Spanning Tree Factorizations of Complete Graphs

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ABSTRACT. We examine decompositions of complete graphs with an even number of vertices into isomorphic spanning trees. We develop a cyclic factorization of K_{2n} into non-symmetric spanning trees. Our factorization method are based on flexible q-labeling and blended labeling, introduced by Froncek. In this paper we presente several infinite classes of non-symmetric trees which have flexible q-labeling or blended labeling.

1. INTRODUCTION

Let G be a graph with at most n vertices. We say that the complete graph K_n has a *G-decomposition* if there are subgraphs $G_0, G_1, G_2, ..., G_s$, all isomorphic to G, such that each edge of K_n belongs to exactly one G_i. Then we say that G divides K_n, and write $G|K_n$. The decomposition is *cyclic* if there exists an ordering $(x_1, x_2, ..., x_n)$ of the vertices of K_n and isomorphisms $\phi_i, i = 1, 2, ..., s$ from G_0 to G_i such that $\phi_i(x_j) = x_{i+j}$ for every $j = 1, 2, ..., n$, where the subscript are taken modulo n. If G has exactly n vertices and none of them is isolated, then G is called a *factor* and the decomposition is called *G-factorization* of K_n.

Decompositions and factorizations of K_n into trees were studied by several authors, but to our surprise, little is known about factorizations of K_{2n} into isomorphic spanning trees, other than Hamiltonian paths. Notice that K_{2n+1} cannot be factorized into spanning trees T_{2n+1}, because $|E(T_{2n+1})| = 2n$ does not divide $|E(K_{2n+1})| = n(2n+1)$.

Many factorization methods are based on graph labelings, where a *labeling* of G with at most $2n+1$ vertices is an injection $\lambda : V(G) \to S, S \subseteq \{0, 1, ..., 2n\}$ and labels of vertices u, v (denote $\lambda(u), \lambda(v)$) induce uniquely the label l of the edge $e = uv$. If a graph G has n vertices and labels of

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vertices are $0, 1, ..., n-1$ then we define the label (also called the *length*) of an edge $e = uv$ as $l(e) = min\{|\lambda(u) - \lambda(v), n - |\lambda(u) - \lambda(v)|\}$.

Eldergill [1] introduced a necessary and sufficient condition for factorization of K_{2n} into symmetric spanning trees. In [1] a symmetric spanning tree is a tree T with an edge $e = uv$ and an automorphism $\alpha : V(T) \to V(T)$ such that $\alpha(u) = v$ and $\alpha(v) = u$. He used *symmetric ρ-labeling*, which is a special case of *ρ-labeling* introduced by Rosa in [5].

We focus our research on factorizations of K_{2n} into non-symmetric spanning trees. We use new types of vertex labelings, namely *flexible q-labeling* (see Definition 2.1) and *blended ρ-labeling* (see Definition 2.3). These labelings were introduced in [2]–[4].

In this article, we investigate the relationship between flexible q-labeling and blended ρ-labeling. We also study an infinite class of the trees called brooms and non-symmetric trees with diameter 4.

2. Definitions and notations

Fronček in [3] proved that every tree on $2n$ vertices with a flexible q-labeling admits *2-cyclic factorization* of K_{2n}, for n odd. For n even a modification of flexible q-labeling has to be used (see [2]). In [4] Fronček proved that every tree with blended ρ-labeling admits *bicyclic factorization* of K_{2n}, again for n odd.

We present here the definitions of these notions.

Since a vertex labeling is an injection, we always identify a vertex $u \in V(G)$ with its label $\lambda(u) \in S$ in this paper.

Definition 2.1. Let G be a graph with $2n-1$ edges and at most $2n$ vertices and
$\lambda : V(G) \to \{0, 1, 2, ..., 2n-1\}$ be an injection. λ is called *flexible q-labeling* if

(i) there is exactly one edge of length n,
(ii) for each m, $1 \le m \le n-1$, there are exactly two edges o length m, and
(iii) if $(r, r+m)$ with $1 \le m \le n-1$ is an edge of G, then the other edge of length m in G is $(r+2s+1, r+m+2s+1)$ for some $s, 0 \le s \le n-1$, where the labels are taken modulo $2n$.

If $(r, r+m)$ is an edge of length m, then the vertex r is called the *origin* and the vertex $r+m$ is called the *terminus*.

Definition 2.2. Let G be a graph with at most n vertices. We say that the complete graph K_n has a *2-cyclic G*-decomposition if there are subgraphs $G_0, G_1, ..., G_s$, all isomorphic to G, such that each edge of K_n belongs to exactly one G_i and there exists an ordering $(x_1, x_2, ..., x_n)$ of vertices of K_n and isomorphism ϕ from G_i to G_{i+1}, $i = 0, 1, ..., s-1$, such that

$\phi(x_j) = x_{2+j}$ for every $j = 1, 2, ..., n$, where the subscripts are taken modulo n.

Definition 2.3. Let G be a graph with $2n - 1 = 4t + 1$ edges, $V(G) = V_0 \cup V_1, V_0 \cap V_1 = \emptyset$ and $|V_0| = |V_1| = 2t + 1$. Let λ be an injection, $\lambda : V_i \to \{0_i, 1_i, 2_i, ..., (2t)_i\}, i = 0, 1$. We define the *pure length* of an edge (x_i, y_i) with $x_i, y_i \in V_i, i \in \{0, 1\}$ as $l_{ii}(x_i, y_i) = min\{|\lambda(x_i) - \lambda(y_i)|, 2t + 1 - |\lambda(x_i) - \lambda(y_i)|\}$ and the *mixed length* of an edge (x_0, y_1) as $l_{01}(x_0, y_1) = \lambda(y_1) - \lambda(x_0)$ modulo $2t + 1$ for $x_0 \in V_0, y_1 \in V_1$. We say that G has a *blended ρ-labeling* if

(i) $\{l_{ii}(x_i, y_i) | (x_i, y_i) \in E(G)\} = \{1, 2, ..., t\}$ for $i = 0, 1$,

(ii) $\{l_{01}(x_0, y_1) | (x_0, y_1) \in E(G)\} = \{0, 1, ..., 2t\}$.

