$
A_5	$\hat{\theta}_1$	$v_g(0) + 1 = v_g(1), e_g(0) + 1 = e_g(1)$	$\bar{\hat{\theta}}_1$	$v_g(0) = v_g(1) + 1, e_g(0) + 1 = e_g(1)$
A_6	$\hat{\phi}_1$	$v_g(0) + 1 = v_g(1), e_g(0) = e_g(1) + 1$	$\bar{\hat{\phi}}_1$	$v_g(0) = v_g(1) + 1, e_g(0) = e_g(1) + 1$
B	μ_2	$v_g(0) + 1 = v_g(1), e_g(0) = e_g(1)$	$\hat{\mu}_1$	$v_g(0) = v_g(1) + 1, e_g(0) = e_g(1)$

Case 5: $v_{g_1}(0)+1=v_{g_1}(1), e_{g_1}(0)+1=e_{g_1}(1)$. Case $P_{t(m+1)}=A_5$ has problem (o).

	$g_1(u_{m+1})=0$		$g_1(u_{m+1})=1$	
$P_{t(m+1)}$	g_2	g	g_2	g
A_1	$\hat{\alpha}_2$	$v_g(0)=v_g(1), e_g(0)+1=e_g(1)$	α_1	$v_g(0)=v_g(1), e_g(0)+1=e_g(1)$
A_2	β_3	$v_g(0)=v_g(1), e_g(0)=e_g(1)$	$\hat{\beta}_4$	$v_g(0)=v_g(1), e_g(0)=e_g(1)$
A_3	$\hat{\gamma}_2$	$v_g(0)=v_g(1), e_g(0)=e_g(1)$	γ_1	$v_g(0)=v_g(1), e_g(0)=e_g(1)$
A_4	$\hat{\delta}_1$	$v_g(0)+1=v_g(1), e_g(0)+1=e_g(1)$	$\bar{\hat{\delta}}_1$	$v_g(0)=v_g(1)+1, e_g(0)+1=e_g(1)$
A_6	$\hat{\phi}_1$	$v_g(0)+1=v_g(1), e_g(0)=e_g(1)$	$\bar{\hat{\phi}}_1$	$v_g(0)=v_g(1)+1, e_g(0)=e_g(1)$
B	μ_2	$v_g(0)+1=v_g(1), e_g(0)+1=e_g(1)$	$\hat{\mu}_1$	$v_g(0)=v_g(1)+1, e_g(0)+1=e_g(1)$
C	$\hat{\xi}_3$	$v_g(0)=v_g(1), e_g(0)=e_g(1)+1$	ξ_1	$v_g(0)=v_g(1), e_g(0)=e_g(1)+1$

Case 6: $v_{g_1}(0)+1=v_{g_1}(1), e_{g_1}(0)=e_{g_1}(1)+1$. Case $P_{t(m+1)}=A_6$ has problem (o).

	$g_1(u_{m+1})=0$		$g_1(u_{m+1})=1$	
$P_{t(m+1)}$	g_2	g	g_2	g
A_1	$\hat{\alpha}_2$	$v_g(0)=v_g(1), e_g(0)=e_g(1)+1$	α_1	$v_g(0)=v_g(1), e_g(0)=e_g(1)+1$
A_2	$\hat{\beta}_2$	$v_g(0)=v_g(1), e_g(0)=e_g(1)$	β_1	$v_g(0)=v_g(1), e_g(0)=e_g(1)$
A_3	γ_3	$v_g(0)=v_g(1), e_g(0)=e_g(1)$	γ_4	$v_g(0)=v_g(1), e_g(0)=e_g(1)$
A_4	$\hat{\delta}_1$	$v_g(0)+1=v_g(1), e_g(0)=e_g(1)+1$	$\bar{\hat{\delta}}_1$	$v_g(0)=v_g(1)+1, e_g(0)=e_g(1)+1$
A_5	$\hat{\theta}_1$	$v_g(0)+1=v_g(1), e_g(0)=e_g(1)$	$\bar{\hat{\theta}}_1$	$v_g(0)=v_g(1)+1, e_g(0)=e_g(1)$
B	μ_2	$v_g(0)+1=v_g(1), e_g(0)=e_g(1)+1$	$\hat{\mu}_1$	$v_g(0)=v_g(1)+1, e_g(0)=e_g(1)+1$
C	$\hat{\xi}_4$	$v_g(0)=v_g(1), e_g(0)+1=e_g(1)$	ξ_2	$v_g(0)=v_g(1), e_g(0)+1=e_g(1)$

Case 7: $v_{g_1}(0) = v_{g_1}(1) + 1, e_{g_1}(0) = e_{g_1}(1)$. Case $P_{t(m+1)} = C$ has problem (∘) or ⊗.

	$g_1(u_{m+1}) = 0$		$g_1(u_{m+1}) = 1$	
$P_{t(m+1)}$	g_2	g	g_2	g
A_1	$\hat{\alpha}_1$	$v_g(1) = v_g(1), e_g(0) = e_g(1)$	$\hat{\alpha}_2$	$v_g(0) = v_g(1), e_g(0) = e_g(1)$
A_2	$\hat{\beta}_1$	$v_g(0) = v_g(1), e_g(0) + 1 = e_g(1)$	$\bar{\hat{\beta}}_2$	$v_g(0) = v_g(1), e_g(0) + 1 = e_g(1)$
A_3	$\hat{\gamma}_1$	$v_g(0) = v_g(1), e_g(0) = e_g(1) + 1$	$\bar{\hat{\gamma}}_2$	$v_g(0) = v_g(1), e_g(0) = e_g(1) + 1$
A_4	$\bar{\delta}_1$	$v_g(0) + 1 = v_g(1), e_g(0) = e_g(1)$	δ_1	$v_g(0) = v_g(1) + 1, e_g(0) = e_g(1)$
A_5	$\bar{\theta}_1$	$v_g(0) + 1 = v_g(1), e_g(0) + 1 = e_g(1)$	θ_1	$v_g(0) = v_g(1) + 1, e_g(0) + 1 = e_g(1)$
A_6	$\bar{\phi}_1$	$v_g(0) + 1 = v_g(1), e_g(0) = e_g(1) + 1$	ϕ_1	$v_g(0) = v_g(1) + 1, e_g(0) = e_g(1) + 1$
B	μ_1	$v_g(0) + 1 = v_g(1), e_g(0) = e_g(1)$	$\hat{\mu}_2$	$v_g(0) = v_g(1) + 1, e_g(0) = e_g(1)$

Case 8: $v_{g_1}(0) = v_{g_1}(1) + 1, e_{g_1}(0) + 1 = e_{g_1}(1)$. Case $P_{t(m+1)} = A_5$ has problem (∘).

	$g_1(u_{m+1}) = 0$		$g_1(u_{m+1}) = 1$	
$P_{t(m+1)}$	g_2	g	g_2	g
A_1	$\hat{\alpha}_1$	$v_g(0) = v_g(1), e_g(0) + 1 = e_g(1)$	α_2	$v_g(0) = v_g(1), e_g(0) + 1 = e_g(1)$
A_2	β_4	$v_g(0) = v_g(1), e_g(0) = e_g(1)$	$\hat{\beta}_3$	$v_g(0) = v_g(1), e_g(0) = e_g(1)$
A_3	$\hat{\gamma}_1$	$v_g(0) = v_g(1), e_g(0) = e_g(1)$	γ_2	$v_g(0) = v_g(1), e_g(0) = e_g(1)$
A_4	$\tilde{\delta}_1$	$v_g(0) + 1 = v_g(1), e_g(0) + 1 = e_g(1)$	δ_1	$v_g(0) = v_g(1) + 1, e_g(0) + 1 = e_g(1)$
A_6	$\tilde{\phi}_1$	$v_g(0) + 1 = v_g(1), e_g(0) = e_g(1)$	ϕ_1	$v_g(0) = v_g(1) + 1, e_g(0) = e_g(1)$
B	μ_1	$v_g(0) + 1 = v_g(1), e_g(0) + 1 = e_g(1)$	$\hat{\mu}_2$	$v_g(0) = v_g(1) + 1, e_g(0) + 1 = e_g(1)$
C	$\hat{\xi}_1$	$v_g(0) = v_g(1), e_g(0) = e_g(1) + 1$	ξ_3	$v_g(0) = v_g(1), e_g(0) = e_g(1) + 1$

Case 9: $v_{g_1}(0) = v_{g_1}(1) + 1, e_{g_1}(0) = e_{g_1}(1) + 1$. Case $P_{t(m+1)} = A_6$ has problem (o).

