d – ANTIMAGIC LABELINGS OF PLANE GRAPHS P_a^b

KM. KATHIRESAN AND R. GANESAN

ABSTRACT. We refer to a labeling of a plane graph as a d – antimagic labeling if the vertices, edges and faces of the graph are labeled in such a way that the label of a face and the labels of the vertices and edges surrounding that face add up to a weight of that face and the weights of all the faces form an arithmetic progression of difference d. This paper describes d – antimagic labelings for special class of plane graphs.

1. INTRODUCTION AND DEFINITIONS

The graphs considered here will be finite, undirected, without loops and multiple edges. The graph G has vertex set V (G) and edge set E(G) and we let $|V(G)| = v$ and $|E(G)| = e$.

A graph G is said to be plane if it is drawn on the Euclidean plane in such a way that its edges do not cross each other except at the vertices of the graph. Assume that all the plane graphs considered in this paper possess no vertices of degree one. For a plane graph G, it makes sense to consider its faces, including the unique face of infinite area. Let $F(G)$ be the face-set of G and $|F(G)| = f$. General references for graph – theoretic notions are [17] and [18].

A labeling of a graph is any mapping that sends some set of graph elements to a set of numbers (usually positive integers). Specially, if we have bijection $\alpha : V(G) \rightarrow \{1,2,3,\ldots,v\}$ or $\beta : E(G) \rightarrow \{1,2,3,\ldots,e\}$ or $\gamma : F(G) \rightarrow \{1,2,3,\ldots,f\}$ then the corresponding labeling is called a vertex labeling or an edge labeling or a face labeling respectively.

A bijection $\phi : V(G) \cup E(G) \cup F(G) \rightarrow \{1,2,3, \ldots, v + e + f\}$ is called a labeling of type (1,1,1) and a labeling of type (1,1,0) is a bijection from the set $\{1,2,3,\ldots, v + e\}$ onto the vertices and edges of plane graph G (V,E,F). The weight of a face under a labeling is the sum of the labels (if present) carried by that face and the edges and the vertices surrounding it.

A labeling of plane graph G is called d – antimagic if for every number s the set of s - sided face weights is $W_s = \{a_s, a_s + d, a_s + 2d, \ldots, a_s + (f_s - 1)d\}$ for some integers a_s and d, $d \geq 0$, where f_s is the number of s - sided faces. We allow different sets W_s for different s. Other types of antimagic labelings were considered by Hartsfield and Ringel [9], Bodendiek and Walther [8]and Simanjuntak, Miller and Bertault [16]. If d=0 then Lih in [13] called such labeling magic (face-magic).Face- magic labelings of type (1,1,0) for wheels, friendship graphs, prisms and some of the platonic polyhedra are given in [13]. Face-magic labelings of type (1,1,1) for m-antiprisms, fans,planar bi-pyramids, ladders and the Mobius ladders are described in [2],[1] and [3]. Baca has described the face-magic labeling of type (1,1,1) for the hexagonal planar maps H_n^m (see [5]) and for the grid graphs G_n^m (see [4]).

If d=1 then d-antimagic labeling is called consecutive. Qu in [15] and Kathiresan et.al. in [10,11,12] studied consecutive labelings for special classes of plane graphs.

d-antimagic labelings of prisms and generalized Petersen graphs P(n,2) can be found in [14] and [7]. Additional known results about face-antimagic labelings are shown in [6].

Let a and b be integers such that $a \geq 2$ and $b \geq 2$. Let $y_1, y_2, \ldots, y_a$ be the a fixed vertices. We connect the vertices y_i and y_{i+1} by means of b internally disjoint paths P_i^j of length i+1 each, $1 \leq i \leq a-1$, $1 \leq j \leq b$. Let $y_i . x_{i,j,1} . x_{i,j,2} \ldots x_{i,j,i} . y_{i+1}$ be the vertices of path P_i^j, where $1 \leq i \leq a-1$ and $1 \leq j \leq b$. The resulting graph embedded in the plane is denoted by P_a^b, where $V(P_a^b) = \{ y_i : 1 \leq i \leq a \} \cup \bigcup_{i=1}^{a-1} \bigcup_{j=1}^{b} \{ x_{i,j,k} : 1 \leq k \leq i \}$ and $E(P_a^b) = \bigcup_{i=1}^{a-1} \{ y_i x_{i,j,1} : 1 \leq j \leq b \} \cup \bigcup_{i=1}^{a-1} \bigcup_{j=1}^{b} \{ x_{i,j,k} x_{i,j,k+1} : 1 \leq k \leq i-1 \} \cup \bigcup_{i=1}^{a-1} \{ x_{i,j,i} y_{i+1} : 1 \leq j \leq b \}$.