The edges (x_i, y_i) are called *pure edges* or *(00)-edges* for $i = 0$ and *(11)-edges* for $i = 1$. The edges (x_0, y_1) are called *mixed edges* or *(01)-edges.*

Definition 2.4. Let G be a graph with at most $4t+2$ vertices. We say that the complete graph K_{4t+2} has a *bicyclic G-decomposition* if there are subgraphs $G_0, G_1, ..., G_s$, all isomorphic to G, such that each edge of K_{4t+2} belongs to exactly one G_i and there exists an ordering $(x_1, x_2, ..., x_{2t+1}, y_1, y_2, ..., y_{2t+1})$ of the vertices of K_{4t+2} and isomorphisms $\phi_i, i = 1, 2, ..., s$ from G_0 to G_i such that $\phi_i(x_j) = x_{i+j}$ and $\phi_i(y_j) = y_{i+j}$ for every $j = 1, 2, ..., 2t+1$, where the subscripts are taken modulo $2k + 1$.

3. Flexible q-labeling and blended ρ-labeling

In this section we present and prove useful lemmas and theorems, which we will need later.

Theorem 3.1. *Let T be a tree on $2n$ vertices, n is odd, which allows flexible q-labeling and $V(T) = V_0 \cup V_1$, where $V_0 = \{0, 2, ..., 2n - 2\}$, $V_1 = \{1, 3, ..., 2n - 1\}$. Then $\sum_{i \in V_0} deg(i) = \sum_{j \in V_1} deg(j) = 2n - 1$.*

Proof. Let

$$(1) \qquad \sum_{j \in V_1} deg(j) = m, \sum_{i \in V_0} deg(i) = k, m \neq k.$$

We know that if T has a flexible q-labeling, then there exists a 2-cyclic T-factorization of K_{2n}. Denote the factors of this factorization $T_0, T_1, ..., T_{n-1}$, where $T_s \cong T$ for every $s \in \{0, 1, ..., n - 1\}$.

If i is an arbitrary vertex from T_0, then the vertex $i + 2s$ is its copy in T_{2s} with the same degree and parity. Further it holds that $\sum_{s=0}^{n-1} deg_{T_s}(i) = 2n - 1 = deg_{K_{2n}}(i)$. It follows that $n \sum_{i \in V_0} deg_T(i) = \sum_{i \in V_0} deg_{K_{2n}}(i)$ and $n \sum_{j \in V_1} deg_T(i) = \sum_{j \in V_1} deg_{K_{2n}}(j)$ and after substitution from (1) we obtain $nk = \sum_{i \in V_0} deg_{K_{2n}}(i)$, $nm = \sum_{j \in V_1} deg_{K_{2n}}(j)$, and therefore

$\sum_{i\in V_0} deg_{K_{2n}}(i) \neq \sum_{j\in V_1} deg_{K_{2n}}(j)$, which is not true, because the sums of degrees of all even and odd vertices in K_{2n} cannot be distinct integers. □

Lemma 3.2. *Every vertex in a tree T on $2n$ vertices with a flexible q-labeling must be incident to at most $(n-1)/2$ edges of even lengths.*

Proof. Let $n = 2t+1$ and $v \in V(T)$ be incident to m edges of even lengths, $m > t$, and let without loss of generality $\lambda(v)$ be even. Since G contains exactly t edges of different even lengths, v has to incident to at least one pair of edges of the same even length but then two edges in $E(T)$ with the same length have origins of the same parity, which is impossible, because T has a flexible q-labeling. □

Theorem 3.3. *A tree T with $2n-1$ edges for an odd n has a flexible q-labeling if and only if it has a blended ρ-labeling.*

Proof. We have T with $2n-1 = 4t+1$ edges and with a blended ρ-labeling.

Let $V(T) = V_0 \cup V_1$, $V_0 = \{0_0, 1_0, ..., (2t)_0\}$ and $V_1 = \{0_1, 1_1, ..., (2t)_1\}$ and l be a length of an edge in $E(T)$.

Now we define a bijection φ on $V(T)$ such that $\varphi(i_0) = 2i$ for $i_0 \in V_0$ and $\varphi(i_1) = 2i+1$. It is obvious that φ is an automorphism. We denote this automorphic tree T'. If we show that T' has a flexible q-labeling, the proof is done.

Now we compare lengths of corresponding edges from $E(T)$ and $E(T')$. Let us note that l' is a length of an edge in T'.

(a) Let $i_0, j_0 \in V_0, i = j+m, m > 0$.

If $m \leq t$, then $l(i_0, j_0) = i - j = i - i + m = m$ and $l'(\varphi(i_0), \varphi(j_0)) = l'(2i, 2(i-m)) = 2m$,

If $m > t$, then $l(i_0, j_0) = l(i_0, (i-m)_0) = 2t+1-(i-(i-m)) = 2t+1-m$ and $l'(\varphi(i_0), \varphi(j_0)) = l'(i, i-m) = 4t+2-(2i-2(i-m)) = 4t+2-2m$.

(b) Let $i_1, j_1 \in V_1, i = j+m, m > 0$.

If $m \leq t$, then $l(i_1, j_1) = i-j = i-(i-m) = m$ and $l'(\varphi(i_1), \varphi(j_1)) = l'(2i+1, 2j+1) = l'(2i+1, 2i-2m+1) = 2i+1-(2i-2m+1) = 2m$.

If $m > t$, then $l(i_1, j_1) = l(i_1, (i-m)_1) = 2t+1-(i-(i-m)) = 2t+1-m$ and $l'(\varphi(i_1), \varphi(j_1)) = l'(2i+1, 2j+1) = l'(2i+1, 2i-2m+1) = 4t+2-(2i+1-(2i-2m+1)) = 4t+2-2m$.

The images of the edges (u, v), where $u, v \in V_0$ (or $u, v \in V_1$), are all edges with the lengths $2, 4, ..., 2t$ and their origins are even (or odd).

(c) Let $x \in V_0, y \in V_1$

We describe every edge (x, y) of length $m, 0 \leq m \leq 2t$ either

$$(i_0, (i+m)_1), i \leq 2t - l \tag{2}$$

or

(3) $$((j-m+2t+1)_0, j_1), j \leq m-1.$$

First we show that an image of an edge of length t from T has length $2t+1$ in T'.

