Self-Dual Modular-Graceful Cyclic Digraphs

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Abstract

In this paper, we introduce, for the first time, the notion of *self-dual modular-graceful labeling* of a cyclic digraph. A cyclic digraph $G(V, E)$ is a digraph whose connected components are directed cycles. The line digraph $G^\wedge(V^\wedge, E^\wedge)$ of the cyclic digraph G is the digraph where $V^\wedge = E$, $E^\wedge = V$, and if α, β are two edges of G which join vertex x to vertex y and vertex y to vertex z respectively, then in the digraph $G^\wedge$, y is the edge joining vertex α to vertex β. A labeling f for a cyclic digraph of order n is a map from V to $\mathbf{Z}_{n+1}$. The labeling f induces a dual labeling $f^\wedge$ for $G^\wedge$ by $f^\wedge(\alpha) = f(x) - f(y)$, where α is an edge of G which joins vertex x to vertex y. A self-dual modular-graceful cyclic digraph G is a cyclic digraph together with a labeling f where the image $f(V) = \mathbf{Z}^*_{n+1}$ and $\langle G^\wedge, f^\wedge\rangle$ is an isomorphic digraph of $\langle G, f\rangle$. We prove the necessary and sufficient conditions for the existence of self-dual modular-graceful cyclic digraphs and connected self-dual modular-graceful cyclic digraphs. We also give some explicit constructions of these digraphs in the case $n+1$ is prime and in the general case where $n+1$ is not prime.

1 Introduction

A vertex labeling for a graph G is an assignment f of labels to the vertices of G. In some cases, a vertex labeling induces a labeling on edges, with the label associated with the edge xy determined only by the label values of the two vertices, that is $f(x)$ and $f(y)$.

Important labeling methods are graceful labelings and harmonious labelings, where the vertex labels are determined by an injective function from the vertex set V to the label set, which is normally $\mathbf{N}$ or $\mathbf{Z}_n$, and the label of an edge xy is defined as $|f(x) - f(y)|$ and $f(x) + f(y)$ respectively. For detailed surveys on graph labeling please refer to [1, 2].

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We introduce a new type of labeling for cyclic digraphs called a *modular-graceful labeling* in which the vertices are labeled by elements of the set $\mathbf{Z}_{n+1}$. A labeling f for a cyclic digraph G induces a dual labeling $f^\wedge$ for the line digraph $G^\wedge$ of G in a natural way: if in the digraph G, α is an edge that joins a vertex x to a vertex y then, in the line digraph $G^\wedge$, the dual labeling $f^\wedge$ assigns α to the value $f^\wedge(\alpha) = f(x) - f(y)$. In this paper, we study the interesting special case where the labeling is self-dual, i.e. the line digraph with the dual labeling is isomorphic to the original digraph with the original labeling, and the label values are all the numbers in the set $\mathbf{Z}^*_{n+1} = \{1, 2, \ldots, n\}$.

We prove that self-dual modular-graceful cyclic digraphs exist for even values of n and give constructions for these digraphs when $n + 1$ is a prime. We also give a construction for general values of n ($n + 1$ not prime).

The paper is organized as follows. In section 2, we give a formal definition of self-dual modular-graceful cyclic digraphs. In section 3, we derive a necessary and sufficient condition for the existence of a self-dual modular-graceful cyclic digraph. Connected self-dual modular-graceful cyclic digraphs are considered in section 4. We introduce the notion of label polynomial of a connected cyclic digraph and give a necessary and sufficient condition on the label polynomial for which a connected self-dual modular-graceful cyclic digraph exists. We also give a construction of a connected self-dual modular-graceful cyclic digraph in the case $n + 1$ is prime. Section 5 deals with self-dual modular-graceful cyclic digraphs in general ($n+1$ is not prime) and gives a construction of these digraphs of even orders. Finally, we conclude the paper with open questions. Included in the Appendix is the list of all self-dual modular-graceful cyclic digraphs of orders from 8 to 16 constructed by several methods in sections 3, 4, and 5.

2 Definition

Let $G(V, E)$ be a digraph (directed graph) with V denotes the set of vertices and E denotes the set of edges. For an edge $\alpha \in E$ and two vertices $x, y \in V$, we write $\alpha = xy$ to denote that the edge α joins the vertex x to the vertex y. The edge α is said to be *incident from* vertex x and *incident to* vertex y.