We observe that $v = \dfrac{ba(a-1)}{2} + a$ and $e = \dfrac{b(a-1)(a+2)}{2}$. The face set $F(P_a^b)$ contains b-1 (2i +2)-sided faces, for $1 \leq i \leq a-1$, and one external infinite face. Here $f = (a-1)(b-1) + 1$.

This paper describes d-antimagic labelings of type (1,1,1) for the plane graph P_a^b, $d \in \{0,1,2,3,4,6\}$.

2. VERTEX AND EDGE LABELINGS

We shall use $\lfloor n \rfloor$ to denote the greatest integer smaller than or equal to n.

If $a \geq 2$ and $b \geq 2$, we construct a vertex labeling α_1 and an edge labeling β_1 of the plane graph P_a^b in the following way.

$\alpha_1(y_i) = \frac{b}{2}(i-1)(i-2)+i$ if $1 \leq i \leq a$.

If $1 \leq i \leq a-1$ and $1 \leq j \leq b$ then

$$\alpha_1(x_{i,j,k}) = \begin{cases} \frac{bi(i-1)}{2}+i+1+\frac{j+1}{2} & \text{if } i \text{ and } j \text{ are odd and } k=1 \\ \frac{bi(i-1)}{2}+\lfloor \frac{b+1}{2} \rfloor+i+1+\frac{j}{2} & \text{if } i \text{ is odd, } j \text{ is even and } k=1 \\ \frac{bi(i-1)}{2}+i+1+j & \text{if } i \text{ is even and } k=1 \\ \frac{bi(i-1)}{2}+kb+i+2-j & \text{if } k \text{ is even, } 2 \leq k \leq i \\ \frac{bi(i-1)}{2}+(k-1)b+i+1+j & \text{if } k \text{ is odd, } 3 \leq k \leq i \end{cases}$$

$\beta_1(y_i x_{i,j,1}) = \frac{b}{2}((i+1)i-2)+j$ if $1 \leq i \leq a-1$ and $1 \leq j \leq b$

If $1 \leq i \leq a-1$ and $1 \leq j \leq b$ then

$$\beta_1(x_{i,j,k} x_{i,j,k+1}) = \begin{cases} \frac{bi(i+1)}{2}+kb+1-j & \text{if } k \text{ is odd, } 1 \leq k < i \\ \frac{bi(i+1)}{2}+(k-1)b+j & \text{if } k \text{ is even, } 2 \leq k < i \end{cases}$$

$\beta_1\ (x_{i,j,i}y_{i+1}) =$

$$\begin{cases} \frac{bi(i+3)}{2} - j + 1 & \text{if } i \text{ is odd} \\ \frac{b}{2}((i+3)i-2) + \frac{j+1}{2} & \text{if } i \text{ is even and } j \text{ is odd} \\ \frac{bi(i+3)}{2} - \lfloor \frac{b}{2} \rfloor + \frac{j}{2} & \text{if } i \text{ and } j \text{ are even} \end{cases}$$

3. THE RESULTS

With the vertex labelings and the edge labelings of the previous section in hand, we investigate d-antimagic labelings of the graph P_a^b.

Theorem 1. For $a \geq 2$ and $b \geq 2$, the plane graph P_a^b has a 0 -antimagic labeling and 2-antimagic labeling of type (1,1,1).

Proof. It is not difficult to check that the values of α_1 are $1,2,3,\ldots, \frac{ba(a-1)}{2} + a$ and that the labeling β_1 uses each integer $1,2,3,\ldots, \frac{b(a-1)(a+2)}{2}$ exactly once.

Label the vertices and the edges of P_a^b by labeling α_1 and labeling $\beta_1 + v$ respectively. The labelings α_1 and $\beta_1 + v$ combine to a labeling of type (1,1,0).