(i) If $(i_0, (i+t)_1) \in E(T)$, then $l'(\varphi(i_0), \varphi((i+t)_1)) = l'(2i, 2i+2t+1) = 2i+2t+1-2i = 2t+1$

If $((j+t+1)_0, j_1) \in E(T)$, then $l'(\varphi((j-t+2t+1)_0), \varphi(j_1)) = l'(2j-2t+4t+2, 2j+1) = l'(2j-2t, 2j+1) = 2j+1-2j+2t = 2t+1.$

Then we show that images of two edges from T of lengths $t+m$ and $t-m$, where $0 < m \leq t$, have the same length in T'.

(ii) Let the edges from T of lengths $t+m$ and $t-m$ be of type (2). Then $(i_0, (i+t+m)_1) \in E(T)$ and $(j_0, (j+t-m)_1) \in E(T)$.

$l'(\varphi(i_0), \varphi(i+t+m)_1) = l'(2i, 2i+2t+2m+1) = 4t+2-(2i+2t+2m+1-2i) = 2t-2m+1.$

$l'(\varphi(j_0), \varphi((j+t-m)_1)) = l'(2j, 2j+2t-2m+1) = 2j+2t-2m+1-2j = 2t-2m+1.$

Let the edges from T of lengths $t+m$ and $t-m$ be of type (3). Then $((i-(t+m)+2t+1)_0, i_1) \in E(T)$ and $((j-(t-m)+2t+1)_0, j_1) \in E(T)$.

$l'(\varphi((i-(t+m)+2t+1)_0), \varphi(i_1)) = l'(2i-2t-2m+4t+2, 2i+1) = l'(2i-(2t+2m), 2i+1) = 4t+2-(2i+1-(2i-(2t+2m))) = 2t-2m+1.$

$l'(\varphi((j-(t-m)+2t+1)_0), \varphi(j_1)) = l'(2j-2t+2m+4t+2, 2j+1) = l'(2j-(2t-2m), 2j+1) = 2j+1-(2j-(2t-2m)) = 2t-2m+1.$

Notice that the images of the edges of length $t+m$ in both cases have an odd origin, because the absolute value of the difference of the labels of the endvertices of an edge $(2i, 2i+2t+2m+1)$ for $m > 0$ is greater than $2t+1$ and therefore its origin is the vertex with greater label, namely $2(i+t+m)+1$. For the other edge, $(2j-(2t+2m), 2j+1)$, the origin is $2j+1$. And the images of the edges of length $t-m$ in both cases have an even origin, because the absolute value of the difference of the labels of the endvertices of an edge $(2i, 2i+2t+1-2m)$ for $m > 0$ is less than $2t+1$ and therefore its origin is the vertex with less label, namely $2i$. For the other edge, $(2j-(2t-2m), 2j+1)$, the origin is $2j-(2t-2m)$. Hence for each length $l' \in \{1, 3, ..., 2t-1\}$ there are exactly two edges in T', which have the length equal to l and moreover these two edges have the origins of the different parities. For $l = 2t+1$ there is exactly one edge in T' of length l.

It follows that a tree T' has flexible q-labeling, which is generated by the blended ρ-labeling in T. Since the mapping φ is an automorphism, there exists $\varphi^{-1} : V(T') \to V(T)$ such that it induces a blended ρ-labeling in T from a flexible q-labeling in T' and the proof is done. □

4. Brooms

A *caterpillar* is a tree in which each edge has at least one end-vertex in a single path (called *spine*).

The caterpillar with n vertices consisting of a star $K_{1,k}$ and a path P_{n-k} (where one endvertex of P_{n-k} is identified with the central vertex of $K_{1,k}$) is called the *broom* and denoted $B(n,k)$. In other words, to construct $B(n,k)$, take a path P_{n-k} and join k isolated vertices to one of the endvertices of P_{n-k}.

Theorem 4.1. *A broom $B(4t+2,s)$, where $s \geq t+2, t \geq 2$, does not allow a flexible q-labeling.*

Proof. Suppose that a broom $B(4t+2,t+2)$ has a flexible q-labeling and $V(B) = V_0 \cup V_1$, where $V_0 = \{0,2,...,4t\}$, $V_1 = \{1,3,...,4t+1\}$. Denote the vertex of degree $t+3$ without loss of generality by 0. Every other vertex in B has a degree either 2 or 1.

According to Theorem 3.1 it holds that $\sum_{i \in V_0} deg(i) = \sum_{j \in V_1} deg(j) = 4t+1$. Thus in B there is one even vertex, namely 0, of degree $t+3$, $t-3$ even vertices and $2t$ odd vertices of degree 2, $t+3$ even vertices and exactly one odd vertex of degree 1.

Vertex 0 has $t+2$ neighbors of degree 1 and at most t of them can be even (see Lemma 3.2). Hence, at least two of them have to be odd, but in B there is exactly one odd vertex of degree 1, which is a contradiction and the proof is done.

For $s > t+2$ the proof is essentialy the same. □

Theorem 4.2. *Every broom $B(4t+2,t)$ admits a blended ρ-labeling for $t \geq 2$.*

Proof.

(i) First suppose that t is odd, $V(B) = V_0 \cup V_1$ and $V_0 = \{0_0, 1_0, ..., (2t)_0\}, V_1 = \{0_1, 1_1, ..., (2t)_1\}$. We construct the path $P = t_0, 0_0, (t-1)_0, 1_0, (t-2)_0, 2_0, ..., (\frac{t+1}{2})_0, (\frac{t-1}{2})_0$, which contains t (00)-edges of lengths $1,2,...,t$ and join it to the path $P' = t_0, (2t)_1, (t+1)_0, (2t-1)_1, (t+2)_0, (2t-2)_1, ..., (\frac{3t-1}{2})_0, (\frac{3t+1}{2})_1, (\frac{3t+1}{2})_0, (\frac{3t-1}{2})_1, ..., (2t-2)_0, (t+2)_1, (2t-1)_0, (t+1)_1, (2t)_0, t_1$ which contains $2t+1$ (01)-edges of lengths $0,1,...,2t-1,2t$.

Further we attach a star with the central vertex in t_1 and the edges

$(t_1, 0_1), (t_1, 1_1), ..., (t_1, (t-1)_1)$
to obtain the broom with a blended ρ-labeling.