Definition 1 *Let n be a positive integer. A* directed cycle *of order n is a digraph with n vertices and n edges such that if x_1, x_2, $\ldots$, x_n denote the n vertices then the n edges are x_1x_2, x_2x_3, $\ldots$, x_nx_1.*

Definition 2 *A* cyclic digraph *is a digraph whose connected subgraphs are directed cycles. In a cyclic digraph, the number of vertices is equal to the number of edges and it is called the* order *of the digraph.*

In this paper, we only consider cyclic digraphs. Figure 1 shows an example of a cyclic digraph of order 11 which contains one directed cycle of order 1, one

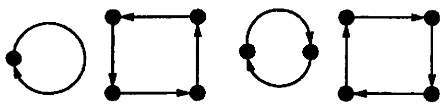

Figure 1: A cyclic digraph of order 11.

directed cycle of order 2 and two directed cycles of order 4.

Let $G(V,E)$ be a cyclic digraph. The *line digraph* of $G(V,E)$, denoted by $G^\wedge(V^\wedge,E^\wedge)$, is the cyclic digraph whose set of vertices $V^\wedge$ is equal to the set of edges, E, of G, and whose set of edges $E^\wedge$ is equal to the set of vertices, V, of G, and if x is a vertex of the digraph G and α and β are two edges of the digraph G such that α is incident to x and β is incident from x, then in the digraph $G^\wedge$, x is the edge that joins the vertex α to the vertex β. It is easy to see that $G^{\wedge\wedge} = G$.

Note that if G is a general digraph then the number of edges of the line digraph $G^\wedge$ is not always equal to the number of vertices of G, so we cannot identify $E^\wedge$ with V. For more details on line digraphs we refer the reader to [3].

Figure 2 shows an example where G has $V = \{x,y,z,t\}$, $E = \{\alpha = tx, \beta = xy, \gamma = yz, \delta = zt\}$, and its line digraph $G^\wedge$ has $V^\wedge = E = \{\alpha,\beta,\gamma,\delta\}$, $E^\wedge = V = \{x = \alpha\beta, y = \beta\gamma, z = \gamma\delta, t = \delta\alpha\}$.

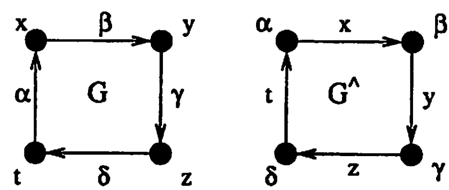

Figure 2: G and $G^\wedge$.

Let $\mathbf{Z}_{n+1} = \{0,1,2,\ldots,n\}$ be the complete set of residues modulo $n+1$ and $\mathbf{Z}^*_{n+1} = \{1,2,\ldots,n\}$ be the set of non-zero residues. A *labeling* for a cyclic digraph G is a function f which maps its vertex set V into the set $\mathbf{Z}_{n+1}$. A labeling f for G induces a labeling $f^\wedge$ for $G^\wedge$ as follows: label $\alpha = xy \in E = V^\wedge$ with $f^\wedge(\alpha) = f(x) - f(y)$. Note that $f^{\wedge\wedge}$ and f are generally two different labelings for G. A cyclic digraph G with a labeling f is denoted by $\langle G, f\rangle$.

Definition 3 *Two labeled cyclic digraphs $\langle G(V,E), f\rangle$ and $\langle G'(V',E'), f'\rangle$ are called* isomorphic, *denoted by $\langle G(V,E), f\rangle \sim \langle G'(V',E'), f'\rangle$, if and only if there exist two bijective functions, $v : V \to V'$ and $e : E \to E'$, satisfying*

1. *for any $\alpha \in E$ and any $x, y \in V$, $\alpha = xy$ if and only if $e(\alpha) = v(x)v(y)$,*
2. *for any $x \in V$, $f(x) = f'(v(x))$.*

Definition 4 *A labeled cyclic digraph $\langle G(V,E), f\rangle$, of order n, is called* self-dual modular-graceful *if and only if*

1. *the image* $f(V) = \mathbf{Z}_{n+1}^{*}$
2. $\langle G(V,E), f\rangle \sim \langle G^{\wedge}(V^{\wedge}, E^{\wedge}), f^{\wedge}\rangle$

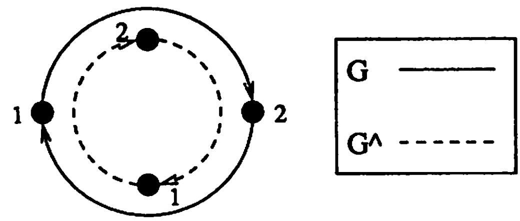

Figure 3: The smallest self-dual modular-graceful cyclic digraph.

