Chordal Bipartite Analogs of 2-Trees and Isolated Failure Immune Networks

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Abstract

2-trees are defined recursively, starting from a single edge, by repeatedly erecting new triangles onto existing edges. These have been widely studied in connection with chordal graphs, series-parallel graphs, and isolated failure immune ('IFI') networks.

A similar family, based on recursively erecting new $K_{2,h}$ subgraphs onto existing edges, is shown to have analogous connections to chordal bipartite graphs, series-parallel graphs, and a notion motivated by IFI networks.

Keywords: 2-trees; Chordal graphs; Series-parallel graphs; Chordal bipartite graphs; Isolated failure immune networks; IFI networks.

1 Introduction

Define *2-trees* inductively as follows:

- K_2 is a 2-tree.
- If G is any 2-tree with $e \in E(G)$ and if $H \cong K_3$ is vertex disjoint from G with $e' \in E(H)$, then the graph formed from G and H by identifying edges e and e' (along with their endpoints) is also a 2-tree.

Such 2-trees have been widely studied; see [2]. A *nontrivial 2-tree* is a 2-tree with ≥ 3 vertices. In the language of [5], the nontrivial 2-trees are precisely the graphs that have 'simplicial tree decompositions into triangles.'

The purpose of the present paper is to present and advocate a bipartite analog of 2-trees, which can be roughly described as being based on *quadrilaterals* (induced cycles of length four), instead of triangles. But this is not done by simply replacing the role of triangles with quadrilaterals, as might be expected from [4, 11]. A somewhat different approach is needed to preserve connections with chordal graphs, series-parallel graphs, and isolated failure immune networks, using the chordal graph connection as a guide.

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Section 2 will first survey 2-trees as chordal graphs, then Section 3 will propose the (chordal) bipartite analog of 2-trees. Section 4 will look at connections with isolated failure immune networks.

2 The chordal approach to 2-trees

A graph is *chordal* if every cycle of length at least four has a chord, that is, every cycle long enough to have a chord, has a chord; see [2, §1.2] or [9, Chapter 2] for details. A vertex v is a *simplicial vertex* in a graph G if its open neighborhood $N(v)$ induces a complete subgraph of G. A graph G has a *perfect vertex elimination ordering* $v_1, \ldots, v_n$ if, for each $i < n = |V(G)|$, v_i is a simplicial vertex in the subgraph of G induced by $\{v_i, \ldots, v_n\}$. A graph is chordal if and only if it has a perfect vertex elimination ordering, and the nontrivial 2-trees are precisely the chordal graphs in which the simplicial vertex v_i has degree two whenever $i < n-1$.

Patil [10] seems to have been the first to mention what corresponds to the following result.

Proposition 1 ([10]) *A graph G is a nontrivial 2-tree if and only if it is an edge-minimal 2-connected chordal graph* (in other words, G is 2-connected and chordal, but deleting any edge would produce a graph that is not).

Another approach to 2-trees involves *series-parallel graphs* [2, §11.2], a well-studied class that is traditionally studied in the context of multigraphs. For our purposes, define *2-connected series-parallel graphs* inductively as follows:

- K_3 is a 2-connected series-parallel graph.
- If G is 2-connected series-parallel with $e \in E(G)$ and if $H \cong K_3$ is vertex disjoint from G with $e' \in E(H)$, then the graph formed from G and H by identifying edges e and e' (along with their endpoints) is also a 2-connected series-parallel graph.
- If G is 2-connected series-parallel with $vw \in E(G)$ and $x \notin V(G)$, then the graph formed by replacing vw with the path vx, xw is also a 2-connected series-parallel graph (i.e., closure under edge bisection).

This is also equivalent to G being 2-connected with no subgraph homeomorphic to K_4 [2].

The following connection between 2-trees and series-parallel graphs corresponds to the result of Wald and Colbourn [12] that series-parallel graphs are precisely the 'partial 2-trees.'

Proposition 2 ([12]) *A graph is a nontrivial 2-tree if and only if it is an edge-maximal 2-connected series-parallel graph* (in other words, G is 2-connected series-parallel, but inserting any new edge would produce a graph that is not).

