The gracefulness of unions of cycles and complete bipartite graphs

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ABSTRACT

A *graceful labeling* of a graph G of size n is an assignment of labels from $\{0, 1, ..., n\}$ to the vertices of G such that when each edge has assigned a weight defined by the absolute difference of its end-vertices, the resulting weights are distinct. The *gracefulness* of a graph G is the smallest positive integer k for which is possible to label the vertices of G with distinct elements from the set $\{0, 1, ..., k\}$ in such a way that distinct edges have distinct weights. In this paper, we determine the gracefulness of the union of cycles and complete bipartite graphs. We also give graceful labelings of unions of complete bipartite graphs.

1. Introduction

A *graceful labeling* of a graph G with m vertices and n edges, is a one-to-one mapping $f : V(G) \to \{0, 1, ..., n\}$, such that for every edge xy of G, f induces a *weight* defined by $|f(x) - f(y)|$ and the set of weights is $\{1, 2, ..., n\}$. In this case, G is said to be a *graceful graph.* When the graceful labeling f has the property that there exists an integer λ such that for each edge xy either $f(x) \leq \lambda < f(y)$ or $f(y) \leq \lambda < f(x)$, f is called an *α-labeling.*

It is known that not every graph is graceful, for instance we can consider the complete graph K_n when $n \geq 5$ and the cycle C_n when $n \equiv 1$ or $2 \pmod 4$. The smallest graph, in order and size, that is not graceful is $C_3 \cup K_{1,1}$. These examples represent three reasons why a graph fails to be graceful: The graph has too many edges (K_n, $n \geq 5$), the graph has not the right parity (C_n, $n \equiv 1$ or $2 \pmod 4$), or the graph has too many vertices and not enough edges ($C_3 \cup K_{1,1}$).

The *gracefulness*, grac(G), of a graph G without isolated vertices, is the smallest positive integer k for which is possible to label the vertices of G with distinct elements from the set $\{0, 1, ..., k\}$ in such a way that distict edges have distinct weights. This parameter is well defined; in fact, since there exists a vertex labeling of G that assigns the integers $2^0, 2^1, ..., 2^{m-1}$

to the m vertices of G. Thus, for every graph G of order m and size n (without isolated vertices,) $n \leq \text{grac}(G) \leq 2^{m-1}$. If G is a graph of size n with $\text{grac}(G) = n$, then G is graceful. Thus, the gracefulness of a graph G is a measure of how close G is to being graceful. For instance, $\text{grac}(K_n)$ is $n(n-1)/2$ when $n = 2, 3, 4$, and 11 when $n = 5$. It is an open problem determine $\text{grac}(K_n)$ for $n \geq 6$. Essentially, this concept was introduced by Golomb [3]. He suggested that the main questions in this topic are determine the relationship between $\text{grac}(G)$ and n, identifying families of graphs for which $\text{grac}(G) = n$ and other for which $\text{grac}(G) > n$, and also to find better bounds for $\text{grac}(G) - n$.

In his dynamic survey [2], Gallian observes that over 300 articles (related with graceful labelings) have been written in the last three decades. Most of them centered in the identification of classes of graphs for which $\text{grac}(G) = n$. We study the gracefulness of a family of graphs in terms of determining precisely the value of the value of this parameter for all menbers of this family.

2. Cycles and Stars

Modifying the permissible vertex labels and/or the weights of a graceful labeling, Rosa [5] introduced the following definition: Let G be a graph of order m and size n and $f : V(G) \rightarrow \{0, 1, ..., n+1\}$ be an injective function, such that the induced weights are $\{1, 2, ..., n-1, n+1\}$ or $\{1, 2, ..., n\}$, Rosa called it *$\widehat{\rho}$-labeling*; later, Frucht called it *nearly graceful labeling*. To avoid confusions, we say that this labeling is of *kind 1* if the set of weights is $\{1, 2, ..., n-1, n+1\}$ and of *kind 2* otherwise. Note that if a graph not satisfies the parity condition (Lemma 1 in [4]), then it cannot have a labeling of kind 2.