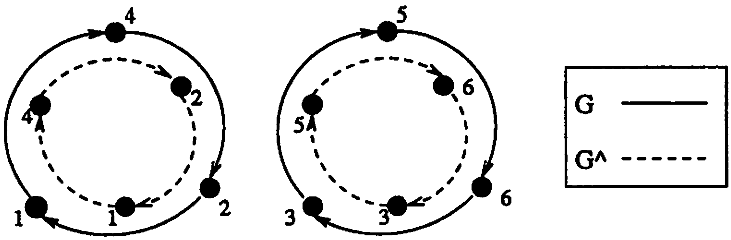

Figure 4: A self-dual modular-graceful cyclic digraph of order 6.

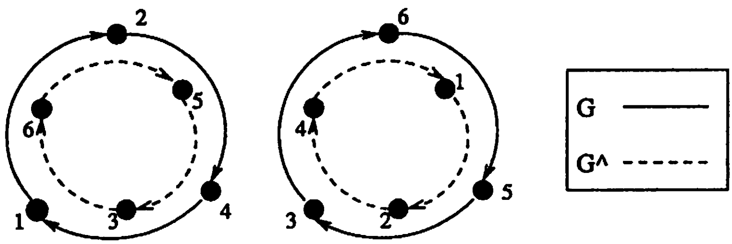

Figure 5: Another self-dual modular-graceful cyclic digraph of order 6.

It is easy to see that no self-dual modular-graceful cyclic digraph contains a cycle component of order 1. Figure 3 shows the smallest self-dual modular-graceful cyclic digraph. This digraph is a directed cycle of order 2. Two self-dual modular-graceful cyclic digraphs of order 6 are shown in Figure 4 and Figure 5.

3 Existence of self-dual modular-graceful cyclic digraphs

We show that self-dual modular-graceful cyclic digraphs exist only for even orders.

Theorem 1 *For a positive integer n, there exists a self-dual modular-graceful cyclic digraph of order n if and only if n is even.*

Proof. Firstly, if $\langle G(V,E), f\rangle$ is a self-dual modular-graceful cyclic digraph of order n then

$$\sum_{x\in V} f(x) = 1+2+\ldots+n = \sum_{\alpha=xy\in V^\wedge} f^\wedge(\alpha) = \sum_{\alpha=xy\in V^\wedge} (f(x)-f(y))$$

$$= \sum_{x\in V} f(x) - \sum_{y\in V} f(y) = 0 \pmod{n+1}.$$

And so

$$\frac{n(n+1)}{2} = 0 \pmod{n+1},$$

which implies that n is even.

Conversely, if n is even, for each i, $1 \le i \le n/2$, let C_i be the directed cycle of order 2 with two vertices labeled by i and $-i$. Let U_n be the union of all C_1, C_2, ..., $C_{\frac{n}{2}}$. We show that U_n is a self-dual modular-graceful cyclic digraph of order n. Indeed, $C_i^\wedge$, the line digraph of C_i, has two vertices labeled by two complement residues $2i$ and $-2i$. Since for any i_1, i_2 such that $1 \le i_1 < i_2 \le n/2$, we have $2i_1 \pm 2i_2 \ne 0 \pmod{n+1}$, when i ranges from 1 to $n/2$, the $n/2$ sets of residues $\{2i, -2i\}$ are pairwise disjoint. It follows that these $n/2$ sets $\{2i, -2i\}$ form a partition of Z^*_{n+1}, and so, they form a permutation of the $n/2$ sets $\{i, -i\}$, $1 \le i \le n/2$. This proves that U_n and $U_n^\wedge$ are isomorphic, and U_n is a self-dual modular-graceful cyclic digraph. ■

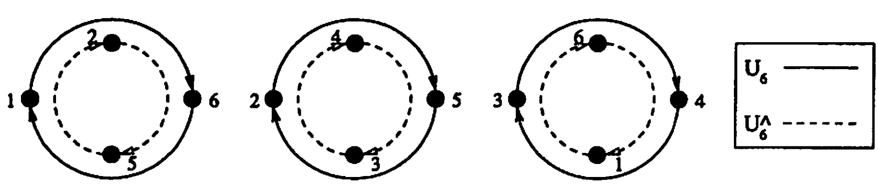

Figure 6: The self-dual modular-graceful cyclic digraph U_6 of order 6.