(ii) For t even the construction is essentialy similar and therefore we describe it very briefly. We construct the path $P = t_0, 0_0, (t-1)_0, 1_0, (t-2)_0, 2_0, ..., (\frac{t-2}{2})_0, (\frac{t}{2})_0$ and the path $P' = t_0, (2t)_1, (t+1)_0, (2t-1)_1, (t+2)_0, (2t-2)_1, ..., (\frac{3t}{2})_1, (\frac{3t}{2})_0, ..., (2t-2)_0, (t+2)_1, (2t-1)_0, (t+1)_1, (2t)_0, t_1$.
Then we attach the star with the central vertex in t_1 and the edges $(t_1, 0_1), (t_1, 1_1), ..., (t_1, (t-1)_1)$. □

Corollary 4.3. *Every broom $B(4t+2, s)$, where $s \leq t$, allows a blended ρ-labeling for $t \geq 2$.*

Proof. We construct the paths P and P' as in the previous proof. We join to t_1 the edge $(t_1, 0_1)$ and we attach to 0_1 the star with the edges $(0_1, (t-1)_1), (0_1, (t-2)_1), ..., (0_1, 1_1)$. We obtain the broom $B(4t+2, t-1)$ with a blended ρ-labeling.

To construct $B(4t+2, t-2)$ we join the edges $(t_1, 0_1), (0_1, (t-1)_1)$ to P and P' and we attach the star with edges $((t-1)_1, 1_1), ..., ((t-1)_1, (t-2)_1)$.

For $B(4t+2, t-3)$ we join to P and P' the edges $(t_1, 0_1), (0_1, (t-1)_1), ((t-1)_1, 1_1)$ and we attach the star with edges $(1_1, (t-2)_1), (1_1, (t-3)_1), ..., (1_1, 2_1)$.

In general if we continue in this algorithm we obtain step by step all brooms $B(4t+2, s)$ for $s = t-1, t-2, ..., 3, 2$. □

Open problem. *Does the broom $B(4t+2, t+1)$ admit a blended ρ-labeling for $t \geq 2$?*

We have constructions of $B(4t+2, t+1)$ for $t = 2, 3, 4, 5, 6$, but we are so far unable to generalize them.

Constructions 4.4.

(i) A broom $B(10, 3)$ contains a star with the edges $(0_0, 1_0), (0_0, 3_0), (0_0, 1_1)$ and path $P = 0_0, 0_1, 2_1, 4_0, 3_1, 4_1, 2_0$.

Remark: This construction was introduced already by Eldergill in [1].

(ii) A broom $B(14, 4)$ contains a star with the edges $(0_0, 3_0), (0_0, 5_0), (0_0, 6_0), (0_0, 1_1)$ and the paths $P = 4_0, 6_1, 1_0, 5_1, 2_0$, $P' = 0_0, 0_1, 4_1, 2_1, 3_1$, which are joined by the edge $(4_0, 3_1)$.

(iii) $B(18, 5)$ contains a star with the edges $(0_0, 5_0), (0_0, 6_0), (0_0, 7_0), (0_0, 8_0), (0_0, 1_1)$ and the paths $P = 4_0, 6_1, 3_0, 7_1, 2_0, 8_1, 1_0$ and $P' = 0_0, 0_1, 5_1, 2_1, 4_1, 3_1$, which are joined by the edge $(4_0, 3_1)$.

(iv) $B(22,6)$ contains a star with the edges $(0_0, 6_0), (0_0, 7_0), \ldots, (0_0, (10)_0)$, $(0_0, 1_1)$ and the paths $P = 1_0, (10)_1, 2_0, 9_1, 3_0, 8_1, 4_0, 7_1, 5_0$ and $P' = 0_0, 0_1, 6_1, 2_1, 5_1, 3_1, 4_1$, which are joined by the edge $(5_0, 4_1)$.

(v) $B(26,7)$ contains a star with the edges $(0_0, 1_0), (0_0, 3_0), (0_0, 5_0), (0_0, 7_0), (0_0, 9_0), (0_0, (11)_0), (0_0, 0_1), (0_0, 1_1)$, the paths $P_1 = (12)_0, 2_1, (10)_0, 4_1, 8_0, 6_1$, $P_2 = (12)_1, 2_0, (10)_1, 4_0, 8_1, 6_0, 5_1$ $P_3 = (12)_1, (11)_1, 7_1, 9_1, 3_1, 6_1$ and extra edge $(0_1, 5_1)$. □

5. Caterpillars with diameter 4.

In this section we present complete characterisation of caterpillars with diameter 4 which allow a blended ρ-labeling.

Before we establish necessary and sufficient conditions for the existence of a blended ρ-labeling for caterpillars with diameter 4, we have to present several lemmas.

Lemma 5.1. *A tree T on $4t+2$ vertices has a blended ρ-labeling with $V(T) = V_0 \cup V_1$, $V_0 = \{0_0, 1_0, \ldots, (2t)_0\}$ and $V_1 = \{0_1, 1_1, \ldots, (2t)_1\}$ if and only if a tree T on $4t+2$ vertices has a blended ρ-labeling with $V(T) = V_0{}' \cup V_1{}'$, $V_0{}' = \{(0+k)_0, (1+k)_0, \ldots, (2t+k)_0\}, V_1{}' = V_1$ for $k \leq 2t$.*

Proof. We denote T' a tree T, where $V(T) = V_0{}' \cup V_1{}'$. It is easy to see that the corresponding edges $(i_0, j_0) \in E(T)$, $((i+k)_0, (j+k)_0) \in E(T')$ and $(i_1, j_1) \in E(T)$, $(i_1, j_1) \in E(T')$ have the same lengths.

Let $(i_0, j_1) \in E(T), ((i+1)_0, j_1) \in E(T^*)$, where $T^* = T'$, for $k = 1$.

(i) If $i < j$, then $l(i_0, j_1) = j - i = m > 0$ and $l((i+1)_0, j_1) = j - i - 1 = m - 1 \geq 0$.
(ii) If $i > j$, then $l(i_0, j_1) = 2t + 1 + (j - i) = m > 0$ and $l((i+1)_0, j_1) = 2t + 1 + (j - i) - 1 = m - 1 \geq 0$.
(iii) If $i = j$, then $l(i_0, j_1) = l(i_0, i_1) = i - i = 0$ and $l((i+1)_0, j_1) = l((i+1)_0, i_1) = i - i - 1 = 2t$.