If $A= b(2i^3+4i^2-i)+6i+2i^2+2+(i+1)a(ba-b+2)$ then the weights of the (2i+2)-sided faces successively attain the values

$$\{ A - b + \lfloor \frac{b+1}{2} \rfloor + j : 1 \leq j \leq b-1 \} \text{ for i odd and}$$

$$\{ A - \lfloor \frac{b}{2} \rfloor + j : 1 \leq j \leq b-1 \} \text{ for i even, } 1 \leq i \leq a-1.$$

If we complete the face labeling with values in the set $\{ v+e+1, v+e+2, \ldots, v+e+(a-1)(b-1)\}$ and $B = b(2i^3+4i^2+2a^2+ia^2-ia-a-1)+5i+2i^2+3 + 3a + 2ai$ then the resulting labeling of type (1,1,1) can be

(i) 0-antimagic with the common weight for all (2i+2)-sided faces equal to $B - b + \lfloor \frac{b+1}{2} \rfloor$ for i odd $B - \lfloor \frac{b}{2} \rfloor$ for i even, $1 \leq i \leq a-1$ or

(ii) 2-antimagic with the weights of (2i+2)-sided faces in the set $\{ B - 2b + \lfloor \frac{b+1}{2} \rfloor + 2j : 1 \le j \le b-1\}$ for i odd and $\{ B - b - \lfloor \frac{b}{2} \rfloor + 2j : 1 \le j \le b-1\}$ for i even, $1 \le i \le a-1$.

The proof is completed.

Theorem 2. For $a \ge 2$ and $b \ge 2$, the plane graph P_a^b has a 1-antimagic labeling and 3-antimagic labeling of type (1,1,1).

Proof. Define a vertex labeling α_2 and an edge labeling β_2 of P_a^b as follows.

$\alpha_2 (y_i) = \alpha_1 (y_i)$

If $1 \le i \le a-1$ and $1 \le j \le b$ then

$$\alpha_2 (x_{i,j,k}) = \begin{cases} \frac{bi(i-1)}{2} + kb + i + 2 - j & \text{if } k \text{ is even}, 2 \le k \le i \\ \frac{bi(i-1)}{2} + (k-1)b + i + 1 + j & \text{if } k \text{ is odd}, 1 \le k \le i. \end{cases}$$

$\beta_2 (y_i x_{i,j,1}) = \beta_1 (y_i x_{i,j,1})$,

$\beta_2 (x_{i,j,k} x_{i,j,k+1}) = \beta_1 (x_{i,j,k} x_{i,j,k+1})$.

If $1 \le i \le a-1$ and $1 \le j \le b$ then

$$\beta_2 (x_{i,j,i} y_{i+1}) = \begin{cases} \frac{bi(i+3)}{2} - j + 1 & \text{if } i \text{ is odd} \\ \frac{b}{2}((i+3)i - 2) + j & \text{if } i \text{ is even.} \end{cases}$$

Label the vertices and the edges of P_a^b by α_2 and $\beta_2 + v$ respectively. It is not difficult to check analogously to the proof of Theorem 1, that in the resulting labeling the weights of (2i+2)-sided faces constitute an arithmetic progression of difference 2 for $1 \le i \le a-1$.

Complete the face labeling of P_a^b such that the values $v+e+1$, $v+e+2$,..., $v+e+(a-1)(b-1)$ will be given to the (2i+2)-sided faces, $1 \le i \le a-1$.

We are able to arrange the face values of (2i+2)-sided faces such that the resulting labeling of type (1,1,1) is

(i) 1- antimagic or

(ii) 3- antimagic.

Theorem 3. For $a \geq 2$ and $b \geq 2$, the plane graph P_a^b has a 4-antimagic labeling and 6-antimagic labeling of type (1,1,1).

Proof: Consider the following bijections

$$\alpha_3 : V(P_a^b) \rightarrow \{\ 1,2,3,\ldots,\frac{b\,a(a-1)}{2}+a\ \} \text{ and}$$

$$\beta_3 : E(P_a^b) \rightarrow \{\ 1,2,3,\ldots,\frac{b\,(a-1)(a+2)}{2}\ \}, \text{where}$$

$\alpha_3\ (y_i) = \alpha_1\ (y_i)$.

If $1 \leq i \leq a-1$ and $1 \leq j \leq b$ then

$\alpha_3\,(x_{i,j,k}) =$

$$\begin{cases} \frac{bi(i-1)}{2}+i+1+\frac{j+1}{2} & \text{if } i \text{ and } j \text{ are odd and } k=1 \\ \frac{bi(i-1)}{2}+\lfloor\frac{b+1}{2}\rfloor+i+1+\frac{j}{2} & \text{if } i \text{ is odd, } j \text{ is even and } k=1 \\ \frac{bi(i-1)}{2}+i+1+j & \text{if } i \text{ is even and } k=1 \\ \frac{bi(i-1)}{2}+kb+i+2-j & \text{if } k \text{ is even, } 4 \leq k \leq i \\ \frac{b}{2}(i^2-i+4)+i+2-j & \text{if } i \text{ is odd and } k=2 \\ \frac{b}{2}(i^2-i+2)+i+1+j & \text{if } i \text{ is even and } k=2 \\ \frac{bi(i-1)}{2}+(k-1)b+i+1+j & \text{if } k \text{ is odd, } 3 \leq k \leq i. \end{cases}$$

$\beta_3\ (y_i x_{i,j,1}) = \beta_1\ (y_i x_{i,j,1})$,

$\beta_3\ (x_{i,j,k}\, x_{i,j,k+1}) = \beta_1\ (x_{i,j,k}\, x_{i,j,k+1})$.