Theorem 1 shows that there exist only self-dual modular-graceful cyclic digraphs of even orders. The second part of the proof gives a construction of the self-dual modular-graceful cyclic digraph U_n. Figure 6 shows the self-dual modular-graceful cyclic digraph U_6 of order 6.

Digraph U_n contains $n/2$ directed cycles of order 2. Since a self-dual modular-graceful cyclic digraph cannot contain cycles of order 1, U_n is a self-dual modular-graceful cyclic digraph with the most number of connected cycles. In contrast, the digraph that has the least number of cycles is the connected digraph. In the next section we derive a condition on connected self-dual modular-graceful cyclic digraphs and give a construction of these digraphs in the case $n+1$ is prime.

4 Connected self-dual modular-graceful cyclic digraphs and their construction when $n+1$ is prime

In this section we will derive a necessary and sufficient condition for the existence of a connected self-dual modular-graceful cyclic digraph of order n and present a construction for the case when $n+1$ is prime.

A connected self-dual modular-graceful cyclic digraph $\langle G, f\rangle$ contains exactly one directed cycle. Let $x_1, x_2, \ldots, x_n$ be its vertices written in the edge-direction order. Then $f(x_1)$, $f(x_2)$, ..., $f(x_n)$ form a permutation of n elements of Z^*_{n+1}. In this section, the index i of x_i is considered in modulo n. That is, x_{n+1} and x_1 are the same vertex.

The condition for $\langle G, f\rangle$ to be self-dual modular-graceful is that the values $f(x_1)$, $f(x_2)$, ..., $f(x_n)$ must satisfy the following equations for some k, $1 \le k < n$,

$$\begin{aligned} f(x_n) - f(x_1) &= f(x_k), \\ f(x_1) - f(x_2) &= f(x_{k+1}), \\ &\vdots \\ f(x_{n-1}) - f(x_n) &= f(x_{k-1}). \end{aligned}$$

We introduce the notion of *label polynomial* for a labeled connected cyclic digraph.

Definition 5 *Let $\langle G, f\rangle$ be a labeled connected cyclic digraph of order n with n vertices x_1, x_2, ..., x_n written in the edge-direction order. The following polynomial in the ring $Z_{n+1}[t]/(t^n - 1)$*

$$\mathcal{G}(t) = f(x_n) + f(x_{n-1})t + f(x_{n-2})t^2 + \ldots + f(x_1)t^{n-1}.$$

is called a label polynomial *of $\langle G, f\rangle$.*

The label polynomial is not unique for a labeled connected cyclic digraph $\langle G, f\rangle$, indeed it has n label polynomials depending on the way x_1 is chosen. Given a label polynomial, it uniquely determines the labeled connected cyclic digraph $\langle G, f\rangle$ up to isomorphism.

Let

$$\mathcal{P}(t) = a_n + a_{n-1}t + a_{n-2}t^2 + \ldots + a_1 t^{n-1}$$

be an arbitrary polynomial in the ring $\mathbf{Z}_{n+1}[t]/(t^n - 1)$. We have

$$\begin{aligned} t^m\mathcal{P}(t) &= a_n t^m + a_{n-1}t^{m+1} + \ldots + a_{m+1}t^{n-1} + a_m t^n + a_{m-1}t^{n+1} + \ldots + a_1 t^{n+m-1} \\ &= a_n t^m + a_{n-1}t^{m+1} + \ldots + a_{m+1}t^{n-1} + a_m + a_{m-1}t + \ldots + a_1 t^{m-1} \\ &= a_m + a_{m-1}t + a_{m-2}t^2 + \ldots + a_{m+1}t^{n-1}, \end{aligned}$$

and so

$$\begin{aligned} (1-t-t^k)\mathcal{P}(t) = (a_n - a_1 - a_k) + (a_{n-1} - a_n - a_{k-1})t + (a_{n-2} - a_{n-1} - a_{k-2})t^2 + \\ \ldots + (a_2 - a_3 - a_{k+2})t^{n-2} + (a_1 - a_2 - a_{k+1})t^{n-1}. \end{aligned}$$