We see that each image of an edge from T with length $m > 0$ has the length $m - 1 \geq 0$ in T^* and an image of an edge with length 0 from T is the longest edge in T^*. Thus T^* has a blended ρ-labeling and if we repeat this procedure k-times we prove that T' has a blended ρ-labeling too.

Let us notice that every implication in this proof holds also in the opposite direction and therefore the proof is complete. □

This result allows us to simplify considerations.

Corollary 5.2. *Let T be a tree with a blended ρ-labeling λ and x, y be arbitrary vertices of T. Then there exists a blended ρ-labeling λ' such that $\lambda'(x) = 0_0$ and $\lambda'(y) = 0_1$.*

Lemma 5.3. *Every caterpillar R on $4t+2$ vertices with a blended ρ-labeling and diameter 4 contains a vertex i such that $deg(i) = 2t + 1$.*

Proof. Let u, v, w be internal vertices of spine of a caterpillar R. Suppose that $deg(u) = 2t - r + 1$, $deg(v) = r + s + 1$, $deg(w) = 2t - s + 1$, where $0 < r < 2t, 0 \leq s < 2t$, and $V(R) = V_0 \cup V_1$, $V_0 = \{0, 2, ..., 4t\}, V_1 = \{1, 3, ..., 4t + 1\}$. If $u, v, w \in V_0$ then $\sum_{x \in V_1} deg(x)$ is less than $4t + 1$ and it contradicts Theorem 3.1. For $u, v, w \in V_1$ the consideration is the same. Hence, precisely two of them have to belong to V_0 and one to V_1 or vice versa.

(a) Let $u, v \in V_0$. $\sum_{y \in V_0} deg(y) = (2t - r + 1) + (r + s + 1) + 2t - 1 = 4t + s + 1$ and so $s = 0$. Thus the vertex w has the degree $2t + 1$. For $v, w \in V_0$ the proof is essentialy the same (in this case it holds that $r = 0$ and $deg(u) = 2t + 1$) and therefore it can be omitted.
For $u, v \in V_1$ (or $v, w \in V_1$) the proof is the same.

(b) Let $u, w \in V_0$. $\sum_{y \in V_0} deg(y)$ (resp.$\sum_{x \in V_1} deg(x)$) $= (2t - r + 1) + (2t - s + 1) + 2t - 1 = 6t - r - s + 1 = 4t + 1$ and therefore $r + s = 2t$. Hence, the vertex v has the degree $2t + 1$.
For $u, w \in V_1$ the proof is the same. □

It follows that for $1 \leq s < 2t$ we can further consider only the caterpillars with the spine u, v, w, which satisfy exactly one of the following necessary conditions:

(i) The vertices u, v belong to V_i and w belongs to V_j for $i \neq j$, where $i, j \in \{0, 1\}$, and $deg(u) = 2t - s + 1, deg(v) = s + 1, deg(w) = 2t + 1$.

(ii) The vertices u, w belong to V_i and the vertex v belongs to V_j for $i \neq j$, where $i, j \in \{0, 1\}$, and $deg(u) = s + 1$, $deg(v) = 2t + 1$, $deg(w) = 2t - s + 1$.

A caterpillar in which one endvertex of the spine is of degree $2t + 1$ will be called $(2t - s + 1, s + 1, 2t + 1)$-*type caterpillar* or *e-type caterpillar.* A caterpillar in which the central vertex is of degree $2t + 1$ will be called $(s + 1, 2t + 1, 2t - s + 1)$-*type caterpillar* or *c-type caterpillar.*

Lemma 5.4. *An e-type caterpillar R, where $s = 1$, does not allow a blended ρ-labeling.*

Eldergill in [1] proved that such caterpillars do not factorize K_{2n} for every $n \geq 3$ and thus also do not allow a blended ρ-labeling.

Lemma 5.5. *Every e-type caterpillar, where $1 < s < 2t$, allows a blended ρ-labeling.*

Proof. By construction.

(a) Let $s > t, w = 0_0, v = 0_1, u = (t + 1)_1$.

(Step 0) Assume that $s = t+1$. Then our caterpillar, called R_1, contains

(i) (00)-edges $(0_0,(t+1)_0),(0_0,(t+2)_0),...,(0_0,(2t)_0)$ with all lengths $t, t-1, ..., 1$.

(ii) (01)-edges $(0_0,0_1),(0_0,1_1),(0_0,2_1),...,(0_0,t_1)$ and $(1_0,0_1),(2_0,0_1), ...,(t_0,0_1)$ with all lengths $1,2,...,t$ and $2t, 2t-1, ..., t+1$,

(iii) (11)-edges $(0_1,(t+1)_1)$ and $((t+1)_1,(t+2)_1),((t+1)_1,(t+3)_1), ...,((t+1)_1,(2t)_1)$ with all lengths t and $1,2,...,t-1$.

We see that we have obtained a caterpillar with a blended ρ-labeling.

In the following steps, where we construct the caterpillars $R_2, R_3, ..., R_{t-1}$ with blended ρ-labelings, the edges from the cases (i) and (ii) will remain the same and we call them *fixed edges*.

(Step k) Let $s = t+k, 1 < k < t$. The caterpillar R_k contains all fixed edges from Step 0 and the edges $(0_1,(t+1)_1),(0_1,(t+2)_1),...,(0_1, (t+k-1)_1)$ with lengths $t, t-1, ..., t-k+2$. Further R_k has the edges $(0_1,(t+k)_1)$ and $((t+k)_1,(t+k+1)_1),((t+k)_1,(t+k+2)_1),...,((t+k)_1,(2t)_1)$ with lengths $t-k+1$ and $1,2,...,t-k$. Thus R_k has a blended ρ-labeling for every k, where $s = t+k, 0 < k < t$.

(b) Let $s \leq t, w = 0_0, u = 0_1$, $0 \leq m \leq t-1/2$ for t odd, $0 \leq m \leq (t-2)/2$ for t even, $v = (t+1)_1$ in Step $2m$ and $v = t_1$ in Step $2m+1$.