	$g(u_{m+1}) = 0$		$g_1(u_{m+1}) = 1$	
$P_{t(m+1)}$	g_2	g	g_2	g
A_1	$\hat{\alpha}_1$	$v_g(0) = v_g(1), e_g(0) = e_g(1) + 1$	α_2	$v_g(0) = v_g(1), e_g(0) = e_g(1) + 1$
A_2	$\hat{\beta}_1$	$v_g(0) = v_g(1), e_g(0) = e_g(1)$	β_2	$v_g(0) = v_g(1), e_g(0) = e_g(1)$
A_3	γ_4	$v_g(0) = v_g(1), e_g(0) = e_g(1)$	$\hat{\gamma}_3$	$v_g(0) = v_g(1), e_g(0) = e_g(1)$
A_4	$\bar{\delta}_1$	$v_g(0) + 1 = v_g(1), e_g(0) = e_g(1) + 1$	δ_1	$v_g(0) = v_g(1) + 1, e_g(0) = e_g(1) + 1$
A_5	$\bar{\theta}_1$	$v_g(0) + 1 = v_g(1), e_g(0) = e_g(1)$	θ_1	$v_g(0) = v_g(1) + 1, e_g(0) = e_g(1)$
B	μ_1	$v_g(0) + 1 = v_g(1), e_g(0) = e_g(1) + 1$	$\hat{\mu}_2$	$v_g(0) = v_g(1) + 1, e_g(0) = e_g(1) + 1$
C	$\hat{\xi}_2$	$v_g(0) = v_g(1), e_g(0) + 1 = e_g(1)$	ξ_4	$v_g(0) = v_g(1), e_g(0) + 1 = e_g(1)$

Part II: T_m is Eulerian and $\mid E(T_m) \mid \not\equiv 2(\text{ mod } 4)$

In this case, $\mid E(T_m) \mid \equiv 0, 1, 3(\text{mod}4)$ and hence T_m is cordial. Let g_1 be a cordial labeling of T_m. Let g be a binary labeling for T_{m+1} as described in Part I. Let e denote $\mid E(T_m) \mid$. We consider three cases.

Case (a):- $e \equiv 0(\text{mod}4)$

Here, $\mid v_{g_1}(0) - v_{g_1}(1) \mid \leq 1$ and $e_{g_1}(0) = e_{g_1}(1)$. Now if $P_{t(m+1)}$ is of Type C, then T_{m+1} is Eulerian and $\mid E(T_{m+1}) \mid \equiv 2(\text{mod}4)$; hence T_{m+1} is not cordial. We make the following sub-cases depending on the vertex label condition in T_m.

Case 1: $v_{g_1}(0) = v_{g_1}(1)$.

	$g_1(u_{m+1}) = 0$		$g_1(u_{m+1}) = 1$	
$P_{t(m+1)}$	g_2	g	g_2	g
A_1	$\hat{\alpha}_1$	$v_g(0)+1 = v_g(1), e_g(0) = e_g(1)$	α_1	$v_g(0) = v_g(1)+1, e_g(0) = e_g(1)$
A_2	$\hat{\beta}_1$	$v_g(0)+1 = v_g(1), e_g(0)+1 = e_g(1)$	β_1	$v_g(0) = v_g(1)+1, e_g(0)+1 = e_g(1)$
A_3	$\hat{\gamma}_1$	$v_g(0)+1 = v_g(1), e_g(0) = e_g(1)+1$	γ_1	$v_g(0) = v_g(1)+1, e_g(0) = e_g(1)+1$
A_4	$\hat{\delta}_1$	$v_g(0) = v_g(1), e_g(0) = e_g(1)$	δ_1	$v_g(0) = v_g(1), e_g(0) = e_g(1)$
A_5	$\hat{\theta}_1$	$v_g(0) = v_g(1), e_g(0)+1 = e_g(1)$	θ_1	$v_g(0) = v_g(1), e_g(0)+1 = e_g(1)$
A_6	$\hat{\phi}_1$	$v_g(0) = v_g(1), e_g(0) = e_g(1)+1$	ϕ_1	$v_g(0) = v_g(1), e_g(0) = e_g(1)+1$
B	μ_2	$v_g(0) = v_g(1), e_g(0) = e_g(1)$	$\hat{\mu}_2$	$v_g(0) = v_g(1), e_g(0) = e_g(1)$

Case 2: $v_{g_1}(0) + 1 = v_{g_1}(1)$.

	$g_1(u_{m+1}) = 0$		$g_1(u_{m+1}) = 1$	
$P_{t(m+1)}$	g_2	g	g_2	g
A_1	$\hat{\alpha}_2$	$v_g(0) = v_g(1), e_g(0) = e_g(1)$	α_1	$v_g(0) = v_g(1), e_g(0) = e_g(1)$
A_2	$\hat{\beta}_2$	$v_g(0) = v_g(1), e_g(0)+1 = e_g(1)$	β_1	$v_g(0) = v_g(1), e_g(0)+1 = e_g(1)$
A_3	$\hat{\gamma}_2$	$v_g(0) = v_g(1), e_g(0) = e_g(1)+1$	γ_1	$v_g(0) = v_g(1), e_g(0) = e_g(1)+1$
A_4	$\hat{\delta}_1$	$v_g(0)+1 = v_g(1), e_g(0) = e_g(1)$	$\bar{\hat{\delta}}_1$	$v_g(0) = v_g(1)+1, e_g(0) = e_g(1)$
A_5	$\hat{\theta}_1$	$v_g(0)+1 = v_g(1), e_g(0)+1 = e_g(1)$	$\bar{\hat{\theta}}_1$	$v_g(0) = v_g(1)+1, e_g(0)+1 = e_g(1)$
A_6	$\hat{\phi}_1$	$v_g(0)+1 = v_g(1), e_g(0) = e_g(1)+1$	$\bar{\hat{\phi}}_1$	$v_g(0) = v_g(1)+1, e_g(0) = e_g(1)+1$
B	μ_2	$v_g(0)+1 = v_g(1), e_g(0) = e_g(1)$	$\hat{\mu}_1$	$v_g(0) = v_g(1)+1, e_g(0) = e_g(1)$

Case 3: $v_{g_1}(0) = v_{g_1}(1) + 1$.

	$g_1(u_{m+1}) = 0$		$g_1(u_{m+1}) = 1$	
$P_{t(m+1)}$	g_2	g	g_2	g
A_1	$\hat{\alpha}_1$	$v_g(0) = v_g(1), e_g(0) = e_g(1)$	α_2	$v_g(0) = v_g(1), e_g(0) = e_g(1)$
A_2	$\hat{\beta}_1$	$v_g(0) = v_g(1), e_g(0) + 1 = e_g(1)$	β_2	$v_g(0) = v_g(1), e_g(0) + 1 = e_g(1)$
A_3	$\hat{\gamma}_1$	$v_g(0) = v_g(1), e_g(0) = e_g(1) + 1$	γ_2	$v_g(0) = v_g(1), e_g(0) = e_g(1) + 1$
A_4	$\tilde{\delta}_1$	$v_g(0) + 1 = v_g(1), e_g(0) = e_g(1)$	δ_1	$v_g(0) = v_g(1) + 1, e_g(0) = e_g(1)$
A_5	$\tilde{\theta}_1$	$v_g(0) + 1 = v_g(1), e_g(0) + 1 = e_g(1)$	θ_1	$v_g(0) = v_g(1) + 1, e_g(0) + 1 = e_g(1)$
A_6	$\tilde{\phi}_1$	$v_g(0) + 1 = v_g(1), e_g(0) = e_g(1) + 1$	ϕ_1	$v_g(0) = v_g(1) + 1, e_g(0) = e_g(1) + 1$
B	μ_1	$v_g(0) + 1 = v_g(1), e_g(0) = e_g(1)$	$\hat{\mu}_2$	$v_g(0) = v_g(1) + 1, e_g(0) = e_g(1)$

Case (b): $e \equiv 1(\mod 4)$.

If $P_{t(m+1)}$ is of Type A_6, then T_{m+1} is Eulerian and $\mid E(T_{m+1}) \mid \equiv 2(\text{mod}4)$; hence T_{m+1} is not cordial. In the remaining cases, we prove that T_{m+1} is cordial. Since T_m is cordial, there exists a labeling g_1 of T_m such that $\mid v_{g_1}(0) - v_{g_1}(1) \mid \leq 1$ and $\mid e_{g_1}(0) - e_{g_1}(1) \mid = 1$. Thus, $e_{g_1}(0) - e_{g_1}(1)$ is either -1 or 1. In fact we prove that it is always possible to have a labeling g_1 such that $e_{g_1}(0) - e_{g_1}(1) = 1$. Firstly observe that as T_m is Eulerian, each block $P_{t(i)}$ of T_m is Eulerian and hence is of one of the types namely Type A_1, A_5, A_6 or C.