Consider the family of disconnected graphs $C_m \cup K_{1,n}$, i.e., the union of cycles and stars, we are intersted in determine its gracefulness. In his survey [2], Gallian summarize the status of this problem. We have extracted from Gallian's work the following information. Choudum and Kishore proved that $C_m \cup K_{1,n}$ is graceful for every $m \geq 7$ and $n \geq 1$. On the other hand, Seoud and Youssef [6] proved that neither $C_3 \cup K_{1,n}$ nor $C_4 \cup K_{1,n}$ (in this case $n \neq 2$) are graceful. However, nothing is said when the cycle is C_5 or C_6. We know that the labels 0 and $m+n$ $(m = 5, 6)$ must be assigned to adjacent vertices; note then, that they cannot be in the star. In fact, without loss of generality, we may assume that 0 was assigned to the central vertex of the star and $m+n$ in a leaf, then the labels $m+n-1, ..., m+1$ must be in the other leaves; remaining the labels $1, ..., m$ to be assigned on the cycle, but with these labels it is impossible to obtain a labeling of C_m

where all the weights are different. Then, 0 and $m+n$ must be assigned on the cycle. With this restriction, it can be checked that $C_5 \cup K_{1,n}$ $(n \geq 2)$ is not graceful and that $C_6 \cup K_{1,n}$ is graceful if and only if n is odd or $n = 2, 4$. We present below a graceful labeling of $C_6 \cup K_{1,2n+1}$.

THEOREM 2.1. *For every non-negative integer n, the graph* $C_6 \cup K_{1,2n+1}$ *is graceful.*

Proof. Let $u_1, ..., u_6$ be consecutive vertices of C_6 and $v_0, v_1, ..., v_{2n+1}$ the vertices of the star, where v_0 is its central vertex. Let $f : V(C_6 \cup K_{1,2n+1}) \rightarrow \{0, 1, ..., n+6)$ defined by

$$f(u_i) = \begin{cases} 0, & i = 1 \\ 2n+6, & i = 2 \\ n+i, & i = 3,4 \\ 2, & i = 5 \\ 2n+7, & i = 6 \end{cases} \text{ and } f(v_j) = \begin{cases} 1, & j = 0 \\ j+2, & 1 \leq j \leq n \\ j+4, & n+1 \leq j \leq 2n+1. \end{cases}$$

The weights on the cycle are $2n+7, 2n+6, 2n+5, n+3, n+2, 1$ and on the star are $2, 3, ..., n+1, n+4, n+5, ..., 2n+4$. Thus, f is a graceful labeling of $C_6 \cup K_{1,2n+1}$. ■

We will represent the labelings of $C_m \cup K_{1,n}$ in the following form:

$$(f(u_1), ..., f(u_m)) \cup (f(v_0); f(v_1), ..., f(v_n)).$$

So, the unique graceful labelings of $C_6 \cup K_{1,2}$ and $C_6 \cup K_{1,4}$ are given by $(0,7,1,6,4,8) \cup (2;3,5)$ and $(0,9,5,3,2,10) \cup (1;4,6,7,8)$, respectively. A graceful labeling of $C_5 \cup K_{1,1}$ is given by $(0,6,2,3,5) \cup (1;4)$ and finally, a graceful labeling of $C_4 \cup K_{1,2}$ is given by $(0,6,1,4) \cup (3;2,5)$.

Since the graphs $C_m \cup K_{1,n}$ are not Eulerian, we may hope that they admit nearly graceful labelings of at least one kind. In fact they admit both kind of nearly graceful labelings, we present these results in the next theorem.