It follows that

$$(1 - t - t^k)\mathcal{P}(t) = 0$$

is equivalent to

$$\begin{aligned} a_n - a_1 &= a_k, \\ a_1 - a_2 &= a_{k+1}, \\ &\vdots \\ a_{n-1} - a_n &= a_{k-1}. \end{aligned}$$

This proves the following theorem about a necessary and sufficient condition to have a connected self-dual modular-graceful cyclic digraph of order n.

Theorem 2 *If $\langle G(V,E), f\rangle$ is a connected self-dual modular-graceful cyclic digraph such that for some k, $1 \leq k < n$,*

$$\begin{aligned} f(x_n) - f(x_1) &= f(x_k), \\ f(x_1) - f(x_2) &= f(x_{k+1}), \\ &\vdots \\ f(x_{n-1}) - f(x_n) &= f(x_{k-1}), \end{aligned}$$

then in $\mathbf{Z}_{n+1}[t]/(t^n - 1)$ its label polynomial $\mathcal{G}(t)$ satisfies

$$(1 - t - t^k)\mathcal{G}(t) = 0.$$

Conversely, if there exist a polynomial in the ring $\mathbf{Z}_{n+1}[t]/(t^n - 1)$,

$$\mathcal{G}(t) = a_n + a_{n-1}t + a_{n-2}t^2 + \ldots + a_1 t^{n-1},$$

where $(a_1, \ldots, a_n)$ is a permutation of elements of $\mathbf{Z}^*_{n+1}$ such that

$$(1 - t - t^k)\mathcal{G}(t) = 0$$

for some k, $1 \leq k < n$, then we can form a connected self-dual modular-graceful cyclic digraph $\langle G(V,E), f\rangle$ as $f(x_i) = a_i$.

4.1 Construction of connected self-dual modular-graceful cyclic digraphs when $n+1$ is prime

Now suppose $n+1 = p$ is an odd prime. Let a be a primitive root modulo p. Consider the following polynomial in $\mathbf{Z}_p[t]/(t^{p-1}-1)$

$$\mathcal{P}(t) = a^{p-2} + a^{p-3}\, t + a^{p-4}\, t^2 + \ldots + a\, t^{p-3} + t^{p-2}$$

We have

$$(t-a)\mathcal{P}(t) = t^{p-1} - a^{p-1} = t^{p-1} - 1 = 0.$$

Since a is primitive, $1-a$ is a non-zero element of $\mathbf{Z}_p$, and for some k, $1 \le k < p-1$, we have

$$1 - a = a^k.$$

This means a is a root of the polynomial $1 - t - t^k$, and so $t-a$ is a factor of $1 - t - t^k$. From $(t-a)\mathcal{P}(t) = 0$, it follows that

$$(1 - t - t^k)\mathcal{P}(t) = 0$$

Therefore, from Theorem 2, we have the following construction of connected self-dual modular-graceful cyclic digraphs of order $p-1$:

Theorem 3 *Let p be an odd prime and a be a primitive element modulo p. Let G be a directed cycle of order $p-1$ with $p-1$ vertices $x_1, x_2, \ldots, x_{p-1}$ written in the edge-direction order. The following labeling $f(x_1) = 1$, $f(x_2) = a$, $f(x_3) = a^2, \ldots, f(x_{p-1}) = a^{p-2}$ makes G become a connected self-dual modular-graceful cyclic digraph.*

Since there are $\varphi(p-1)$ primitive elements modulo p, using Theorem 3, we can construct $\varphi(p-1)$ different connected self-dual modular-graceful cyclic digraphs of order $p-1$. Figure 7 shows two such digraphs of order 6 which correspond to the case $p = 7$ and $a = 3, 5$.