Let the number of blocks of Type C be $2q_1 + s_1, s_1 = 0, 1$; the number of blocks of Type A_5 be $4q_2 + s_2, 0 \leq s_2 \leq 3$; and the number of blocks of Type A_6 be $4q_3 + s_3, 0 \leq s_3 \leq 3$. We do not need to know the number of blocks of Type A_1 as in each such block, the number of edges is congruent to $0(\mod 4)$. The labeling g_1 for T_m is done in two steps. We first label the end vertices $u_1, u_2, \cdots, u_m, u_{m+1}$, as $g(u_i) = 1$, if i is odd and $g(u_i) = 0$, if i is even . Secondly, to label the intermediate vertices, we label each block by a different labeling as described below. The restriction of the labeling g_1 then, to each of these blocks, is precisely the labeling so chosen. Due to the condition already imposed on the end vertices by g_1, for the odd blocks, we choose the labeling as mentioned, but for the even blocks, we take the inversion of the labeling mentioned therein.

For each of the blocks of Type A_1, use the labeling α_1 listed in the summary. For q_1 blocks of Type C, use ξ_1 and for the next q_1 use ξ_2. In

$2q_2$ blocks of Type A_5, use the labeling $\hat{\theta_1}$, in q_2 blocks of Type A_5, use θ_1 and in the next q_2, use the alternate labeling θ_2. In $2q_2$ blocks of Type A_6, use the labeling $\hat{\phi_1}$, in q_2 blocks of Type A_6, use ϕ_1 and in the next q_2, use the alternate labeling ϕ_2. There now remain s_1 blocks of Type C, s_2 blocks of Type A_5 and s_3 blocks of Type A_6. As $e \equiv 1(\text{mod}4)$, only the following cases will arise. The choice of the labeling made in each case is indicated alongside. Where there is more than one block of the same type to be labeled, the various labelings used are mentioned in the same cell.

s_1	s_2	s_3	Edge label condition
0	0	$1(\phi_1)$	$e_{g_1}(0) = e_{g_1}(1) + 1$
0	$1(\theta_1)$	$2(\phi_1, \hat{\phi_1})$	$e_{g_1}(0) = e_{g_1}(1) + 1$
0	$2(\theta_1, \theta_1)$	$3(\phi_1, \hat{\phi_1}, \hat{\phi_1})$	$e_{g_1}(0) = e_{g_1}(1) + 1$
0	$3(\hat{\theta_1}, \hat{\theta_1}, \theta_2)$	0	$e_{g_1}(0) = e_{g_1}(1) + 1$
$1(\xi_2)$	0	$3(\phi_1, \phi_1, \hat{\phi_1})$	$e_{g_1}(0) = e_{g_1}(1) + 1$
$1(\xi_1)$	$1(\theta_1)$	0	$e_{g_1}(0) = e_{g_1}(1) + 1$
$1(\xi_1)$	$2(\theta_1, \hat{\theta_1})$	$1(\phi_1)$	$e_{g_1}(0) = e_{g_1}(1) + 1$
$1(\xi_1)$	$3(\theta_1, \theta_1, \theta_1)$	$2(\hat{\phi_1}, \hat{\phi_1})$	$e_{g_1}(0) = e_{g_1}(1) + 1$

From the above table, it is evident that for the labeling $g_1, e_{g_1}(0) = e_{g_1}(1)+1$. However, $| v_{g_1}(0) - v_{g_1}(1) | \leq 1$. We thus have the following three cases:-

Case 1:- $v_{g_1}(0) = v_{g_1}(1), e_{g_1}(0) = e_{g_1}(1) + 1$.

	$g_1(u_{m+1}) = 0$		$g_1(u_{m+1}) = 1$	
$P_{i(m+1)}$	g_2	g	g_2	g
A_1	$\hat{\alpha_1}$	$v_g(0) + 1 = v_g(1), e_g(0) = e_g(1) + 1$	α_1	$v_g(0) = v_g(1) + 1, e_g(0) = e_g(1) + 1$
A_2	$\hat{\beta_1}$	$v_g(0) + 1 = v_g(1), e_g(0) = e_g(1)$	β_1	$v_g(0) = v_g(1) + 1, e_g(0) = e_g(1)$
A_3	γ_4	$v_g(0) + 1 = v_g(1), e_g(0) = e_g(1)$	$\hat{\gamma_4}$	$v_g(0) = v_g(1) + 1, e_g(0) = e_g(1)$
A_4	$\hat{\delta_1}$	$v_g(0) = v_g(1), e_g(0) = e_g(1) + 1$	δ_1	$v_g(0) = v_g(1), e_g(0) = e_g(1) + 1$
A_5	$\hat{\theta_1}$	$v_g(0) = v_g(1), e_g(0) = e_g(1)$	θ_1	$v_g(0) = v_g(1), e_g(0) = e_g(1)$
B	μ_2	$v_g(0) = v_g(1), e_g(0) = e_g(1) + 1$	$\hat{\mu_2}$	$v_g(0) = v_g(1), e_g(0) = e_g(1) + 1$
C	$\hat{\xi_2}$	$v_g(0) + 1 = v_g(1), e_g(0) + 1 = e_g(1)$	ξ_2	$v_g(0) = v_g(1) + 1, e_g(0) + 1 = e_g(1)$

Case 2:- $v_{g_1}(0) + 1 = v_{g_1}(1), e_{g_1}(0) = e_{g_1}(1) + 1$.

	$g_1(u_{m+1}) = 0$		$g_1(u_{m+1}) = 1$	
$P_{t(m+1)}$	g_2	g	g_2	g
A_1	$\hat{\alpha}_2$	$v_g(0) = v_g(1), e_g(0) = e_g(1) + 1$	α_1	$v_g(0) = v_g(1), e_g(0) = e_g(1) + 1$
A_2	$\hat{\beta}_2$	$v_g(0) = v_g(1), e_g(0) = e_g(1)$	β_1	$v_g(0) = v_g(1), e_g(1) = e_g(1)$
A_3	γ_3	$v_g(0) = v_g(1), e_g(0) = e_g(1)$	$\hat{\gamma}_4$	$v_g(0) = v_g(1), e_g(0) = e_g(1)$
A_4	$\hat{\delta}_1$	$v_g(0) + 1 = v_g(1), e_g(0) = e_g(1) + 1$	$\bar{\hat{\delta}}_1$	$v_g(0) = v_g(1) + 1, e_g(0) = e_g(1) + 1$
A_5	$\hat{\theta}_1$	$v_g(0) + 1 = v_g(1), e_g(0) = e_g(1)$	$\bar{\hat{\theta}}_1$	$v_g(0) = v_g(1) + 1, e_g(0) = e_g(1)$
B	μ_2	$v_g(0) + 1 = v_g(1), e_g(0) = e_g(1) + 1$	$\hat{\mu}_1$	$v_g(0) = v_g(1) + 1, e_g(0) = e_g(1) + 1$
C	ξ_4	$v_g(0) = v_g(1), e_g(0) + 1 = e_g(1)$	ξ_2	$v_g(0) = v_g(1), e_g(0) + 1 = e_g(1)$

Case 3:- $v_{g_1}(0) = v_{g_1}(1) + 1, e_{g_1}(0) = e_{g_1}(1) + 1$.

	$g_1(u_{m+1}) = 0$		$g_1(u_{m+1}) = 1$	
$P_{t(m+1)}$	g_2	g	g_2	g
A_1	$\hat{\alpha}_1$	$v_g(0) = v_g(1), e_g(0) = e_g(1) + 1$	α_2	$v_g(0) = v_g(1), e_g(0) = e_g(1) + 1$
A_2	$\hat{\beta}_1$	$v_g(0) = v_g(1), e_g(0) = e_g(1)$	β_2	$v_g(0) = v_g(1), e_g(0) = e_g(1)$
A_3	γ_4	$v_g(0) = v_g(1), e_g(0) = e_g(1)$	$\hat{\gamma}_3$	$v_g(0) = v_g(1), e_g(0) = e_g(1)$
A_4	$\bar{\delta}_1$	$v_g(0) + 1 = v_g(1), e_g(0) = e_g(1) + 1$	δ_1	$v_g(0) = v_g(1) + 1, e_g(0) = e_g(1) + 1$
A_5	$\bar{\theta}_1$	$v_g(0) + 1 = v_g(1), e_g(0) = e_g(1)$	θ_1	$v_g(0) = v_g(1) + 1, e_g(0) = e_g(1)$
B	μ_1	$v_g(0) + 1 = v_g(1), e_g(0) = e_g(1) + 1$	$\hat{\mu}_2$	$v_g(0) = v_g(1) + 1, e_g(0) = e_g(1) + 1$
C	$\hat{\xi}_2$	$v_g(0) = v_g(1), e_g(0) + 1 = e_g(1)$	ξ_4	$v_g(0) = v_g(1) + 1, e_g(0) + 1 = e_g(1)$

Case (c):- $e \equiv 3(\text{mod}4)$.