THEOREM 2.2. *The graph* $C_m \cup K_{1,n}$ *is nearly graceful for every positive integer n and m=3,4,5,6.*

Proof. It is enough to show a nearly graceful labeling $C_m \cup K_{1,n}$ in every case. For this purpose, we will use the notation introduced above; the first labeling is of kind 1 and the second of kind 2. Thus, when $m = 3$, we have $(1, 2, n+3) \cup (n+4; 0, 4, 5, ..., n+2)$ and $(0, 1, n+3) \cup (n+4; 3, 4, ..., n+2)$. When $m = 4$, $(1, 3, 2, n+4) \cup (n+5; 0, 4, 5, ..., n+2)$ and $(0, n+4, n+5, 2) \cup (1; 4, 5, ..., n+3)$. When $m = 5$, $(0, n+6, n+3, n+2, n+4)$

$\cup$ ($n+5$;2,3,...,$n+1$) and (0,$n+5$,3,$n+6$,$n+4$) $\cup$ (1;2,4,5,...,$n+2$). Finally, when $m = 6$, we have (0,$n+7$,$n+3$,$n+4$,$n+2$,$n+5$) $\cup$ ($n+6$;2,3,...,$n+1$) and (0,$n+6$,5,$n+4$,2,$n+5$) $\cup$ (3;1,4,6,7,...,$n+1$,$n+3$,$n+7$) when $n \geq 4$ and $(0,9,5,7,2,8) \cup (3;4,6,10)$ when $n = 3$. ■

To conclude this section, we summarize these results in the next table.

G	grac(G)
$C_3 \cup K_{1,n}$	$n+4$ for $n \geq 1$
$C_4 \cup K_{1,n}$	$n+5$ for $n \neq 2$ and $n+4$ for $n = 2$
$C_5 \cup K_{1,n}$	$n+6$ for $n \geq 1$
$C_6 \cup K_{1,n}$	$n+6$ for n odd or $n = 2,4$ $n+7$ for n even $n \geq 6$
$C_m \cup K_{1,n}$	$n+m$ for $m \geq 7$ and $n \geq 1$

3. Cycles and complete bipartite graphs

In this section we study the parameter grac($C_m \cup K_{n_1,n_2}$), giving its exact value when the cycle is graceful and also a sharper upper bound when the cycle is nearly graceful. Seoud and Youssef [6] have considered some small cases, they proved that $C_3 \cup K_{n_1,n_2}$ is graceful if and only if n_1 and n_2 are greater than 1, $C_4 \cup K_{n_1,n_2}$ is graceful if and only if n_1 and n_2 are greater than 1 or $(n_1, n_2) = (1, 2)$, and finaly they proved that C_7 or C_8 union K_{n_1,n_2} is graceful for all n_1 and n_2.

Let G be a graph of size n. Any one-to-one assignment f of nonnegative integers to the vertices of G such that the weights induced are $1, 2, ..., n$ is called a *complete labeling* of G. Thus, any graceful labeling of G is also a complete labeling of G. In the next theorem, we will construct complete labelings of the cycle C_m where $m \equiv 0$ or $3 (\mathrm{mod}\, 4)$, $m \geq 11$, such that the labels used are are in the set $\{0, 1, ..., m-1, m+1\}$.

THEOREM 3.1. *Let C_m be a graceful cycle of size m greater than 8. Then, there exists a complete labeling of C_m with labels taken from $\{0, 1, ..., m-1, m+1\}$.*

Proof. We break the proof in two cases.

Case 1: If $m \equiv 3(\mathrm{mod}\, 4)$. That is $C_m = C_{4k-1}$ and $k \geq 3$.

First at all, we decompose C_{4k-1} into four paths when $k \equiv 0$ or $1(\mathrm{mod}\, 3)$ and five paths when $k \equiv 2(\mathrm{mod}\, 3)$. Let P_t be any of these paths, its consecutive vertices are $v_0, v_1, ..., v_{t-1}$. The labeling of the vertices of P_t will be denoted by f. Consider the path P_{2k+2}, the labeling f is defined by:

$$f(v_0) = 3k+1, \quad f(v_{2k+1}) = 4k,$$
$$f(v_{2i}) = 3k-i, \quad 1 \le i \le k$$
$$f(v_{2i+1}) = k+i, \; 0 \le i \le k-1.$$

The distribution of weights is $2k+1, 2k-1, 2k-2, ..., 1, 2k$. The end-vertices of this path are labeled $3k+1$ and $4k$. Now, we distinguish three subcases.