Theorems 3 and 4 are simple consequences of these two Propositions that will have bipartite analogs in Section 3.

Theorem 3 *A graph is a nontrivial 2-tree if and only if it is a 2-connected series-parallel chordal graph.*

Proof. Every nontrivial 2-tree is easily seen to be a 2-connected series-parallel chordal graph.

Conversely, suppose G is a 2-connected series-parallel chordal graph. Delete edges from G in order to produce an edge-minimal 2-connected chordal graph G^-, which is a 2-tree by Proposition 1. Insert edges into G in order to produce an edge-maximal 2-connected series-parallel graph G^+, which is a 2-tree by Proposition 2. A simple inductive argument shows that every 2-tree must have exactly $2|V| - 3$ edges, so $G^- = G^+ = G$ is a nontrivial 2-tree. □

Define the *sum of cycles* to be the symmetric difference of the cycles' edge sets, as in the usual treatments of cycle spaces.

Theorem 4 *A 2-connected graph is a 2-tree if and only if every k-cycle is uniquely the sum of $k - 2$ triangles* (in other words, is the sum of a unique set of $k - 2$ triangles).

Proof. The 'only if' direction follows by a straightforward induction argument, paralleling the recursive definition of 2-trees.

For the 'if' direction, suppose that G is 2-connected with every k-cycle uniquely the sum of $k-2$ triangles. Jamison's characterization [7] of chordal graphs as those graphs in which every k-cycle is the sum of $k - 2$ triangles shows that G is chordal. Every edge of G is either in only one triangle (in which case its removal would destroy being 2-connected) or is in at least two triangles (in which case it would be the unique chord of a 4-cycle, and so its removal would destroy being chordal). Thus, G is edge-minimal 2-connected chordal, and the theorem follows from Proposition 1. □

3 The chordal bipartite approach to 2∗-trees

A graph is *chordal bipartite* if it is a bipartite graph in which every cycle of length at least six had a chord, that is, every cycle long enough to have a chord, has a chord; see [2, §3.3, 5.9] or [9, §7.3] for details and other characterizations.

An edge vw is a *bisimplicial edge* in a graph G if $N(v) \cup N(w)$ induces a complete bipartite subgraph of G. A bipartite graph G has a *perfect edge elimination ordering* $e_1, \ldots, e_m$ (called a 'perfect edge-without-vertex elimination ordering' in [2]) if, for each $i < m = |E(G)|$, e_i is a bisimplicial edge in the subgraph of G consisting of $\{e_i, \ldots, e_m\}$. For convenience, delete every degree-one vertex that is formed along the way. A graph is chordal bipartite if and only if it has a perfect edge elimination ordering; see [1, 2].

Define *2∗-trees* inductively as follows:

- K_2 is a 2∗-tree.
- If G is any 2∗-tree with $e \in E(G)$ and if $H \cong K_{2,h}$ $(h \geq 2)$ is vertex disjoint from G with $e' \in E(H)$, then the graphs formed from G and H by identifying edges e and e' (along with their endpoints) are also 2∗-trees.

As shown in Figure 1, two graphs can be produced in the recursive step when $h > 2$, depending on which end of e is identified with which end of e'. A *nontrivial 2∗-tree* is a 2∗-tree with ≥ 3 vertices. In the language of [5], the nontrivial 2∗-trees are precisely the 'simplicial tree decompositions into $K_{2,b}$s.'

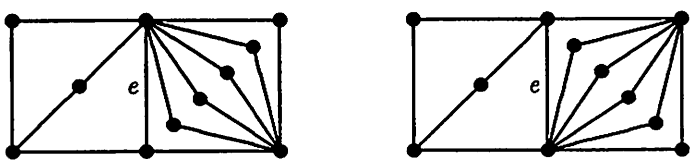

Figure 1: The two 2∗-trees built from one $K_{2,3}$ and one $K_{2,6}$.

Notice that a graph is a 2∗-tree if and only if it has a perfect edge elimination ordering $e_1, \ldots, e_m$ such that each $e_i = v_i w_i$ has $N(v_i) \cup N(w_i) \cong K_{2,h}$ $(h \geq 2)$ whenever $i < m - 2$. This is the bipartite analog of a graph being a 2-tree if and only if it has a perfect vertex elimination ordering $v_1, \ldots, v_n$ such that each v_i has $N(v_i) \cong K_2$ in the subgraph of G induced by $\{v_i, \ldots, v_n\}$ whenever $i < n - 1$.