If $P_{t(m+1)}$ is of the Type A_5, then T_{m+1} is Eulerian with $\mid E(T_{m+1}) \mid \equiv 2(\text{mod}4)$; hence T_{m+1} is not cordial. In all the remaining cases we prove that T_{m+1} is cordial. Now, there exists a labeling g_1 of T_m such that $\mid v_{g_1}(0) - v_{g_1}(1) \mid \leq 1$ and $\mid e_{g_1}(0) - e_{g_1}(1) \mid = 1$. In this case we can prove similarly as in Case (b) above that we can always give a cordial labeling g_1 of T_m such that $e_{g_1}(0) + 1 = e_{g_1}(1)$. However, $\mid v_{g_1}(0) - v_{g_1}(1) \mid \leq 1$. We thus have the following three cases:-

Case 1: $v_{g_1}(0) = v_{g_1}(1), e_{g_1}(0) + 1 = e_{g_1}(1)$.

	$g_1(u_{m+1}) = 0$		$g_1(u_{m+1}) = 1$	
$P_{t(m+1)}$	g_2	g	g_2	g
A_1	$\hat{\alpha}_1$	$v_g(0) + 1 = v_g(1), e_g(0) + 1 = e_g(1)$	α_1	$v_g(0) = v_g(1) + 1, e_g(0) + 1 = e_g(1)$
A_2	β_4	$v_g(0) + 1 = v_g(1), e_g(0) = e_g(1)$	$\hat{\beta}_4$	$v_g(0) = v_g(1) + 1, e_g(0) = e_g(1)$
A_3	$\hat{\gamma}_1$	$v_g(0) + 1 = v_g(1), e_g(0) = e_g(1)$	γ_1	$v_g(0) = v_g(1) + 1, e_g(0) = e_g(1)$
A_4	$\hat{\delta}_1$	$v_g(0) = v_g(1), e_g(0) + 1 = e_g(1)$	δ_1	$v_g(0) = v_g(1), e_g(0) + 1 = e_g(1)$
A_6	$\hat{\phi}_1$	$v_g(0) = v_g(1), e_g(0) = e_g(1)$	ϕ_1	$v_g(0) = v_g(1), e_g(0) = e_g(1)$
B	μ_2	$v_g(0) = v_g(1), e_g(0) + 1 = e_g(1)$	$\hat{\mu}_2$	$v_g(0) = v_g(1), e_g(0) + 1 = e_g(1)$
C	$\hat{\xi}_1$	$v_g(0) + 1 = v_g(1), e_g(0) = e_g(1) + 1$	ξ_1	$v_g(0) = v_g(1) + 1, e_g(0) = e_g(1) + 1$

Case 2: $v_{g_1}(0) + 1 = v_{g_1}(1), e_{g_1}(0) + 1 = e_{g_1}(1)$.

	$g_1(u_{m+1}) = 0$		$g_1(u_{m+1}) = 1$	
$P_{t(m+1)}$	g_2	g	g_2	g
A_1	$\hat{\alpha}_2$	$v_g(0) = v_g(1), e_g(0) + 1 = e_g(1)$	α_1	$v_g(0) = v_g(1), e_g(0) + 1 = e_g(1)$
A_2	β_3	$v_g(0) = v_g(1), e_g(0) = e_g(1)$	$\hat{\beta}_4$	$v_g(0) = v_g(1), e_g(0) = e_g(1)$
A_3	$\hat{\gamma}_2$	$v_g(0) = v_g(1), e_g(0) = e_g(1)$	γ_1	$v_g(0) = v_g(1), e_g(0) = e_g(1)$
A_4	$\hat{\delta}_1$	$v_g(0) + 1 = v_g(1), e_g(0) + 1 = e_g(1)$	$\bar{\hat{\delta}}_1$	$v_g(0) = v_g(1) + 1, e_g(0) + 1 = e_g(1)$
A_6	$\hat{\phi}_1$	$v_g(0) + 1 = v_g(1), e_g(0) = e_g(1)$	$\bar{\hat{\phi}}_1$	$v_g(0) = v_g(1) + 1, e_g(0) = e_g(1)$
B	μ_2	$v_g(0) + 1 = v_g(1), e_g(0) + 1 = e_g(1)$	$\hat{\mu}_1$	$v_g(0) = v_g(1) + 1, e_g(0) + 1 = e_g(1)$
C	$\hat{\xi}_3$	$v_g(0) = v_g(1), e_g(0) = e_g(1) + 1$	ξ_1	$v_g(0) = v_g(1), e_g(0) = e_g(1) + 1$

Case 3:- $v_{g_1}(0) = v_{g_1}(1) + 1, e_{g_1}(0) + 1 = e_{g_1}(1)$.

	$g_1(u_{m+1}) = 0$		$g_1(u_{m+1}) = 1$	
$P_{t(m+1)}$	g_2	g	g_2	g
A_1	$\bar{\alpha_1}$	$v_g(0) = v_g(1), e_g(0) + 1 = e_g(1)$	α_2	$v_g(0) = v_g(1), e_g(0) + 1 = e_g(1)$
A_2	β_4	$v_g(0) = v_g(1), e_g(0) = e_g(1)$	$\hat{\beta_3}$	$v_g(0) = v_g(1), e_g(0) = e_g(1)$
A_3	$\hat{\gamma_1}$	$v_g(0) = v_g(1), e_g(0) = e_g(1)$	γ_2	$v_g(0) = v_g(1), e_g(0) = e_g(1)$
A_4	$\bar{\delta_1}$	$v_g(0) + 1 = v_g(1), e_g(0) + 1 = e_g(1)$	δ_1	$v_g(0) = v_g(1) + 1, e_g(0) + 1 = e_g(1)$
A_6	$\bar{\phi_1}$	$v_g(0) + 1 = v_g(1), e_g(0) = e_g(1)$	ϕ_1	$v_g(0) = v_g(1) + 1, e_g(0) = e_g(1)$
B	μ_1	$v_g(0) + 1 = v_g(1), e_g(0) + 1 = e_g(1)$	$\hat{\mu_2}$	$v_g(0) = v_g(1) + 1, e_g(0) + 1 = e_g(1)$
C	$\hat{\xi_1}$	$v_g(0) = v_g(1), e_g(0) = e_g(1) + 1$	ξ_3	$v_g(0) = v_g(1), e_g(0) = e_g(1) + 1$

Part III: T_m is Eulerian and $e \equiv 2(\text{mod}4)$.

If $P_{t(m+1)}$ is of the Type A_1, then T_{m+1} will be Eulerian with $| E(T_{m+1}) | \equiv 2(\text{mod}4)$. Hence, in this case, T_{m+1} is not cordial. We prove that in all the remaining cases, T_{m+1} is cordial. Since T_m is not cordial,$i(T_m) = 2$. In fact, we have obtained two labelings for T_m, one in which $e_g(0) + 2 = e_g(1)$ and the other in which $e_g(0) = e_g(1) + 2$. Denote the first labeling by h_1 and the latter by h_2. Further depending on $| V(T_m) |$ either $v_g(0) = v_g(1)$ or $v_g(0) + 1 = v_g(1)$. We make the following cases, depending on the existing vertex condition in T_m. Let g_1 be the labeling for T_m, g_2 the labeling for $P_{t(m+1)}$ and g the resulting labeling for T_{m+1}. The choice of g_1 and g_2 in each case is indicated in the table below.

Case 1:- $v_{g_1}(0) = v_{g_1}(1)$.

		$g_1(u_{m+1}) = 0$		$g_1(u_{m+1}) = 1$	
g_1	$P_{t(m+1)}$	g_2	g	g_2	g
h_2	A_2	$\hat{\beta}_1$	$v_g(0)+1 = v_g(1), e_g(0) = e_g(1)+1$	β_1	$v_g(0) = v_g(1)+1, e_g(0) = e_g(1)+1$
h_1	A_3	$\hat{\gamma}_1$	$v_g(0)+1 = v_g(1), e_g(0)+1 = e_g(1)$	γ_1	$v_g(0) = v_g(1)+1, e_g(0)+1 = e_g(1)$
h_2	A_4	δ_2	$v_g(0) = v_g(1), e_g(0) = e_g(1)$	$\hat{\delta}_2$	$v_g(0) = v_g(1), e_g(0) = e_g(1)$
h_2	A_5	$\hat{\theta}_1$	$v_g(0) = v_g(1), e_g(0) = e_g(1)+1$	θ_1	$v_g(0) = v_g(1), e_g(0) = e_g(1)+1$
h_1	A_6	$\hat{\phi}_1$	$v_g(0) = v_g(1), e_g(0)+1 = e_g(1)$	ϕ_1	$v_g(0) = v_g(1), e_g(0)+1 = e_g(1)$
h_1	B	$\hat{\mu}_3$	$v_g(0) = v_g(1), e_g(0) = e_g(1)$	μ_3	$v_g(0) = v_g(1), e_g(0) = e_g(1)$
h_1	C	$\hat{\xi}_1$	$v_g(0)+1 = v_g(1), e_g(0) = e_g(1)$	ξ_1	$v_g(0) = v_g(1)+1, e_g(0) = e_g(1)$

Case 2:- $v_{g_1}(0) + 1 = v_{g_1}(1)$.