Subcase 1.1: When $k \equiv 0 \pmod 3$.

First, take the path $P_{\frac{2k}{3}}$, the labeling f is defined on its vertices by:

$$f(v_0) = 4k,$$
$$f(v_{2i}) = 4k-1-3i, \; 1 \le i \le \tfrac{k-3}{3}$$
$$f(v_{2i+1}) = 1+3i, \quad 0 \le i \le \tfrac{k-3}{3}$$

The distribution of weights is $4k-1, 4k-5, 4k-8, ..., 2k+4$. The end-vertices of this path are labeled $4k$ and $k-2$.

Second, take the path $P_{\frac{2k}{3}+1}$, the labeling f is defined on its vertices by:

$$f(v_0) = k-2,$$
$$f(v_{2i}) = k-3i, \quad 1 \le i \le \tfrac{k}{3}$$
$$f(v_{2i+1}) = 3k+3i, \; 0 \le i \le \tfrac{k-3}{3}$$

The weight distribution is $2k+2, 2k+3, 2k+6, ..., 4k-3$. The end-vertices of this path are labeled $k-2$ and 0.

Third, take the path $P_{\frac{2k}{3}}$, the labeling f is defined on its vertices by:

$$f(v_0) = 0,$$
$$f(v_{2i}) = 3i-1, \quad 1 \le i \le \tfrac{k-3}{3}$$
$$f(v_{2i+1}) = 4k-2-3i, \; 0 \le i \le \tfrac{k-3}{3}$$

The weight distribution is $4k-2, 4k-4, 4k-7, ..., 2k+5$. The end-vertices of this path are labeled 0 and $3k+1$.

Subcase 1.2: When $k \equiv 1 \pmod 3$.

First, take the path $P_{\frac{2}{3}(k-1)+1}$, the labeling f is defined on its vertices by:

$$f(v_0) = 4k,$$
$$f(v_{2i}) = 4k-1-3i, \; 1 \le i \le \tfrac{k-1}{3}$$
$$f(v_{2i+1}) = 1+3i, \quad 0 \le i \le \tfrac{k-4}{3}$$

The distribution of weights is $4k-1, 4k-5, 4k-8, ..., 2k+3$. The end-vertices of this path are labeled $4k$ and $3k$.

Second, take the path $P_{\frac{2}{3}(k-1)+1}$, the labeling f is defined on its vertices by:

$$f(v_0) = 3k,$$
$$f(v_{2i}) = 3k + 3i - 1, \ \ 1 \le i \le \tfrac{k-1}{3}$$
$$f(v_{2i+1}) = k - 2 - 3i, \ 0 \le i \le \tfrac{k-4}{3}$$

The weight distribution is $2k + 2, 2k + 4, 2k + 7, ..., 4k - 4$. The end-vertices of this path are labeled $3k$ and $4k - 2$.

Third, take the path $P_{\frac{2}{3}(k-1)+1}$, the labeling f is defined on its vertices by:

$$f(v_0) = 4k - 2,$$
$$f(v_{2i}) = 4k - 3i, \ 1 \le i \le \tfrac{k-1}{3}$$
$$f(v_{2i+1}) = 3i, \quad 0 \le i \le \tfrac{k-4}{3}$$

The weight distribution is $4k - 2, 4k - 3, 4k - 6, ..., 2k + 5$. The end-vertices of this path are labeled $4k - 2$ and $3k + 1$.

Subcase 1.3: When $k \equiv 2 (\bmod 3)$.

First, take the path P_9 whose vertices are labeled $4k, 1, 4k - 3, 0, 4k - 2, 3, 4k - 5, 2,$ and $4k - 4$ respectively. The weights induced by this labeling are $4k - 8, 4k - 7, ..., 4k - 1$. The end-vertices are labeled $4k$ and $4k - 4$.

Note that the smallest cycle in this subcase is C_{19}, for $k = 5$, which has been completely labeled. So, in the follow we assume that $k \ge 8$.