(Step $2m$) The caterpillar R_{2m} contains fixed (00)-edges and (01)-edges (these edges are independent to m) $(0_0,t_0),(0_0,(t+2)_0),(0_0,(t+3)_0),..., (0_0,(2t)_0)$ with lengths $t, t-1,...,1$, and $(0_0,1_1),(0_0,2_1),..., (0_0,t_1),(0_0,(t+1)_1),((t+1)_0,(t+1)_1$, with lengths $1,2,...,t,t+1,0$, and $(1_0,0_1),(2_0,0_1),...,((t-1)_0,0_1)$ with lengths $2t, 2t-1, ..., t+2$, and finally "moving" (11)-edges (these edges are dependent to m)

(i) $(0_1,(t+1)_1),(0_1,(t+2)_1),...,(0_1,(t+m+1)_1)$ with lengths $t, t-1,...,t-m$,

(ii) $(0_1,(2t)_1),...,(0_1,(2t-m+1)_1)$ with lengths $1,2,...,m$,

(iii) $((t+1)_1,(t+m+2)_1),((t+1)_1,(t+m+3)_1),...,((t+1)_1, (2t-m)_1)$ with lengths $m+1, m+2, ..., t-m-1$.

Remark :

- the edges (ii) have to be omitted for $m = 0$.
- the edges (iii) have to be omitted for $m > (t-2)/2$.

(Step $2m+1$) In this step there remain all fixed edges from the previous step, except for the edges $(0_0,t_0)$ and $((t+1)_0,(t+1)_1)$. They are replaced by the edges $(0_0,(t+1)_0)$ and (t_0,t_1).

Further the caterpillar R_{2m+1} contains "moving" (11)-edges

(iv) $(0_1, t_1), (0_1, (t+2)_1), (0_1, (t+3)_1), ..., (0_1, (t+m)_1)$ and $(0_1, (t+m+1)_1)$ with lengths $t, t-1, t-2, ..., t-m$,

(v) $(0_1, (2t)_1), (0_1, (2t-1)_1), ..., (0_1, (2t-m)_1)$ with lengths $1, 2, ..., m+1$, and

(vi) $(t_1, (t+m+2)_1), (t_1, (t+m+3)_1), ..., (t_1, (2t-m-1)_1)$ with lengths $m+2, m+3, ..., t-m-1$.

Remark :

- an edge $(0_1, (t+m+1)_1)$ from (iv) has to be omitted for $m = 0$.
- the edges (vi) have to be omitted for $m > (t-3)/2$.

Hence, every caterpillar R_{2m} and R_{2m+1} has a blended ρ-labeling for each m. Finally for t odd the last step of our construction is $2m$, where $t = 2m+1$ and for t even the last step is $2m+1$, where $t = 2(m+1)$. Hence, the edges (iii) and (vi) are always omitted in the last step of our construction. Now the proof is complete. □

Lemma 5.6. *Every c-type caterpillar allows a blended ρ-labeling.*

Proof. By construction.

We have a c-type caterpillar R with the spine $P = u, v, w$ and $deg(u) = s+1$, $deg(v) = 2t+1$, $deg(w) = 2t-s+1$.

Let $u = s_1, v = 0_0$ and $w = 0_1$.

First we construct the edges of the star with the central vertex 0_0.

(i) (00)-edges: $(0_0, (2t)_0), (0_0, (2t-1)_0), ..., (0_0, (t+1)_0)$.

(ii) (01)-edges: $(0_0, 0_1), (0_0, 1_1), ..., (0_0, (t-1)_1), (0_0, t_1)$.

(00)-edges have all lengths $1, 2, ..., t$ and (01)-edges have all lengths $0, 1, 2, ..., t$.

Further we construct the edges of the star with the center 0_1.

(iii) (01)-edges: $(0_1, 1_0), (0_1, 2_0), ..., (0_1, t_0)$,

(iv) (11)-edges: $(0_1, (2t)_1), (0_1, (2t-1)), ..., (0_1, (t+s+1)_1)$. (01)-edges have all lengths $2t, 2t-1, ..., t+1$ and (11)-edges have all lengths $1, 2, ..., t-s$.

At last we construct the remaining (11)-edges $(s_1, (t+s)_1), (s_1, (t+s-1)_1), ..., (s_1, (t+1)_1)$ which have all lengths $t, t-1, ..., t-s+1$.

Remark :

- the edges (iv) must be omitted for $s = t$.

Thus every c-type caterpillar with diameter 4 allows a blended ρ-labeling. □

The following theorem is a direct consequence of the previous lemmas.

Theorem 5.7. *Let R be a caterpillar with $4t+2$ vertices and diameter 4. Let v be the central vertex of the spine and u, w the endvertices of the spine. Then R allows a blended ρ-labeling if and only if either* $\deg(u) = 2t+1, \deg(v) = s+1, \deg(w) = 2t-s+1$ *and* $2 \leq s < 2t$ *or* $\deg(u) = s+1, \deg(v) = 2t+1, \deg(w) = 2t-s+1$ *and* $1 \leq s < 2t$.

6. Other results

In the previous section we have completely characterized all caterpillars with diameter 4, which allow a blended ρ-labeling, but we know nothing about the trees of diameter 4, which are not caterpillars. These trees are called *lobsters.*

A *lobster* is a tree such that after deleting all vertices of degree 1 we obtain a caterpillar, which is distinct from a path.

Therefore a lobster with diameter 4 is basically a star $K_{1,m}$, where $m > 2$, where to the vertices of degree 1 there are joined the stars K_{1,r_1}, $K_{1,r_2}, ..., K_{1,r_m}$ and there exist at least three different subscripts $i, j, k \in \{1, 2, ..., m\}$ such that $r_i, r_j, r_k > 0$.

Hence, if we consider a lobsters with diameter 4 on $2n$ vertices and n is odd, then $n \geq 5$.

Let us have lobsters on $4t+2$ vertices and with diameter 4, where $r \leq r_1 \leq r_2 \leq ... \leq r_m \leq r+1$ and $4t+2-(m+1) = rm+q$ and thus $2n = 4t+2 = (r+1)m+q+1$. We see that these lobsters have "regular distribution" of vertices to the joining stars. These lobsters are called *m-balanced lobsters.*

Hence, the terminal stars joined to the endvertices of the central star $K_{1,m}$ are either $K_{1,r}$ or $K_{1,r+1}$, where number of the stars $K_{1,r+1}$ is q. Thus an m-balanced lobster contains one vertex of degree m, q vertices of degree $r+2$, $m-q$ vertices of degree $r+1$ and $rm+q$ vertices of degree 1.