		$g_1(u_{m+1}) = 0$		$g_1(u_{m+1}) = 1$	
g_1	$P_{t(m+1)}$	g_2	g	g_2	g
h_2	A_2	$\hat{\beta}_2$	$v_g(0) = v_g(1), e_g(0) = e_g(1)+1$	β_1	$v_g(0) = v_g(1), e_g(0) = e_g(1)+1$
h_1	A_3	$\hat{\gamma}_2$	$v_g(0) = v_g(1), e_g(0)+1 = e_g(1)$	γ_1	$v_g(0) = v_g(1), e_g(0)+1 = e_g(1)$
h_2	A_4	δ_2	$v_g(0)+1 = v_g(1), e_g(0) = e_g(1)$	$\hat{\delta}_2$	$v_g(0) = v_g(1)+1, e_g(0) = e_g(1)$
h_2	A_5	$\hat{\theta}_1$	$v_g(0)+1 = v_g(1), e_g(0) = e_g(1)+1$	$\bar{\hat{\theta}}_1$	$v_g(0) = v_g(1)+1, e_g(0) = e_g(1)+1$
h_1	A_6	$\hat{\phi}_1$	$v_g(0)+1 = v_g(1), e_g(0)+1 = e_g(1)$	$\bar{\hat{\phi}}_1$	$v_g(0) = v_g(1)+1, e_g(0)+1 = e_g(1)$
h_1	B	$\hat{\mu}_3$	$v_g(0)+1 = v_g(1), e_g(0) = e_g(1)$	μ_3	$v_g(0) = v_g(1)+1, e_g(0) = e_g(1)$
h_1	C	$\hat{\xi}_3$	$v_g(0) = v_g(1), e_g(0) = e_g(1)$	ξ_1	$v_g(0) = v_g(1), e_g(0) = e_g(1)$

Part IV: Problematic Cases:

In this part, we deal with the problematic cases that arose in Part I.

Problems of Type (o).

For these cases vertex balance is maintained but $\mid e_f(0) - e_f(1) \mid\not\leq 1$. All such cases are the following:

T_m is non-Eulerian		
$e_{g_1}(0)+1=e_{g_1}(1)$	$g_1(u_{m+1})=0$	Type A_5
	$g_1(u_{m+1})=1$	Type A_5
$e_{g_1}(0)=e_{g_1}(1)+1$	$g_1(u_{m+1})=0$	Type A_6
	$g_1(u_{m_1})=1$	Type A_6
$e_{g_1}(0)=e_{g_1}(1)$	$g_1(u_{m+1})=0$	Type C
	$g_1(u_{m+1})=1$	Type C

Since T_m is non-Eulerian, it is cordial, hence there exists a cordial labeling f of T_m. Let the restriction of f to each block $P_{t(i)}$ in T_m be denoted by f_i. Let k be the maximum positive integer such that $P_{t(k)}$ is non-Eulerian, that is $P_{t(k)}$ is of one of the types A_2, A_3, A_4 or B. For each $r > k, P_{t(r)}$ will be Eulerian. Throughout this paper, since we have used only labelings of type A for Eulerian single $t-$ply graphs, therefore in each such block, either $f(u_r)=1, f(u_{r+1})=0$, or $f(u_r)=0, f(u_{r+1})=1$. In fact, if $m-k$ is even,
$f(u_{k+1})=1, f(u_{k+2})=0, f(u_{k+3})=1,\cdots, f(u_m)=0, f(u_{m+1})=1$ or
$f(u_{k+1})=0, f(u_{k+2})=1, f(u_{k+3})=0,\cdots, f(u_m)=1, f(u_{m+1})=0$.
If $m-k$ is odd, then
$f(u_{k+1})=1, f(u_{k+2})=0, f(u_{k+3})=1,\cdots, f(u_m)=1, f(u_{m+1})=0$ or
$f(u_{k+1})=0, f(u_{k+2})=1, f(u_{k+3})=0,\cdots, f(u_m)=0, f(u_{m+1})=1$.

Thus $v_f(0) = \sum_{i=1, i\neq k}^{m} v_{f_i(0)} - x_1 - x_2 + v_{f_k}(0), \cdots\cdots\cdots\cdots (i)$

where x_1 = number of vertices in $\{u_2,\cdots,u_k\}$ which receive the label 0 and x_2 = the number of vertices in $\{u_{k+1},\cdots,u_m\}$ which receive the label 0. Observe that x_1, x_2 together give us the number of "shared" zeroes. Clearly,

$$\begin{aligned} x_2 &= (m-k)/2; \quad \text{when } m-k \text{is even} \\ &= (m-k-1)/2; \quad \text{when } m-k \text{ is odd and } f(u_{k+1})=1 \\ &= (m-k+1)/2; \quad \text{when } m-k \text{ is odd and} f(u_{k+1})=0. \end{aligned}$$

Further, we have $v_f(1) = \sum_{i=1, i\neq k}^{m} v_{f_i(1)} - y_1 - y_2 + v_{f_k}(1), \cdots\cdots\cdots (ii)$

where $y_1 = k - 1 - x_1$ and $y_2 = (m-k) - x_2$. Also,

$$\begin{aligned} e_f(0) &= \sum_{i=1, i\neq k}^{m} e_{f_i}(0) + e_{f_k}(0), \\ e_f(1) &= \sum_{i=1, i\neq k}^{m} e_{f_i}(1) + e_{f_k}(1). \end{aligned}$$

We now give an alternate labeling F for T_m as follows: For each block $P_{t(i)}$, define a new labeling F_i as

$$\begin{aligned} F_i &= f_i, 1 \leq i < k, \\ &= \tilde{f}_i, (k+1) \leq i \leq m. \end{aligned}$$

For $i = k$, if f_k is a labeling of type A(respectively B), then we choose F_k to be an appropriate labeling of type B(respectively A) as described later.

Then $v_F(0) = \sum_{i=1, i\neq k}^{m} v_{f_i(0)} - x_1 - X_2 + v_{F_k}(0), \cdots\cdots\cdots (iii)$

where $v_{F_k}(0)$ depends on F_k and

$$\begin{aligned} X_2 &= (m-k)/2;\ m-k \ \textit{even} \\ &= (m-k-1)/2;\ m-k \ \textit{odd},\ f(u_{k+1}) = 0 \\ &= (m-k+1)/2;\ m-k \ \textit{odd},\ f(u_{k+1}) = 1. \end{aligned}$$

$v_F(1) = \sum_{i=1, i\neq k}^{m} v_{f_i(1)} - y_1 - Y_2 + v_{F_k}(1), \cdots\cdots\cdots (iv)$

where $Y_2 = (m-k) - X_2$. Also,

$$\begin{aligned} e_F(0) &= \sum_{i=1, i\neq k}^{m} e_{f_i}(0) + e_{F_k}(0), \\ e_F(1) &= \sum_{i=1, i\neq k}^{m} e_{f_i}(1) + e_{F_k}(1). \end{aligned}$$

From the above it follows that

$$\begin{aligned} v_F(0) - v_f(0) &= v_{F_k}(0) - v_{f_k}(0) + (x_2 - X_2), \\ v_F(1) - v_f(1) &= v_{F_k}(1) - v_{f_k}(1) + (y_2 - Y_2), \\ e_F(0) - e_f(0) &= e_{F_k}(0) - e_{f_k}(0), \\ e_F(1) - e_f(1) &= e_{F_k}(1) - e_{f_k}(1). \end{aligned}$$

Suppose $P_{t(k)}$ is a graph of **Type** A_2. If f_k is a labeling of type A, then $e_{f_k}(0)+1 = e_{f_k}(1)$. In that case, we have to take F_k to be a labeling of type B, but in any labeling of type B for graphs of Type A_2 (see summary), the number of edges with the label 0 is one more than the number of edges with the label 1. Hence $e_{F_k}(0) = e_{F_k}(1)+1$. Then:
$e_F(0)-e_f(0) = [e_{F_k}(1)+1]-e_{f_k}(0), e_F(1)-e_f(1) = e_{F_k}(1)-e_{f_k}(0)-1$
Hence $e_F(0)-e_F(1) = e_f(0)-e_f(1)+2 \cdots\cdots\cdots (I)$

Suppose f_k is a labeling of type B, then $e_{f_k}(0) = e_{f_k}(1)+1$. In that case, $e_{F_k}(0)+1 = e_{F_k}(1)$. Then:
$e_F(0)-e_f(0) = e_{F_k}(0)-[e_{f_k}(1)+1]$, $e_F(1)-e_f(1) = [e_{F_k}(0)+1]-e_{f_k}(1)$
Hence $[e_F(0)-e_F(1)]+2 = [e_f(0)-e_f(1)] \cdots\cdots\cdots (II)$