If $1 \leq i \leq a-1$ and $1 \leq j \leq b$ then

$\beta_3\ (x_{i,j},_i\ y_{i+1}) =$

$$\begin{cases} \frac{b}{2}(i^2 + 3i - 2) + j & \text{if i is odd} \\ \frac{b}{2}((i+3)i - 2) + \frac{j+1}{2} & \text{if i is even and j is odd} \\ \frac{bi(i+3)}{2} - \lfloor \frac{b}{2} \rfloor + \frac{j}{2} & \text{if i and j are even.} \end{cases}$$

It is a matter of routine checking to see that combining the vertex labeling α_3 and the edge labeling $\beta_3 + v$ gives a 5-antimagic labeling of type (1,1,0).

If we complete the face labeling of P_a^b with values in the set $\{ v+e+1, v+e+2, \ldots v+e+(a-1)(b-1)\}$ then we are able to obtain

(i) 4-antimagic labeling of type (1,1,1) or

(ii) 6-antimagic labeling of type (1,1,1).

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REFERENCES

[1] M. Baca, *On magic and consecutive labelings for the special classes of plane graphs*, Utilitas Math. 32 (1987), 59-65.

[2] M. Baca, *Labelings of m-antiprisms*, Ars Combinatoria 28 (1989), 242-245.

[3] M. Baca, *On magic labelings of Mobius ladders*, J. Franklin Inst. 326 (1989), 885-888.

[4] M. Baca, *On magic labelings of grid graphs*, Ars Combinatoria 33 (1990), 295 – 299.

[5] M. Baca, *On magic labelings of honeycomb*, Discrete Math. 105 (1992), 305-311.

[6] M. Baca, J.A. MacDougall, M. Miller, Slamin and W.D. Wallis, *Survey of certain valuations of graphs*, Discuss Math. Graph Theory 20 (2000), 219-229.

[7] M. Baca, S. Jendrol, M. Miller and J. Ryan, *Antimagic labelings of generalized Petersen graphs*, Ars Combin., to appear.

[8] R. Bodendiek and G. Walther, *On number theoretical methods in graph labelings*, Res. Exp. Math. 21 (1995), 3-25.

[9] N. Hartsfield and G. Ringel, *Pearls in Graph Theory*, Academic Press, Boston – San Diego - New York - London , 1990.

[10] KM. Kathiresan, S. Muthuvel and V.N. Nagasubbu, *Cosecutive labelings for two classes of plane graphs*, Utilitas Math. 55 (1999), 237-241.
[11] KM. Kathiresan and S. Gokulakrishnan, *On magic labelings of type (1,1,1) for the special classes of plane graphs*, Utilitas Math., to appear.
[12] KM. Kathiresan and R. Ganesan, *A labeling problem on the plane graphs* $P_{a,b}$, Ars Combin., to appear.
[13] Ko-Wei Lih, *On magic and consecutive labelings of plane graphs*, Utilitas Math. 24 (1983), 165-197.
[14] Y. Lin, Slamin, M. Baca and M. Miller, *On d-antimagic labelings of prisms*, Ars Combin., to appear.
[15] A.J. Qu, *On complementary consecutive labelings of octahedra*, Ars Combin. 51 (1999), 287-294.
[16] R. Simanjuntak, M. Miller and F. Bertault, *Two new (a,d) - antimagic graph labelings*, Proc. AWOCA (2000), 179-189.
[17] W.D. Wallis, *Magic Graphs*, Birkhauser, Boston - Basel - Berlin, 2001.
[18] D.B. West, *An Introduction to Graph Theory*, Prentice - Hall, 1996.

KM. KATHIRESAN, DEPARTMENT OF MATHEMATICS
AYYA NADAR JANAKI AMMAL COLLEGE, SIVAKASI - 626 124, INDIA.
E-mail address : kathir2esan@yahoo.com

R. GANESAN, DEPARTMENT OF MATHEMATICS
RAJA COLLEGE OF ENGINEERING AND TECHNOLOGY, MADURAI - 625 020, INDIA.