While every 2-tree satisfies $|E| = 2|V| - 3$, Figure 2 shows that the number of vertices of a 2∗-tree does not determine the number of edges. This

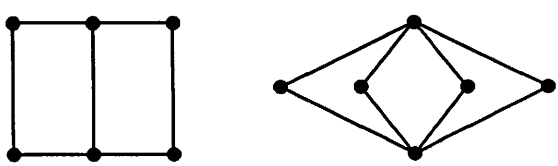

Figure 2: Two 2∗-trees with the same number of vertices, but not of edges.

is one of several important differences between the studies of 2-trees and

2∗-trees. Another difference is that, although Corollary 6 will be the analog of Proposition 1, Figure 3 shows there is no direct analog to Proposition 2: The graph there is edge-maximal 2-connected series-parallel and bipartite, yet is not a 2∗-tree.

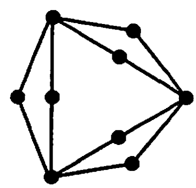

Figure 3: An edge-maximal 2-connected series-parallel bipartite graph that is not a 2∗-tree.

In spite of the failure of a Proposition 2 analog, Theorem 5 is the bipartite analog of Theorem 3; Corollary 6 and Theorem 7 are the bipartite analogs of Proposition 1 and Theorem 4, respectively.

Theorem 5 *A graph is a nontrivial 2∗-tree if and only if it is a 2-connected series-parallel chordal bipartite graph.*

Proof. The 'only if' direction follows by a straightforward inductive argument, paralleling the recursive definition of 2∗-trees.

Conversely, suppose G is 2-connected series-parallel and chordal bipartite. Suppose vw is a bisimplicial edge of G and H is the complete bipartite subgraph of G induced by $N(v) \cup N(w)$. Since G is 2-connected and series-parallel, G cannot contain a $K_3, 3$ subgraph, and so either $|N(v)| = 2$ or $|N(w)| = 2$. Without loss of generality, suppose $H \cong K_{2,h}$ $(h \geq 2)$ where $N(w) = \{v, w'\}$ and $h = |N(v)|$. If $H = G$, then G is a 2∗-tree. Otherwise, assume (inductively) that every chordal bipartite, 2-connected series-parallel proper subgraph of G is 2∗-tree. Because G is 2-connected and series-parallel, G cannot contain a subgraph homeomorphic to K_4, and so no two of vertices in $N(v) - \{w\}$ can have degree greater than 2. If every vertex in $N(v) - \{w\}$ has degree 2, then (since w' cannot be a cut vertex) $G \cong H \cong K_{2,h}$, and so G is a 2∗-tree. So suppose $v' \in N(v)$ such that $N(v') \geq 3$ and every vertex in $N(v) - \{v'\}$ has degree 2. Then $H \cong K_{2,h}$ shares only the edge $v'w'$ in common with the graph G^- induced by $(V(G) - N[v]) \cup \{v'\}$, where the inductive hypothesis implies that G^- is a 2∗-tree. Therefore, G is a nontrivial 2∗-tree. □

Corollary 6 *A graph is a nontrivial 2∗-tree if and only if it is an edge-minimal 2-connected chordal bipartite graph.*

Proof. The 'only if' direction follows by a straightforward inductive argument, paralleling the recursive definition of 2∗-trees.