Second, take the path $P_{\frac{2}{3}(k-2)}$, the labeling f is defined on its vertices by:

$$f(v_0) = 4k - 4, \qquad f(v_{\frac{2}{3}(k-5)+1}) = k - 2$$
$$f(v_{2i}) = 4k - 5 - 3i, \ 1 \le i \le \tfrac{k-5}{3}$$
$$f(v_{2i+1}) = 5 + 3i, \quad 0 \le i \le \tfrac{k-8}{3}$$

The distribution of weights is $4k - 9, 4k - 13, 4k - 16, ..., 2k + 3$, and $2k + 2$. The end-vertices of this path are labeled $4k - 4$ and $k - 2$.

Third, take the path $P_{\frac{2}{3}(k-5)}$, the labeling f is defined on its vertices by:

$$f(v_0) = k - 2,$$
$$f(v_{2i}) = k - 2 - 3i, \quad 1 \le i \le \tfrac{k-8}{3}$$
$$f(v_{2i+1}) = 3k - 1 + 3i, \ 0 \le i \le \tfrac{k-5}{3}$$

The weight distribution is $2k + 4, 2k + 7, ..., 4k - 12$. The end-vertices of this path are labeled $k - 2$ and $4k - 6$.

Fourth, take the path $P_{\frac{2}{3}(k-5)+1}$, the labeling f is defined on its vertices bv:

$f(v_0) = 4k - 6,$
$f(v_{2i}) = 4k - 4 - 3i,\ 1 \le i \le \frac{k-5}{3}$
$f(v_{2i+1}) = 1 + 3i, \quad 0 \le i \le \frac{k-5}{3}$

The weight distribution is $4k-10, 4k-11, 4k-14, ..., 2k+5$. The end-vertices of this path are labeled $4k-6$ and $3k+1$.

Now, for each subcase consider the corresponding paths together with the path P_{2k+2}, and connect them identifying the end-vertices with the same labels. Thus, we have constructed a complete labeling of the cycle C_{4m-1} for $m \ge 3$.

Before we consider the last case, note that the following subsets of $V(C_m)$ form a partition of $V(C_m)$: $V_1 = \{v \in V(C_m) : 0 \le f(v) \le 2k-1\}$, $V_2 = \{v \in V(C_m) : f(v) = 2k\}$, and $V_3 = \{v \in V(C_m) : 2k+1 \le f(v) \le 4k\}$.

Case 2: If $m \equiv 0 (\mathrm{mod}\, 4)$. That is $C_m = C_{4k}$ and $k \ge 3$.

In order to construct the complete labeling of C_{4k} we need C_{4k-1} labeled as in Case 1. The weight 1 will be obtained on the edge with end-vertices labeled $2k-1$ and $2k$. Replace this edge by the path P_3 labeled $2k-1, 2k+1, 2k$ (weights 1 and 2), and also add 1 to the label of every vertex of V_3. So the weights of the edges connecting vertices of V_1 and V_2 with vertices of V_3 are increased one unit. Hence, we have C_{4k} with the complete labeling required. ■

LEMMA 3.1. *Let G be a graph of size n and let f be a complete labeling of G, such that the labels assigned by f are taken from $\{0, 1, ..., n-1, n+1\}$. Then, when n_1, $n_2 \ge 2$ the graph $H = G \cup K_{n_1,n_2}$ is graceful.*

Proof. Let $\{A, B\}$ be the partition of K_{n_1,n_2}, where $|A| = n_1$, $|B| = n_2$. Without loss of generality, we may assume that $n_1 \le n_2$. Let $g : V(H) \to \{0, 1, ..., n+n_1n_2\}$ be a one-to-one mapping such that g assigns the integers $0, 1, ..., n_1-1$ on the vertices of A, the integers $n+n_1, n+2n_1, ..., n+n_1n_2$ on the vertices of B (inducing the weights $n+1, n+2, ..., n+n_1n_2$.) And for every $v \in V(G)$, we have $g(v) = f(v) + n_1$ (inducing the weights $1, 2, ..., n$.)