Now we investigate which of these lobsters satisfy the necessary condition that was introduced in Theorem 3.1.

Let a vertex of degree m belong to the set V_0 and let x be the number of vertices of degree $r+2$ and y be the number of vertices of degree $r+1$ from set V_1. Then

$$x(r+2) + y(r+1) + \frac{(r+1)m+q+1}{2} - x - y = (r+1)m + q.$$

Hence

$$2xr + 2x + 2yr = (r+1)m + q + 1 - 2.$$

Since

$$(r+1)m + q + 1 = 2n$$

we get

$$(r+1)x + ry = n - 1. \tag{4}$$

We have now proved the following theorem.

Theorem 6.1. *Let a vertex of degree m belong to V_0. Then for every m-balanced lobster with diameter 4 and with a blended ρ-labeling it holds that $(r+1)x + ry = n - 1$, where x is the number of vertices of degree $r+2$ and y is the number of vertices of degree $r+1$ from V_1.*

Notice that the left hand side of the equation (4) determines the number of the vertices with degree 1 that are adjacent to the vertices of degree $r+1$ and $r+2$ from V_1.

Now we introduce two easy lemmas.

Lemma 6.2. *In every m-balanced lobster L on $2n$ vertices it holds that*

$$\frac{2}{m}(n-m) \leq r \leq \frac{2n-(m+1)}{m}.$$

Proof. If r is maximal then in L there does not exist a vertex of degree $r+2$ and therefore $rm = 2n - (m+1)$. Thus $r = \frac{2n-(m+1)}{m}$.

If r is minimal then in L there exists precisely one vertex of degree $r+1$ and therefore $(r+1)(m-1) + r = 2n - (m+1)$ which leads to $rm + m - 1 = 2n - m - 1$ and $r = \frac{2n-2m}{m}$. □

Lemma 6.3. *Let L be an m-balanced lobster on $2n$ vertices with a blended ρ-labeling and let a vertex of degree m belong to V_0. Then the number of vertices of degree $r+1$ and $r+2$ in V_1 is at most $m-1$.*

Proof. If in V_1 there are m vertices of degree $r+1$ and $r+2$ then the number of the vertices of degree 1, which are adjacent to them is $2n-(m+1)$ and $2n-(m+1) > n-1$ for every $n > m$, which contradicts Theorem 6.1. □

Let us show now that the necessary condition (4) from section 6 for m-balanced lobsters is also sufficient for $m = 3, 4$.

Construction 6.4. Let L be a 3-balanced lobster on $2n$ vertices, where n is odd, $V(L) = V_0 \cup V_1, V_0 = \{0_0, 1_0, ..., (n-1)_0\}, V_1 = \{0_1, 1_1, ..., (n-1)_1\}$, and the vertex of degree 3 belongs to V_0.

First we show that our lobster satisfies necessary condition (4) only for $n = 5, 7$.

From Lemma 6.2 it follows that $r \leq 2(n-2)/3$ and thus $r + 1 \leq \frac{2}{3}(n-2) + 1 = \frac{2n-1}{3} < n - 1$ for every n. Therefore in V_1 there have to be at least two vertices of degree $r + 1$ or $r + 2$. Hence, the number of vertices of degree one that are adjacent to the vertices of degree $r+1$ and $r+2$ is at least $2r$, but $2r \geq \frac{4}{3}(n-3) > n - 1$ for every $n > 9$. Notice that for $n = 9$ there has to be in V_1 at least one vertex of degree $r + 2 = 6$ and therefore our 3-balanced lobster does not satisfy condition (4) also for $n = 9$.

Now we must show by construction that a 3-balanced lobster admits a blended ρ-labeling for $n = 5, 7$.

(i) If $n = 5$ then our balanced lobster contains the following edges:

$(0_0, 1_0), (1_0, 4_0)$ of lengths $1, 2$, $(0_0, 0_1), (0_0, 1_1), (1_0, 3_1), (3_0, 1_1), (2_0, 1_1)$ of lengths $0, 1, 2, 3, 4$ and $(0_1, 4_1), (0_1, 2_1)$ of lengths $1, 2$ and

(ii) if $n = 7$ then it contains the edges:

$(0_0, 1_0), (1_0, 6_0), (1_0, 4_0)$ of lengths $1, 2, 3$, $(0_0, 0_1), (0_0, 1_1), (1_0, 3_1), (1_0, 5_1)$ of lengths $0, 1, 2, 4$, $(5_0, 1_1), (3_0, 1_1), (2_0, 1_1)$ of lengths $3, 5, 6$ and $(0_1, 6_1), (0_1, 2_1), (0_1, 4_1)$ of lengths $1, 2, 3$. □

Construction 6.5. Let us have 4-balanced lobster L on $2n$ vertices, where n is odd, $V(L) = V_0 \cup V_1, V_0 = \{0_0, 1_0, ..., (n-1)_0\}, V_1 = \{0_1, 1_1, ..., (n-1)_1\}$, and the vertex of degree 4 belongs to V_0.

Notice that each 4-balanced lobster has precisely one vertex of degree $r + 2$ and three vertices of degree $r + 1$ for every n odd.

Hence, $2n = 2(2r + 3)$ and $n = 2r + 3$. When we substitute n in condition (1) from section 6 by $2r + 3$ we obtain $(r + 1)x + ry = 2r + 2$ and possible values of x are either 0 or 1. Therefore we look for the solutions of two equations either

$$ry = 2r + 2 \tag{5}$$

or

$$ry = r + 1. \tag{6}$$

The equation (5) leads to $y = 2 + \frac{2}{r}$ and from Lemma 6.3. it follows that only one of two integral solutions of (5) is admissible, namely $r = 2$ and $y = 3$. The equation (6) leads to $y = 1 + \frac{1}{r}$ and therefore (6) has also precisely one integral solution, $r = 1$ and $y = 2$.

Since necessary condition (4) has solution only for $r = 1, 2$, our 4-balanced lobster satisfies this condition only for $n = 5, 7$.

Finally we show that a 4-balanced lobster on $2n$ vertices allows a blended ρ-labeling for $n = 5$ and $n = 7$.