Conversely, suppose G is a graph with as few vertices as possible such that G is an edge-minimal 2-connected chordal bipartite graph that is not series-parallel (arguing toward a contradiction with Theorem 5). Then G is not itself complete bipartite and contains a subgraph H that is homeomorphic to K_4; assume H has the minimum number of vertices among such subgraphs and (since induced subgraphs of chordal bipartite graphs are chordal bipartite, and using the assumed minimality of $|V(G)|$) assume H spans G. Let vw be any bisimplicial edge of G. Since G is 2-connected but not complete bipartite, there must be vertices $v' \in N(v) - \{w\}$ and $w' \in N(w) - \{v\}$ that have neighbors outside of H, and so such that $v'w'$ is an edge in a cycle of G that contains no other vertex of $N(v) \cup N(w)$. But then the subgraph of G induced by $V(G) - \{v, w\}$ would also contain a subgraph homeomorphic to K_4 with two fewer vertices than H, contradicting the assumed minimality of $|V(H)|$. □

Theorem 7 *A 2-connected graph is a 2∗-tree if and only if every k-cycle is uniquely the sum of* $\frac{k}{2} - 1$ *quadrilaterals.*

Proof. The 'only if' direction follows by a straightforward induction argument, paralleling the recursive definition of 2∗-trees.

For the 'if' direction, suppose that G is 2-connected with every k-cycle uniquely the sum of $k/2 - 1$ quadrilaterals. The characterization in [8] of chordal bipartite graphs as those graphs in which every k-cycle is the sum of $k/2 - 1$ quadrilaterals (which implies that k must be even and so that the graph is bipartite) shows that G is chordal bipartite. Every edge of G is either in only one quadrilateral (in which case its removal would destroy being 2-connected) or is in at least two quadrilaterals (in which case it would be the unique chord of a 6-cycle, and so its removal would destroy being chordal bipartite). Thus, G is edge-minimal 2-connected chordal bipartite, and the theorem follows from Corollary 6. □

Observe that each 2∗-tree can be simply altered to become a 2-tree by inserting one edge into each $K_{2,h}$ factor ($h \geq 2$) so as to make it into a $K_{1,1,h}$. Since a simple inductive argument shows that a 2∗-tree contains exactly $2|V(G)| - |E(G)| - 3$ many $K_{2,h}$ factors, exactly that many edges need to be inserted to form the 2-tree.

4 Chordal (bipartite) versions of IFI networks

A set S of *elements*—meaning $S \subseteq V(G) \cup E(G)$—is a *separating set* of a connected graph G if the removal of all the elements in S leaves a subgraph that is not connected. (Removing the elements of S includes deleting all the edges incident with each vertex in S, but not deleting an endpoint of an edge in S unless that vertex is also in S.) A set S of elements is *isolated* [6] if:

- No two vertices in S are incident with a common edge of G.
- No two edges in S are incident with a common vertex of G.
- No vertex v and edge e in S are incident with, respectively, an edge e' and vertex v' of G such that v' is incident with e' (and so v is not incident with e).

In [6], Farley introduced *isolated failure immune—or IFI—networks* as graphs in which no isolated set of elements is a separating set. Recent papers on IFI networks include [3, 13]. Observe that a straightforward inductive argument shows that *every chordal graph is an IFI network.* The intimate connection between IFI networks and 2-trees includes Proposition 8.

Proposition 8 ([6]) *Every 2-connected graph that has a spanning 2-tree is an IFI network.*

Figure 4 shows that the converse to Proposition 8 fails (contrary to a misstatement in [13]). The graph in Figure 4 is IFI by [6, Theorem 1], but any spanning 2-tree would have to contain all four 'corner' triangles, and so the other four triangles as well.

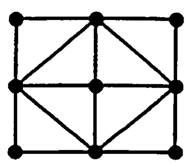

Figure 4: A graph that is an isolated failure immune network, but has no spanning 2-tree; it is also edge-minimal IFI, but not edge-minimum.

An IFI network is *edge-minimum* if it has the minimum possible number of edges, $2|V|-3$; it is *edge-minimal* if deleting each edge would leave a non-IFI network. Farley [6] also showed that every 2-tree is an edge-minimum (and so edge-minimal) IFI network. Wald and Colbourn [12] then showed that every edge-minimum IFI network is a 2-tree—but not every edge-minim*al* IFI network is, as Figure 4 also shows (contrary to a misstatement in [3]). These results combine to give Proposition 9.

Proposition 9 ([6, 12]) *A graph is a 2-tree if and only if it is an edge-minimum IFI network.*

Theorem 10 is similar to Proposition 9, but is stated in terms that will have an analog for 2∗-trees in Theorem 11 (using edge-minim*al* instead of edge-minim*um*, as is necessary because the number of vertices does not determine the number of edges in a 2∗-tree).