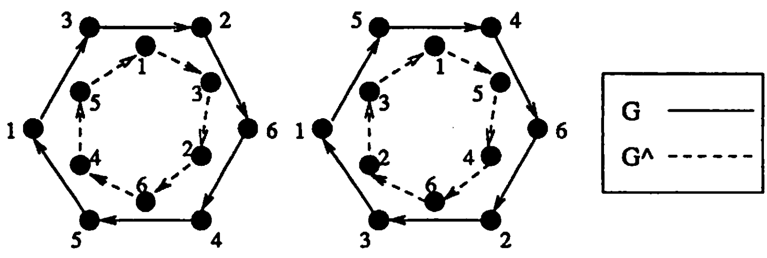

Figure 7: Two connected self-dual modular-graceful cyclic digraphs of order 6.

An open question is whether, in the case $n+1$ is prime, there exist other labelings than those given by Theorem 3. Also, in the case $n+1$ is not prime, the question is whether there exists at all a labeling that results in a connected self-dual modular-graceful cyclic digraph. We have written a computer program to search for connected self-dual modular-graceful cyclic digraphs of orders up to 20. If $n+1$ is not a prime number then our program finds no labelings, and if $n+1$ is prime then the outputs of our program are only those specified in Theorem 3. This strongly suggests that there exist connected self-dual modular-graceful cyclic digraphs of order n if and only if $n+1$ is prime, and there are exactly $\varphi(n)$ non-isomorphic connected self-dual modular-graceful cyclic digraphs as described in Theorem 3.

5 Construction of self-dual modular-graceful cyclic digraphs in the general case

Let n be an even positive integer. Let a be an integer such that

$$\gcd(a(a-1), n+1) = 1.$$

Consider the multiplicative group $\langle a \rangle = \{1, a, a^2, a^3, \ldots\}$ generated by a modulo $n+1$. Since $\gcd(a, n+1) = 1$, for any $z \in \mathbf{Z}^*_{n+1}$ and any $g = a^i \in \langle a \rangle$, we have $gz \in \mathbf{Z}^*_{n+1}$. Therefore, the map $\langle a \rangle \times \mathbf{Z}^*_{n+1} \to \mathbf{Z}^*_{n+1}$ sending $(g, z) \in \langle a \rangle \times \mathbf{Z}^*_{n+1}$ to $gz \in \mathbf{Z}^*_{n+1}$ defines an action of the group $\langle a \rangle$ on the set $\mathbf{Z}^*_{n+1}$.

The set $\mathbf{Z}^*_{n+1}$ is partitioned into orbits under this group action. Take an arbitrary element $z \in \mathbf{Z}^*_{n+1}$, if m is the least positive integer such that $z(a^m - 1) = 0$, then it is easy to see that the orbit of z under this action is $\langle a \rangle z = \{z, za, za^2, \ldots, za^{m-1}\}$.

Suppose $\mathbf{Z}^*_{n+1}$ is partitioned into s orbits

$$\begin{aligned} \langle a \rangle z_1 &= \{1, a, a^2, \ldots, a^{m_1 - 1}\}, (\text{here } z_1 = 1), \\ \langle a \rangle z_2 &= \{z_2, z_2 a, z_2 a^2, \ldots, z_2 a^{m_2 - 1}\}, \\ &\vdots \\ \langle a \rangle z_s &= \{z_s, z_s a, z_s a^2, \ldots, z_s a^{m_s - 1}\}. \end{aligned}$$

For each i, $1 \le i \le s$, let M_i be the directed cycle with m_i vertices labeled by z_i, $z_i a$, $\ldots$, $z_i a^{m_i - 1}$ in the direction of edges. And let G be the union of all M_i. We will show that G is a self-dual modular-graceful cyclic digraph of order n.

Indeed, since $\gcd(1-a, n+1) = 1$, the following m_i residues, $(1-a)z_i$, $(1-a)z_i a$, $(1-a)z_i a^2$, $\ldots$, $(1-a)z_i a^{m_i - 1}$ are distinct. Therefore, the orbit of $(1-a)z_i$ is $\langle a \rangle (1-a) z_i = \{(1-a)z_i, (1-a)z_i a, (1-a)z_i a^2, \ldots, (1-a)z_i a^{m_i - 1}\}$.