Suppose $P_{t(k)}$ is a graph of **Type** A_3. If f_k is a labeling of type A, then $e_{f_k}(0) = e_{f_k}(1)+1$. In that case, $e_{F_k}(0)+1 = e_{F_k}(1)$. Then:
$e_F(0)-e_f(0) = e_{F_k}(0)-[e_{f_k}(1)+1]$, $e_F(1)-e_f(1) = [e_{F_k}(0)+1]-e_{f_k}(1)$
Hence $[e_F(0)-e_F(1)]+2 = e_f(0)-e_f(1) \cdots\cdots\cdots (III)$

Suppose f_k is a labeling of type B, then $e_{f_k}(0)+1 = e_{f_k}(1)$. In that case, $e_{F_k}(0) = e_{F_k}(1)+1$. Then:
$e_F(0)-e_f(0) = [e_{F_k}(1)+1]-e_{f_k}(0)$, $e_F(1)-e_f(1) = e_{F_k}(1)-[e_{f_k}(0)+1]$
Hence $e_F(0)-e_F(1) = [e_f(0)-e_f(1)]+2 \cdots\cdots\cdots (IV)$

Suppose $P_{t(k)}$ is a graph of **Type** A_4. If f_k is a labeling of type A, then $e_{f_k}(0) = e_{f_k}(1)$. In that case, $e_{F_k}(0)+2 = e_{F_k}(1)$. Then:
$e_F(0)-e_f(0) = e_{F_k}(0)-e_{f_k}(0)$, $e_F(1)-e_f(1) = [e_{F_k}(0)+2]-e_{f_k}(0)$
Hence $[e_F(0)-e_F(1)]+2 = e_f(0)-e_f(1) \cdots\cdots\cdots (V)$

Suppose f_k is a labeling of type B, then
$e_F(0)-e_F(1) = [e_f(0)-e_f(1)]+2 \cdots\cdots\cdots (VI)$

Suppose $P_{t_{(k)}}$ is a graph of **Type** B. If f_k is a labeling of type B for it, then $e_{f_k}(0) = e_{f_k}(1)$. In that case, $e_{F_k}(0) = e_{F_k}(1)+2$. Then:
$e_F(0)-e_f(0) = [e_{F_k}(1)+2]-e_{f_k}(0)$, $e_F(1)-e_f(1) = e_{F_k}(1)-e_{f_k}(0)$
Hence $e_F(0)-e_F(1) = [e_f(0)-e_f(1)]+2 \cdots\cdots\cdots (VII)$

Suppose f_k is a labeling of type A for it, then $e_{f_k}(0) = e_{f_k}(1)+2$. In that case, $e_{F_k}(0) = e_{F_k}(1)$. Then:
$e_F(0)-e_f(0) = e_{F_k}(1)-e_{f_k}(1)-2$, $e_F(1)-e_f(1) = e_{F_k}(1)-e_{f_k}(1)$
Hence $[e_F(0)-e_F(1)]+2 = e_f(0)-e_f(1) \cdots\cdots\cdots (VIII)$

In the problematic case when $P_{t(m+1)}$ is of Type A_5, in T_m the existing edge label condition is $e_f(0)+1 = e_f(1)$. Then $e_F(1) = e_F(0)+3$ or $e_F(0) = e_F(1)+1$. In the first case, choose the labeling g for A_5 in which $e_g(0) = e_g(1)+3$, while in the latter choose g such that $e_g(0)+1 = e_g(1)$.

In the case when $P_{t(m+1)}$ is of Type A_6, we have $e_f(0) = e_f(1)+1$. Then $e_F(1)+3 = e_F(0)$ or $e_F(0)+1 = e_F(1)$. In the first case, choose the labeling g for A_6 in which $e_g(0)+3 = e_g(1)$, while in the latter choose g such that $e_g(0) = e_g(1)+1$.

In the case when $P_{t(m+1)}$ is of Type C, we have $e_f(0) = e_f(1)$. Then $e_F(1)+2 = e_F(0)$ or $e_F(0)+2 = e_F(1)$.

In the first case, choose the labeling g for C in which $e_g(0)+2=e_g(1)$, while in the latter choose g such that $e_g(0)=e_g(1)+2$.

In defining the alternate labeling F for T_m, we may in all likelihood, have disturbed the original vertex label balance so that $| v_F(0)-v_F(1) | \leq 1$ may NOT be satisfied. We now determine $v_F(0)-v_F(1)$ in order to restore the vertex label balance.

From equations $(i),(ii),(iii)$ and (iv) we have

$v_F(0)-v_F(1)=[v_f(0)-v_f(1)]+(x_2-X_2)+(Y_2-y_2)+[v_{F_k}(0)-v_{F_k}(1)]$
$-[v_{f_k}(0)-v_{f_k}(1)]\cdots\cdots\cdots\cdots(A)$

We now use this equation in what follows.

Case I : $P_{t(k)}$ is of Type A_2 There are following possibilities:

Case 1: $f(u_{k+1})=1$

Case 1(a) If $m-k$ is even, then it is immediate that $x_2=X_2$ and $y_2=Y_2$, so that from equation (A) it follows that

$v_F(0)-v_F(1)=[v_f(0)-v_f(1)]+[v_{F_k}(0)-v_{F_k}(1)]-[v_{f_k}(0)-v_{f_k}(1)]\cdots(B)$

Case 1(a)(i): $f(u_k)=0$

Case 1(a)(i)(α): $v_{f_k}(0)=v_{f_k}(1)$.

In $P_{t(k)}$ we choose F_k such that $v_{F_k}(0)=v_{F_k}(1), F_k(u_k)=0=F_k(u_{k+1})$. Then from (B), we have $v_F(0)-v_F(1)=v_f(0)-v_f(1)$. Hence the original vertex label balance is maintained.

Case 1(a)(i)(β): $v_{f_k}(0)=v_{f_k}(1)+2$.

In $P_{t(k)}$ we choose F_k such that $v_{F_k}(0)=v_{F_k}(1), F_k(u_k)=0=F_k(u_{k+1})$. Then from (B), we have $v_F(0)-v_F(1)=v_f(0)-v_f(1)$. Hence the original vertex label balance is maintained.

Case 1(a)(i)(γ): $v_{f_k}(0)+2=v_{f_k}(1)$.

In $P_{t(k)}$ we choose F_k such that $v_{F_k}(0)=v_{F_k}(1)+2, F_k(u_k)=0=F_k(u_{k+1})$. Then from (B), we have $v_F(0)-v_F(1)=v_f(0)-v_f(1)+2$. Here the original vertex label balance is not maintained. We come back to this case later.

Case 1(a)(ii): $f(u_k)=1$

Case 1(a)(ii)(α): $v_{f_k}(0)=v_{f_k}(1)$.

In $P_{t(k)}$ we choose F_k such that $v_{F_k}(0)=v_{F_k}(1), F_k(u_k)=1, F_k(u_{k+1})=0$. Then from (B), we have $v_F(0)-v_F(1)=v_f(0)-v_f(1)$. Hence the original vertex label balance is maintained.

Case 1(a)(ii)(β): $v_{f_k}(0)+2=v_{f_k}(1)$.

In $P_{t(k)}$ we choose F_k such that $v_{F_k}(0)+2=v_{F_k}(1), F_k(u_k)=1, F_k(u_{k+1})=0$. Then from (B), we have $v_F(0)-v_F(1)=v_f(0)-v_f(1)$. Hence the original vertex label balance is maintained.

Case 1(b): $m-k$ is odd.

If $m-k$ is odd, then $x_2=Y_2$ and $y_2=X_2$, so that from equation (A) it follows that

$v_F(0)-v_F(1)=[v_f(0)-v_f(1)]+[v_{F_k}(0)-v_{F_k}(1)]-[v_{f_k}(0)-v_{f_k}(1)]-2\cdots(C)$.

Case 1(b)(i): $f(u_k)=0$

Case 1(b)(i)(α): $v_{f_k}(0) = v_{f_k}(1)$.
In $P_{t(k)}$ we choose F_k such that $v_{F_k}(0) = v_{F_k}(1)+2, F_k(u_k) = 0 = F_k(u_{k+1})$. Then from (C), we have $v_F(0) - v_F(1) = v_f(0) - v_f(1)$. Hence the original vertex label balance is maintained.
Case 1(b)(i)(β): $v_{f_k}(0) = v_{f_k}(1) + 2$.
In $P_{t(k)}$ we choose F_k such that $v_{F_k}(0) = v_{F_k}(1)+2, F_k(u_k) = 0 = F_k(u_{k+1})$. Then from (C), we have $v_F(0)-v_F(1) = v_f(0)-v_f(1)-2$. Here the original vertex label balance is not maintained. We come back to this case later.
Case 1(b)(i)(γ): $v_{f_k}(0) + 2 = v_{f_k}(1)$.
In $P_{t(k)}$ we choose F_k such that $v_{F_k}(0) = v_{F_k}(1), F_k(u_k) = 0 = F_k(u_{k+1})$. Then from (C), we have $v_F(0) - v_F(1) = v_f(0) - v_f(1)$. Hence the original vertex label balance is maintained.