Theorem 10 *A graph is a 2-tree if and only if it is chordal and edge-minimal with respect to every separating set of elements containing two elements from a common triangle.*

Proof. The 'only if' direction follows by a straightforward inductive argument, paralleling the recursive definition of 2-tree.

Conversely, suppose G is chordal and is edge-minimal with respect to every separating set of elements containing two elements from a common triangle, yet G is not a 2-tree (arguing toward a contradiction); moreover, among all such G, assume G has a minimum number of vertices. Then, since G is chordal, there exists a simplicial vertex v of G (meaning that $N(v)$ induces a complete subgraph K_a in G). By G's assumed vertex-minimality as a non 2-tree, $a \geq 3$. Let $G^- = G - v$. By G's edge-minimality, there exists a separating set S^- of elements of G^-, no two of whose elements lie in a common triangle of G^-; assume as well that S^- is element-minimal (in other words, no proper subset of S^- has the property of being a separating set of G^- with no two elements in a common triangle of G^-).

CASE 1: S^- is also a separating set for G. Then G being edge-minimal ensures that some two elements x and y of S^- must lie in a common triangle of G; since there can be no such triangle in G^-, that common triangle must also contain v. But then $a \geq 3$ would imply that $N(v)$ already contained a triangle in G^- that contained x and y (a contradiction).

CASE 2: S^- is not a separating set for G. Yet $S^- \cup \{v\}$ is a separating set for G. Since some two elements of $S^- \cup \{v\}$—but no two elements of S^-—are in a common triangle, v must be in a common triangle with some element x of S^-; indeed x will be the unique element of S^- inside the subgraph induced by $N(v)$. If x is an edge, then $a \geq 3$ would imply that the endpoints of x are still connected inside $N(v)$ in G^-, and so $S^- - \{x\}$ would also be a separating set of G^- (contradicting S^- being element-minimal). If x is a vertex, then $S^- \cup \{v\}$ being a separating set for G would imply the same for S^- (contradicting the premise of Case 2). □

Theorem 11 *A graph is a 2∗-tree if and only if it is chordal bipartite and edge-minimal with respect to every separating set of elements containing two elements from a common quadrilateral.*

Proof. The 'only if' direction follows by a straightforward inductive argument, paralleling the recursive definition of 2∗-trees.

Conversely, suppose G is chordal bipartite and is edge-minimal with respect to every separating set of elements containing two elements from a common quadrilateral, yet G is not a 2∗-tree (arguing toward a contradiction); moreover, among all such G, assume G has a minimum number of vertices. Then, since G is chordal bipartite, there exists a bisimplicial edge vw of G where $N(v) \cup N(w)$ induces a $K_{a,b}$ in G. By G's assumed assumed vertex-minimality as a non 2∗-tree, both $a, b \geq 3$. Let $G^- = G - vw$. By G's edge-minimality, there exists a separating set S^- of elements of G^-, no two of whose elements lie in a common quadrilateral of G^-; assume as well that S^- is element-minimal.

CASE 1: S^- is also a separating set for G. Then G being edge-minimal ensures that some two elements x and y of S^- must lie in a common quadrilateral of G; since there can be no such quadrilateral in G^-, that common quadrilateral must also contain edge vw. But then $a, b \geq 3$ would imply that $N(v) \cup N(w)$ also contained a quadrilateral in G^- that contained x and y (a contradiction).

CASE 2: S^- is not a separating set for G. Yet $S^- \cup \{vw\}$ is a separating set for G. Since some two elements of $S^- \cup \{vw\}$—but no two elements of S^-—are in a common quadrilateral, vw must be in a common quadrilateral with some element x of S^-; indeed x will be the unique element of S^- inside the subgraph induced by $N(v) \cup N(w)$. If x is an edge, then $a, b \geq 3$ would imply that the endpoints of x are still connected inside $N(v) \cup N(w)$ in G^-, and so $S^- - \{vw\}$ would also be a separating set of G^- (contradicting that S^- is element-minimal). If x is a vertex, then $S^- \cup \{vw\}$ being a separating set for G would imply the same for S^- (contradicting the premise of Case 2). □

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