If z and z' are in different orbits then $(1-a)z$ and $(1-a)z'$ are also in different orbits. It follows that $\langle a \rangle (1-a) z_1$, $\langle a \rangle (1-a) z_2$, $\ldots$, $\langle a \rangle (1-a) z_s$ are all distinct orbits, and hence, they form a permutation of the orbits $\langle a \rangle z_1$, $\langle a \rangle z_2$, $\ldots$, $\langle a \rangle z_s$.

Since vertices of the line digraph $M_i^\wedge$ are labeled with $(1-a)z_i$, $(1-a)z_ia$, $(1-a)z_ia^2$, ..., $(1-a)z_ia^{m_i-1}$ in this order, it proves that G and $G^\wedge$ are isomorphic, and we have the following theorem:

Theorem 4 *Let n be an even positive integer. Let a be an integer such that* $\gcd(a(a-1), n+1) = 1$. *Under the group action $\langle a\rangle \times \mathbf{Z}_{n+1}^* \to \mathbf{Z}_{n+1}^*$, $(g,z) \mapsto gz$, let $\mathbf{Z}_{n+1}^*$ be partitioned into the following orbits*

$$\begin{aligned}
\langle a\rangle z_1 &= \{1, a, a^2, \ldots, a^{m_1-1}\}, (\text{here } z_1 = 1),\\
\langle a\rangle z_2 &= \{z_2, z_2a, z_2a^2, \ldots, z_2a^{m_2-1}\},\\
&\vdots\\
\langle a\rangle z_s &= \{z_s, z_sa, z_sa^2, \ldots, z_sa^{m_s-1}\}.
\end{aligned}$$

Let G be the labeled cyclic digraph that formed by s directed cycles M_i, where in each directed cycle M_i, vertices are labeled by z_i, z_ia, ..., z_ia^{m-1} in the direction of edges, then G is a self-dual modular-graceful cyclic digraph of order n.

Theorem 4 is a generalization of Theorem 1 and Theorem 3. Indeed, in Theorem 4, if we choose $a = n$ then we have the digraph described in Theorem 1. If $n+1$ is prime, choose a to be a primitive element, then the group action in Theorem 4 has only one orbit, and so the digraph it generates is connected as in Theorem 3.

Example. Let $n = 8$ and $a = 5$, $\gcd(a(a-1), n+1) = 1$,

$$\langle 5\rangle = \{1, 5, 7, 8, 4, 2\}.$$

$\mathbf{Z}_9^*$ is partitioned into two orbits

$$\begin{aligned}
\langle 5\rangle 1 &= \{1, 5, 7, 8, 4, 2\},\\
\langle 5\rangle 3 &= \{3, 6\}.
\end{aligned}$$

Figure 8 shows a self-dual modular-graceful cyclic digraph of order 8 which contains two directed cycles whose vertices are labeled cyclically by elements of the above two orbits.

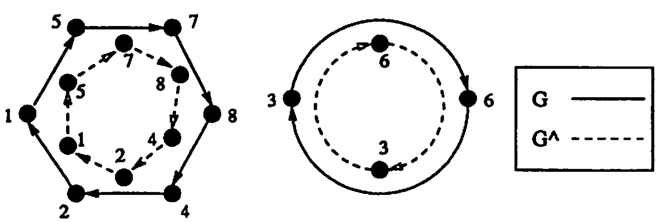

Figure 8: A self-dual modular-graceful cyclic digraph of order 8.

Included in the Appendix is the list of all self-dual modular-graceful cyclic digraphs of orders from 8 to 16 constructed by methods in sections 3, 4, and 5. For instance, the first line $n = 10$ is the digraph that has five directed cycles of order 2 labeled by (1,10), (2,9), (3,8), (4,7), (5,6) respectively. This digraph is generated from Theorems 1 and Theorems 4 with $a = 10$. The asterisk next to the order $n = 10$ indicates that $n + 1$ is a prime.

6 Open Problems

Below is a list of interesting open problems about self-dual modular-graceful cyclic digraph:

1. Suppose $n + 1 = p$ is prime. Does there exist any connected self-dual modular-graceful cyclic digraph other than the $\varphi(p-1)$ digraphs generated by Theorem 3?
2. Does there exist any connected self-dual modular-graceful cyclic digraph of order n where $n + 1$ is not prime?
3. Does there exist any self-dual modular-graceful cyclic digraph other than the digraphs generated by Theorem 4?