Case 1(b)(ii): $f(u_k) = 1$
Case 1(b)(ii)(α): $v_{f_k}(0) = v_{f_k}(1)$.
In $P_{t(k)}$ we choose F_k such that $v_{F_k}(0) = v_{F_k}(1), F_k(u_k) = 1, F_k(u_{k+1}) = 0$. Then from (C), we have $v_F(0) - v_F(1) = v_f(0) - v_f(1) - 2$. Hence the original vertex label balance is not maintained. As before we come back to this case later.
Case 1(b)(ii)(β): $v_{f_k}(0) + 2 = v_{f_k}(1)$.
In $P_{t(k)}$ we choose F_k such that $v_{F_k}(0) = v_{F_k}(1), F_k(u_k) = 1, F_k(u_{k+1}) = 0$. Then from (C), we have $v_F(0) - v_F(1) = v_f(0) - v_f(1)$. Hence the original vertex label balance is maintained.
Case 2: $f(u_{k+1}) = 0$
Case 2(a): If $m - k$ is even, we use equation (B) as before.
Case 2(a)(i): $f(u_k) = 0$
Case 2(a)(i)(α): $v_{f_k}(0) = v_{f_k}(1)$.
In $P_{t(k)}$ we choose F_k such that $v_{F_k}(0) = v_{F_k}(1), F_k(u_k) = 0, F_k(u_{k+1}) = 1$. Then from (B), we have $v_F(0) - v_F(1) = v_f(0) - v_f(1)$. Hence the original vertex label balance is maintained.
Case 2(a)(i)(β): $v_{f_k}(0) = v_{f_k}(1) + 2$.
In $P_{t(k)}$ we choose F_k such that $v_{F_k}(0) = v_{F_k}(1)+2, F_k(u_k) = 0, F_k(u_{k+1}) = 1$. Then from (B), we have $v_F(0)-v_F(1) = v_f(0)-v_f(1)$. Hence the original vertex label balance is maintained.
Case 2(a)(ii): $f(u_k) = 1$
Case 2(a)(ii)(α): $v_{f_k}(0) = v_{f_k}(1)$.
In $P_{t(k)}$ we choose F_k such that $v_{F_k}(0) = v_{F_k}(1), F_k(u_k) = 1, F_k(u_{k+1}) = 1$. Then from (B), we have $v_F(0) - v_F(1) = v_f(0) - v_f(1)$. Hence the original vertex label balance is maintained.
Case 2(a)(i)(β): $v_{f_k}(0) = v_{f_k}(1) + 2$.
In $P_{t(k)}$ we choose F_k such that $v_{F_k}(0) = v_{F_k}(1), F_k(u_k) = 1 = F_k(u_{k+1})$. Then from (B), we have $v_F(0)-v_F(1) = v_f(0)-v_f(1)-2$. Here the original vertex label balance is not maintained. We come back to this case later.

Case 2(a)(ii)(γ): $v_{f_k}(0) + 2 = v_{f_k}(1)$.
In $P_{t(k)}$ we choose F_k such that $v_{F_k}(0)+2 = v_{F_k}(1), F_k(u_k) = 1 = F_k(u_{k+1})$. Then from (B), we have $v_F(0) - v_F(1) = v_f(0) - v_f(1)$. Hence the original vertex label balance is maintained.
Case 2(b): If $m - k$ is odd, from equation (A), it follows that
$v_F(0) - v_F(1) = [v_f(0) - v_f(1)] + [v_{F_k}(0) - v_{F_k}(1)] - [v_{f_k}(0) - v_{f_k}(1)] + 2 \cdots (D)$.
Case 2(b)(i): $f(u_k) = 0$
Case 2(b)(i)(α): $v_{f_k}(0) = v_{f_k}(1)$.
In $P_{t(k)}$ we choose F_k such that $v_{F_k}(0)+2 = v_{F_k}(1), F_k(u_k) = 0, F_k(u_{k+1}) = 1$. Then from (D), we have $v_F(0)-v_F(1) = v_f(0)-v_f(1)$. Hence the original vertex label balance is maintained.
Case 2(b)(i)(β): $v_{f_k}(0) = v_{f_k}(1) + 2$.
In $P_{t(k)}$ we choose F_k such that $v_{F_k}(0) = v_{F_k}(1), F_k(u_k) = 0, F_k(u_{k+1}) = 1$. Then from (D), we have $v_F(0) - v_F(1) = v_f(0) - v_f(1)$. Here the original vertex label balance is maintained.
Case 2(b)(ii): $f(u_k) = 1$
Case 2(b)(ii)(α): $v_{f_k}(0) = v_{f_k}(1)$.
In $P_{t(k)}$ we choose F_k such that $v_{F_k}(0)+2 = v_{F_k}(1), F_k(u_k) = 1 = F_k(u_{k+1})$. Then from (D), we have $v_F(0) - v_F(1) = v_f(0) - v_f(1)$. Hence the original vertex label balance is maintained.
Case 2(a)(i)(β): $v_{f_k}(0) = v_{f_k}(1) + 2$.
In $P_{t(k)}$ we choose F_k such that $v_{F_k}(0) = v_{F_k}(1), F_k(u_k) = 1 = F_k(u_{k+1})$. Then from (D), we have $v_F(0) - v_F(1) = v_f(0) - v_f(1)$. Hence the original vertex label balance is maintained.
Case 2(b)(ii)(γ): $v_{f_k}(0) + 2 = v_{f_k}(1)$.
In $P_{t(k)}$ we choose F_k such that $v_{F_k}(0)+2 = v_{F_k}(1), F_k(u_k) = 1 = F_k(u_{k+1})$. Then from (D), we have $v_F(0)-v_F(1) = v_f(0)-v_f(1)+2$. Hence the original vertex label balance is not maintained. We come back to this case later.
Case II: If $P_{t(m+1)}$ is of Type A_3
The discussion is similar to that of Case I.
Case III: $P_{t(k)}$ is of Type A_4
Case 1: $f(u_{k+1}) = 1$
Case 1(a): If $m - k$ is even, we can use equation (B).
Case 1(a)(i): $f(u_k) = 0$.
Case 1(a)(i)(α): $v_{f_k}(0) = v_{f_k}(1) + 1$.
In $P_{t(k)}$ we choose F_k such that $v_{F_k}(0) = v_{F_k}(1)+1, F_k(u_k) = 0 = F_k(u_{k+1})$. Then from (B), we have $v_F(0) - v_F(1) = v_f(0) - v_f(1)$.
Case 1(a)(i)(β): $v_{f_k}(0) + 1 = v_{f_k}(1)$.
In $P_{t(k)}$ we choose F_k such that $v_{F_k}(0) = v_{F_k}(1)+1, F_k(u_k) = 0 = F_k(u_{k+1})$. Then from (B), we have $v_F(0)-v_F(1) = v_f(0)-v_f(1)+2$. Hence the original vertex label balance is not maintained. We come back to this case later.
Case 1(a)(ii): $f(u_k) = 1$