Hence, we just need to check that all the labels used are different. The labels on G are in the set $\{n_1, n_1+1, ..., n+n_1-1, n+n_1+1\}$, the labels on A are $0, 1, ..., n_1-1$ and the labels on B are $n+n_1, n+2n_1, ...n+n_1n_2$. Using the fact that $n_1 \ge 2$ we have that g is a graceful labeling of H. ■

Now, we are able to prove our main theorem.

THEOREM 3.2. *For any $m \equiv 0$ or $3 (\mathrm{mod}\, 4)$, $m \ge 11$ and n_1, $n_2 \ge 2$, the graph $C_m \cup K_{n_1,n_2}$ is graceful.*

Proof. From the previous theorem we know that under these conditions, C_m has a complete labeling f with labels taken from $\{0, 1, ..., m-1, m+1\}$. Hence, applying Lemma 1 with this labeling, we have that $C_m \cup K_{n_1,n_2}$ is graceful. ■

A slightly different version of the Lemma 1 is presented now.

LEMMA 3.2. *Let G be a graceful graph of size n and let f be a graceful labeling of G, such that the labels assigned by f are taken from $\{0, 1, ..., n-2, n\}$. Then, when n_1, $n_2 \geq 2$ the graph $H = G \cup K_{n_1,n_2}$ is also graceful.*

Proof. Let g be defined on the vertices of K_{n_1,n_2} as in Lemma 1, and for every $v \in V(G)$, we have now $g(v) = f(v) + 1 + n_1$. So, the labels used on G are in the set $\{1 + n_1, 2 + n_1, ..., n - 1 + n_1, n + 1 + n_1\}$. Then, the labeling g is a graceful labeling of H. ■

For example, the labeling of K_3 with labels $0, 1, 3$ or the labeling of K_4 with labels $0, 1, 4, 6$, may be used to construct graceful labelings of K_3 or K_4 union K_{n_1,n_2} with n_1, $n_2 \geq 2$.

Consider the complete bipartite graphs K_{m_1,n_1} and K_{m_2,n_2} where $2 \leq m_i < n_i$ $(i = 1, 2)$, with partition $\{A_i, B_i\}$ and $|A_i| = m_i$, $|B_i| = n_i$. Let f be the labeling of K_{m_1,n_1} that assigns on the vertices of A_1 the integers $0, 1, ..., m_1 - 1$ and on the vertices of B_1 the integers $m_1, 2m_1, ..., m_1n_1$. Since $n_1 > 2$ we have that f satisfies the conditions of Lemma 2; therefore, $K_{m_1,n_1} \cup K_{m_2,n_2}$ is graceful. Note that the greatest labels used are $m_1n_1 + m_2n_2 - m_2$ and $m_1n_1 + m_2n_2$; so, using the fact that $m_2 \geq 2$ it can be proved that $K_{m_1,n_1} \cup K_{m_2,n_2} \cup K_{m_3,n_3}$, where $2 \leq m_3 < n_3$, is graceful. Then, by recurrence we may prove the following theorem.

THEOREM 3.3. *Let $2 \leq m_i < n_i$ for $1 \leq i \leq t$ then, the union $\bigcup_{i=1}^{t} K_{m_i,n_i}$ is graceful.*

Combining the results of Theorem 4 and Lemma 2, the next theorem can be proved.

THEOREM 3.4. *For any $m \equiv 0$ or $3 (\text{mod}\, 4)$, $m \geq 11$ and $2 \leq m_i \leq n_i$, the union of C_m and $\bigcup_{i=1}^{t} K_{m_i,n_i}$ is graceful.*

REMARK 3.1. All the graceful labelings of bipartite graphs constructed here satisfy the conditions to be α-labelings.

Suppose now that $m \equiv 1$ or $2 (\text{mod}\, 4)$ and that n_1 and n_2 are even, by the parity condition we know that $C_m \cup K_{n_i,n_2}$ cannot be graceful, however we do not know what happens when n_1 or n_2 is odd. In the following theorem we show a nearly graceful labeling of C_m that is the complementary labeling of C_m that the author gave in [1]. We will use this labeling in the next theorem.