References

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[2] J.A. Gallian, A Dynamic Survey of Graph Labeling, *Electronic J. of Combinatorics,* Nov 2002. (http://www.combinatorics.org/Surveys/ds6.pdf)

[3] D. B. West, *Introduction to Graph Theory,* Prentice Hall, 1996.

Appendix

n	Digraph	Thm#(a =#)
8	(1,8)(2,7)(3,6)(4,5)	1, 4(8)
	(1,2,4,8,7,5)(3,6)	4(2)
	(1,5,7,8,4,2)(3,6)	4(5)
10*	(1,10)(2,9)(3,8)(4,7)(5,6)	1, 4(10)
	(1,2,4,8,5,10,9,7,3,6)	3(2), 4(2)
	(1,6,3,7,9,10,5,8,4,2)	3(6), 4(6)
	(1,7,5,2,3,10,4,6,9,8)	3(7), 4(7)
	(1,8,9,6,4,10,3,2,5,7)	3(8), 4(8)
	(1,3,9,5,4)(2,6,7,10,8)	4(3)
	(1,4,5,9,3)(2,8,10,7,6)	4(4)
	(1,5,3,4,9)(2,10,6,8,7)	4(5)
	(1,9,4,3,5)(2,7,8,6,10)	4(9)
12*	(1,12)(2,11)(3,10)(4,9)(5,8)(6,7)	1, 4(12)
	(1,2,4,8,3,6,12,11,9,5,10,7)	3(2), 4(2)
	(1,6,10,8,9,2,12,7,3,5,4,11)	3(6), 4(6)
	(1,7,10,5,9,11,12,6,3,8,4,2)	3(7), 4(7)
	(1,11,4,5,3,7,12,2,9,8,10,6)	3(11), 4(11)
	(1,3,9)(2,6,5)(4,12,10)(7,8,11)	4(3)
	(1,4,3,12,9,10)(2,8,6,11,5,7)	4(4)
	(1,5,12,8)(2,10,11,3)(4,7,9,6)	4(5)
	(1,8,12,5)(2,3,11,10)(4,6,9,7)	4(8)
	(1,9,3)(2,5,6)(4,10,12)(7,11,8)	4(9)
	(1,10,9,12,3,4)(2,7,5,11,6,8)	4(10)
14	(1,14)(2,13)(3,12)(4,11)(5,10)(6,9)(7,8)	1, 4(14)
	(1,2,4,8)(3,6,12,9)(5,10)(7,14,13,11)	4(2)
	(1,8,4,2)(3,9,12,6)(5,10)(7,11,13,14)	4(8)
16*	(1,16)(2,15)(3,14)(4,13)(5,12)(6,11)(7,10)(8,9)	1, 4(16)
	(1,3,9,10,13,5,15,11,16,14,8,7,4,12,2,6)	3(3), 4(3)
	(1,5,8,6,13,14,2,10,16,12,9,11,4,3,15,7)	3(5), 4(5)
	(1,6,2,12,4,7,8,14,16,11,15,5,13,10,9,3)	3(6), 4(6)
	(1,7,15,3,4,11,9,12,16,10,2,14,13,6,8,5)	3(7), 4(7)
	(1,10,15,14,4,6,9,5,16,7,2,3,13,11,8,12)	3(10), 4(10)
	(1,11,2,5,4,10,8,3,16,6,15,12,13,7,9,14)	3(11), 4(11)
	(1,12,8,11,13,3,2,7,16,5,9,6,4,14,15,10)	3(12), 4(12)
	(1,14,9,7,13,12,15,6,16,3,8,10,4,5,2,11)	3(14), 4(14)
	(1,2,4,8,16,15,13,9)(3,6,12,7,14,11,5,10)	4(2)
	(1,4,16,13)(2,8,15,9)(3,12,14,5)(6,7,11,10)	4(4)
	(1,8,13,2,16,9,4,15)(3,7,5,6,14,10,12,11)	4(8)
	(1,9,13,15,16,8,4,2)(3,10,5,11,14,7,12,6)	4(9)
	(1,13,16,4)(2,9,15,8)(3,5,14,12)(6,10,11,7)	4(13)
	(1,15,4,9,16,2,13,8)(3,11,12,10,14,6,5,7)	4(15)