Case 1(a)(ii)(α): $v_{f_k}(0)+1=v_{f_k}(1)$.
In $P_{t(k)}$ we choose F_k such that $v_{F_k}(0)+1=v_{F_k}(1), F_k(u_k)=1, F_k(u_{k+1})=0$. Then from (B), we have $v_F(0)-v_F(1)=v_f(0)-v_f(1)$.
Case 1(b): If $m-k$ is odd, we can use equation C.
Case 1(b)(i): $f(u_k)=0$
Case 1(b)(i)(α): $v_{f_k}(0)=v_{f_k}(1)+1$.
In $P_{t(k)}$ we choose F_k such that $v_{F_k}(0)=v_{F_k}(1)+1, F_k(u_k)=0=F_k(u_{k+1})$. Then from (C), we have $v_F(0)-v_F(1)=v_f(0)-v_f(1)-2$. Hence the original vertex label balance is not maintained. We come back to this case later.
Case 1(b)(i)(β): $v_{f_k}(0)+1=v_{f_k}(1)$.
In $P_{t(k)}$ we choose F_k such that $v_{F_k}(0)=v_{F_k}(1)+1, F_k(u_k)=0=F_k(u_{k+1})$. Then from (C), we have $v_F(0)-v_F(1)=v_f(0)-v_f(1)$.
Case 1(b)(ii): $f(u_k)=1$
Case 1(b)(ii)(α): $v_{f_k}(0)+1=v_{f_k}(1)$.
In $P_{t(k)}$ we choose F_k such that $v_{F_k}(0)=v_{F_k}(1)+1, F_k(u_k)=1, F_k(u_{k+1})=0$. Then from (C), we have $v_F(0)-v_F(1)=v_f(0)-v_f(1)$.
Case 2: $f(u_{k+1})=0$.
Case 2(a): If $m-k$ is even, we use equation (B) as before.
Case 2(a)(i): $f(u_k)=0$.
Case 2(a)(i)(α): $v_{f_k}(0)=v_{f_k}(1)+1$.
In $P_{t(k)}$ we choose F_k such that $v_{F_k}(0)=v_{F_k}(1)+1, F_k(u_k)=0, F_k(u_{k+1})=1$. Then from (B), we have $v_F(0)-v_F(1)=v_f(0)-v_f(1)$. Hence the original vertex label balance is maintained.
Case 2(a)(ii): $f(u_k)=1$.
Case 2(a)(ii)(α): $v_{f_k}(0)=v_{f_k}(1)+1$.
In $P_{t(k)}$ we choose F_k such that $v_{F_k}(0)+1=v_{F_k}(1), F_k(u_k)=1, F_k(u_{k+1})=1$. Then from (B), we have $v_F(0)-v_F(1)=v_f(0)-v_f(1)-2$. Hence the original vertex label balance is not maintained and we come back to this case later.
Case 2(a)(ii)(β): $v_{f_k}(0)+1=v_{f_k}(1)$.
In $P_{t(k)}$ we choose F_k such that $v_{F_k}(0)+1=v_{F_k}(1), F_k(u_k)=1=F_k(u_{k+1})$. Then from (B), we have $v_F(0)-v_F(1)=v_f(0)-v_f(1)$. Hence the original vertex label balance is maintained.
Case 2(b): If $m-k$ is odd, we use equation (D).
Case 2(b)(i): $f(u_k)=0$
Case 2(b)(i)(α): $v_{f_k}(0)=v_{f_k}(1)+1$.
In $P_{t(k)}$ we choose F_k such that $v_{F_k}(0)+1=v_{F_k}(1), F_k(u_k)=0, F_k(u_{k+1})=1$. Then from (D), we have $v_F(0)-v_F(1)=v_f(0)-v_f(1)$. Hence the original vertex label balance is maintained.
Case 2(b)(ii): $f(u_k)=1$.
Case 2(b)(ii)(α): $v_{f_k}(0)=v_{f_k}(1)+1$.
In $P_{t(k)}$ we choose F_k such that $v_{F_k}(0)+1=v_{F_k}(1), F_k(u_k)=1=F_k(u_{k+1})$. Then from (D), we have $v_F(0)-v_F(1)=v_f(0)-v_f(1)$. Hence the original

vertex label balance is maintained.
Case 2(b)(ii)(β): $v_{f_k}(0) + 1 = v_{f_k}(1)$.
In $P_{t(k)}$ we choose F_k such that $v_{F_k}(0)+1 = v_{F_k}(1), F_k(u_k) = 1 = F_k(u_{k+1})$. Then from (D), we have $v_F(0)-v_F(1) = v_f(0)-v_f(1)+2$. Hence the original vertex label balance is not maintained. We come back to this case later.
Case IV: If $P_{t(m+1)}$ is of Type B
The discussion is similar to that of Case III.

We now take a look at those problems where we were unable to restore the original vertex label condition. The problematic cases can be classified as: (1) $v_F(0) - v_F(1) = v_f(0) - v_f(1) - 2, F(u_{m+1}) = 1$.
(2) $v_F(0) - v_F(1) = v_f(0) - v_f(1) + 2, F(u_{m+1}) = 0$.
In problem (1),
(i) If $v_f(0) = v_f(1)$, then $v_F(0) + 2 = v_F(1)$.
(ii) If $v_f(0) = v_f(1) + 1$, then $v_F(0) + 1 = v_F(1)$.
(iii) If $v_f(0) + 1 = v_f(1)$, then $v_F(0) + 3 = v_F(1)$.
Likewise in problem (2),
(i) If $v_f(0) = v_f(1)$, then $v_F(0) = v_F(1) + 2$.
(ii) If $v_f(0) = v_f(1) + 1$, then $v_F(0) = v_F(1) + 3$.
(iii) If $v_f(0) + 1 = v_f(1)$, then $v_F(0) = v_F(1) + 1$.

Rather than trying to restore the vertex label balance for the above cases in T_m, we choose an appropriate labeling for $P_{t(m+1)}$.
We will be in a position to do this if we have the following labelings for graphs of Type A_5, A_6 and C.

Labelings required for graphs of Type A_5.
(i) $v_f(0) = v_f(1) + 1, e_f(0) + 1 = e_f(1), f(u) = 1$
(ii) $v_f(0) + 1 = v_f(1), e_f(0) + 1 = e_f(1), f(u) = 0$.
(iii) $v_f(0) = v_f(1) + 1, e_f(0) = e_f(1) + 3, f(u) = 1$
(iv) $v_f(0) + 1 = v_f(1), e_f(0) = e_f(1) + 3, f(u) = 0$.

Labelings required for graphs of Type A_6.
(i) $v_f(0) = v_f(1) + 1, e_f(0) = e_f(1) + 1, f(u) = 1$
(ii) $v_f(0) + 1 = v_f(1), e_f(0) = e_f(1) + 1, f(u) = 0$.
(iii) $v_f(0) = v_f(1) + 1, e_f(0) + 3 = e_f(1), f(u) = 1$
(iv) $v_f(0) + 1 = v_f(1), e_f(0) + 3 = e_f(1), f(u) = 0$.

Labelings required for graphs of Type C.
(i) $v_f(0) = v_f(1) + 2, e_f(0) = e_f(1) + 2, f(u) = 1$
(ii) $v_f(0) = v_f(1) + 2, e_f(0) + 2 = e_f(1), f(u) = 1$
(iii) $v_f(0) + 2 = v_f(1), e_f(0) = e_f(1) + 2, f(u) = 0$
(iv) $v_f(0) + 2 = v_f(1), e_f(0) + 2 = e_f(1), f(u) = 0$.

The required labelings for graphs of Types A_5, A_6 and C are already available. Hence in these cases also, there is a cordial labeling of $P_{t(m+1)}$.

Problems of Type $\otimes$.

We now deal with those cases in which $| v_f(0) - v_f(1) | > 1$,
$| e_f(0) - e_f(1) | > 1$. We list these cases below:

Condition on vertex and edge labels in $T_m(g_1)$	$g(u_{m+1})$	$P_{t(m+1)}$
T_m is non-Eulerian		
$v_{g_1}(0)+1=v_{g_1}(1)$ $e_{g_1}(0)=e_{g_1}(1)$	0	C
$v_{g_1}(0)=v_{g_1}(1)+1$ $e_{g_1}(0)=e_{g_1}(1)$	1	C

In the above case that is when T_m is non-Eulerian and $P_{t(m+1)}$ is of Type C, then use the alternate labeling F of T_m. Then either
$v_F(0)-v_F(1)=v_f(0)-v_f(1)$ or
$v_F(0)-v_F(1)=v_f(0)-v_f(1)-2, F(u_{m+1})=1$; or
$v_F(0)-v_F(1)=v_f(0)-v_f(1)+2, F(u_{m+1})=0$.
To resolve these problems we need the following labelings for graphs of Type C : (i) $v_f(0)=v_f(1), e_f(0)=e_f(1)+2$
(ii) $v_f(0)=v_f(1), e_f(0)+2=e_f(1)$
(iii) $v_f(0)=v_f(1)+2, e_f(0)=e_f(1)+2$
(iv) $v_f(0)=v_f(1)+2, e_f(0)+2=e_f(1)$
(v) $v_f(0)+2=v_f(1), e_f(0)=e_f(1)+2$
(vi) $v_f(0)+2=v_f(1), e_f(0)+2=e_f(1)$
Since, we have these labelings for graphs of Type C, in this case too, $P_{t(m+1)}$ is cordial. Hence the proof.

References:

[1] I Cahit, Cordial Graphs: A weaker version of graceful and harmonious graphs, Ars Combinatoria, 23(1987), 201-207.

[2] K. R. Parthasarathy, Basic Graph Theory, Tata McGraw-Hill Publishing Co. 1994

[3] Mahesh Andar, Samina Boxwala and N.B.Limaye, On the Cordiality of the t-ply graph, To appear in Ars Combinatoria

[4] S.C.Shee, Y.S. Ho, The cordiality of one point union of n copies of a graph, Discrete Math, 117(1993), 225-243.

Samina Boxwala
Department of Mathematics
N. Wadia College,
Pune,411001.INDIA

N. B. Limaye
Department of Mathematics
University of Mumbai
Vidyanagari, Mumbai 400098,INDIA
limaye@math.mu.ac.in