THEOREM 3.5. *When $m \equiv 1$ or $2 \pmod 4$, the cycle C_m has a nearly graceful labeling f of kind 1.*

We omit the proof and just give the labeling f. Let C_m be described by a circuit $v_1, v_2, ..., v_m, v_1$.

When $m \equiv 1 \pmod 4$,

$$f(v_i) = \begin{cases} m+1, & \text{if } i = 1 \\ m+1-(i+1)/2, & \text{if } i = 3, 5, ..., (m-3)/2 \\ m+1-(i+3)/2, & \text{if } i = (m+1)/2, (m+5)/2, ..., m \\ i/2-1, & \text{if } i \text{ is even.} \end{cases}$$

When $m \equiv 2 \pmod 4$,

$$f(v_i) = \begin{cases} m+1, & \text{if } i = 1 \\ m+1-(i+1)/2, & \text{if } i = 3, 5, ..., (m-4)/2 \text{ and } m \geq 10 \\ m+1-(i+3)/2, & \text{if } i = m/2, (m+4)/2, ..., m-1 \text{ for every } m \\ i/2-1, & \text{if } i \text{ is even.} \end{cases}$$

This labeling never assigns the label m on the vertices of C_m. Then, we may apply the idea of Lemma 1 to find a labeling of $C_m \cup K_{n_1,n_2}$ whose greatest label is $m + n_1 n_2 + 1$.

THEOREM 3.6. *For every $m \equiv 1$ or $2 \pmod 4$ and $n_1, n_2 \geq 2$, the graph $C_m \cup K_{n_1,n_2}$ has a labeling with maximum label $m + n_1 n_2 + 1$ and whose induced weights are all different.*

Proof. Let $\{A, B\}$ be the partition of K_{n_1,n_2}. The vertices of A and B are labeled as in the proof of Lemma 1, but now we increase one unit every label of B. Thus, the weights induced on K_{n_1,n_2} are $m+2, m+3, ..., m+n_1 n_2+1$. The labels on A are $0, 1, ..., n_1 - 1$ and the labels on B are $m + n_1 + 1, m + 2n_1 + 1, ..., m + n_1 n_2 + 1$.

Suppose now that the vertices of C_m have been labeled using the function f of Theorem 7. Then, adding the constant $n_1 + 1$ on every vertex labeled, we have a labeling of C_m with weights $1, 2, ..., m-1$ and $m+1$. The labels on C_m are now in the set $\{n_1 + 1, n_1 + 2, ..., m + n_1 + 2\}$. Thus, every induced weigh appears exactly once and there are not overlapping of labels. Since the greatest label assigned is $m+n_1n_2+1$, this is the labeling required. ■

As consequence of this theorem we have that grac($C_m \cup K_{n_1,n_2}$) is $m+n_1n_2+1$ when both n_1 and n_2 are even; and that is at most $m+n_1n_2+1$ otherwise. However, the author believes that when n_1 or n_2 is odd, the graph $C_m \cup K_{n_1,n_2}$ is graceful. We have some examples that support this idea; for instance, from Section 2 we know that $C_m \cup K_{1,n}$ is graceful for every $m \geq 7$ and $n \geq 1$; we also know that $C_m \cup K_{2,n}$ is graceful for $m \in \{5,6,9,10\}$ and n odd greater than 1. Thus, is an open problem to determine grac($C_m \cup K_{n_1,n_2}$) for the cases where $m \equiv 1$ or $2 \pmod 4$ and n_1 or n_2 is odd greater than 1.

To conclude, we summarize the results in the next table.

G	grac(G)
$C_m \cup K_{n_1,n_2}$	$m+n_1n_2$ for $m \equiv 0$ or $3 \pmod 4$ and $n_1, n_2 \geq 2$
$C_m \cup K_{n_1,n_2}$	$m+n_1n_2+1$ for $m \equiv 1$ or $2 \pmod 4$ and n_1, n_2 even
$C_m \cup K_{n_1,n_2}$	$\leq m+n_1n_2$ otherwise

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