(i) Let $n = 5$. Then the 4-balanced lobster contains the edges:

$(0_0, 1_0), (1_0, 4_0)$ of lengths $1, 2$, $(0_0, 0_1), (0_0, 1_1), (0_0, 3_1)$ of lengths $0, 1, 3$, $(3_0, 0_1), (2_0, 1_1)$ of lengths $2, 4$ and $(0_1, 2_1), (3_1, 4_1)$ of lengths $2, 1$.

(ii) Let $n = 7$. Then our balanced lobster has the edges:

$(0_0, 1_0), (1_0, 6_0), (1_0, 5_0)$ of lengths $1, 2, 3$, $(0_0, 0_1), (0_0, 1_1), (0_0, 2_1), (1_0, 4_1)$ of lengths $0, 1, 2, 3$ $(3_0, 0_1), (4_0, 2_1), (2_0, 1_1)$ of lengths $4, 5, 6$ and $(0_1, 6_1), (1_1, 3_1), (2_1, 5_1)$ of lengths $1, 2, 3$. □

At last we introduce examples of two infinite classes of "totally imbalanced" lobsters with diameter 4 on $4t + 2$, $t \geq 2$, vertices, which admit blended ρ-labelings.

Construction 6.6. Let us have a double star which contains the stars $K_{1,2t}$ and $K_{1,2t-2}$ joined by an extra edge, and the extra edge is incident to the central vertices of these stars. Further exactly two vertices are joined to two distinct endvertices of the star $K_{1,2t-2}$.

Let such imbalanced lobster contain the following edges:

$(0_0, (t-1)_0), (0_0, (t+1)_0), (0_0, (t+3)_0), (0_0, (t+4)_0), ..., (0_0, (2t)_0)$ of lengths $t-1, t, t-2, t-3, ..., 1$,

$(0_0, 0_1), (0_0, 1_1), ..., (0_0, t_1), (1_0, (t+3)_1)$ of lengths $0, 1, 2, ..., t, t+2$,

$(2_0, 0_1), (3_0, 0_1), ..., ((t-2)_0, 0_1), (t_0, 0_1), ((t+2)_0, (t+1)_1)$ of lengths $2t-1, 2t-2, ..., t+3, t+1, 2t$ and

$(0_1, (t+1)_1), (0_1, (t+2)_1), ..., (0_1, (2t)_1)$ of lengths $t, t-1, ..., 1$.
Then it has a blended ρ-labeling.

□

Construction 6.7.

(i) Let t be odd. Now we consider an imbalanced lobster, which contains two stars $K_{1,2t-1}$ and $K_{1,(3t+1)/2}$ joined by an extra edge and the extra edge is incident to the central vertices of the stars. Further, exactly $(t-1)/2$ vertices are joined to $(t-1)/2$ endvertices of the star $K_{1,(3t+1)/2}$.

Let us show now that our imbalanced lobster admits a blended ρ-labeling.

Let L have the edges: $(0_0, 1_0), (0_0, 2_0), ..., (0_0, (\frac{t-1}{2})_0)$ of lengths $1, 2, ..., (t-1)/2$,

$(0_0, (t+1)_0), (0_0, (t+2)_0), \ldots, (0_0, (\frac{3t+1}{2})_0)$ of lengths $t, t-1, \ldots, (t+1)/2$,

$(0_0, 0_1), (0_0, 1_1), (0_0, 2_1), \ldots, (0_0, (t-1)_1), (t_0, (2t)_1)$ of lengths $0, 1, 2, \ldots, t-1, t$,

$((\frac{t+1}{2})_0, 0_1), ((\frac{t+3}{2})_0, 0_1), \ldots, (t_0, 0_1)$ of lengths $(3t+1)/2, (3t-1)/2, \ldots, t+1$,

$((2t)_0, (2t-1)_1), ((2t-1)_0, (2t-3)_1), ((2t-2)_0, (2t-5)_1), \ldots, ((\frac{3t+5}{2})_0, (t+4)_1), ((\frac{3t+3}{2})_0, (t+2)_1)$ of lengths $2t, 2t-1, 2t-3, \ldots, (3t+5)/2, (3t+3)/2$, and

$(0_1, t_1), (t_1, (t+1)_1), (0_1, (t+2)_1), (0_1, (t+3)_1), \ldots, (0_1, (2t-2)_1), (0_1, (2t-1)_1)$ of lengths $t, 1, t-1, t-2, \ldots, 3, 2$.

Then L has a blended ρ-labeling.

(ii) For t even the construction is very similar. We have again an imbalanced lobster which contains two stars $K_{1,2t-1}$ and $K_{1,(3t+1)/2}$ joined by an extra edge and the extra edge is incident to the central vertices of the stars. Further, exactly $(t-2)/2$ vertices are joined to $(t-2)/2$ endvertices of the star $K_{1,(3t+1)/2}$.
This lobster has the following edges:
$(0_0, 1_0), (0_0, 2_0), \ldots, (0_0, (\frac{t-2}{2})_0)$ of lengths $1, 2, \ldots, (t-2)/2$,

$(0_0, (t+1)_0), (0_0, (t+2)_0), \ldots, (0_0, (\frac{3t+2}{2})_0)$ of lengths $t, t-1, \ldots, t/2$,

$(0_0, 0_1), (0_0, 1_1), (0_0, 2_1), \ldots, (0_0, (t-1)_1), (t_0, (2t)_1$ of lengths $0, 1, 2, \ldots, t-1, t$,

$((\frac{t}{2})_0, 0_1), (\frac{t+2}{2})_0, 0_1), \ldots, (t_0, 0_1)$ of lengths $(3t+2)/2, 3t/2, \ldots, t+1$,

$((2t)_0, (2t-1)_1), ((2t-1)_0, (2t-3)_1), ((2t-2)_0, (2t-5)_1), \ldots, ((\frac{3t+6}{2})_0, (t+5)_1), ((\frac{3t+4}{2})_0, (t+3)_1)$ of lengths $2t, 2t-1, 2t-3, \ldots, (3t+6)/2, (3t+4)/2$, and

$(0_1, t_1), (t_1, (t+1)_1), (0_1, (t+2)_1), (0_1, (t+3)_1), \ldots, (0_1, (2t-2)_1), (0_1, (2t-1)_1)$ of lengths $t, 1, t-1, t-2, \ldots, 3, 2$.

Then L has a blended ρ-labeling. □

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