ON LABELING THE UNION OF TWO CYCLES

JERRETT DUMOUCHEL AND SAAD I. EL-ZANATI
4520 MATHEMATICS DEPARTMENT
ILLINOIS STATE UNIVERSITY
NORMAL, ILLINOIS 61790–4520, U.S.A.

ABSTRACT. It is conjectured that any 2-regular graph G with n edges has a ρ-labeling (and thus divides K_{2n+1} cyclically). In this note we show that the conjecture holds when G has at most two components.

1. INTRODUCTION

If a and b are integers we denote $\{a, a+1, \ldots, b\}$ by $[a, b]$. Let $\mathbb{N}$ denote the set of nonnegative integers and $\mathbb{Z}_n$ the group of integers modulo n. For a graph G, let $V(G)$ and $E(G)$ denote the vertex set of G and the edge set of G, respectively. Let $V(K_v) = \mathbb{Z}_v$ and let G be a subgraph of K_v. By *clicking* G, we mean applying the isomorphism $i \to i+1$ to $V(G)$. Let K and G be graphs such that G is a subgraph of K. A *G-decomposition* of K is a set $\Gamma = \{G_1, G_2, \ldots, G_t\}$ of subgraphs of K each of which is isomorphic to G and such that the edge sets of the graphs G_i form a partition of the edge set of K. If K is K_v, a G-decomposition Γ of K is *cyclic* if clicking is a permutation of Γ. If G is a graph and r is a positive integer, rG denotes the vertex disjoint union of r copies of G.

For any graph G, an injective function $h : V(G) \to \mathbb{N}$ is called a *labeling* (or a *valuation*) of G. In [15], Rosa introduced a hierarchy of labelings. We add a few items to this hierarchy. Let G be a graph with n edges and no isolated vertices and let h be a labeling of G. Let $h(V(G)) = \{h(u) : u \in V(G)\}$. Define a function $\bar{h} : E(G) \to \mathbb{Z}^+$ by $\bar{h}(e) = |h(u) - h(v)|$, where $e = \{u, v\} \in E(G)$. Let $\bar{E}(G) = \{\bar{h}(e) : e \in E(G)\}$. Consider the following conditions:

(a) $h(V(G)) \subseteq [0, 2n]$,
(b) $h(V(G)) \subseteq [0, n]$,
(c) $\bar{E}(G) = \{x_1, x_2, \ldots, x_n\}$, where for each $i \in [1, n]$ either $x_i = i$ or $x_i = 2n + 1 - i$,
(d) $\bar{E}(G) = [1, n]$.

If in addition G is bipartite, then there exists a bipartition (A, B) of $V(G)$ (with every edge in G having one endvertex in A and the other in B) such that

(e) for each $\{a, b\} \in E(G)$ with $a \in A$ and $b \in B$, we have $h(a) < h(b)$,
(f) there exists an integer λ such that $h(a) \leq \lambda$ for all $a \in A$ and $h(b) > \lambda$ for all $b \in B$.

Then a labeling satisfying the conditions:

(a), (c) is called a *ρ-labeling*;
(a), (d) is called a *σ-labeling*;
(b), (d) is called a *β-labeling*.

A β-labeling is necessarily a σ-labeling which in turn is a ρ-labeling. If G is bipartite and a ρ, σ or β-labeling of G also satisfies (e), then the labeling is *ordered* and is denoted by ρ^+, σ^+ or β^+, respectively. If in addition (f) is satisfied, the labeling is *uniformly-ordered* and is denoted by ρ^{++}, σ^{++} or β^{++}, respectively.

A β-labeling is better known as a *graceful* labeling and a uniformly-ordered β-labeling is an *α-labeling* as introduced in [15].

Labelings are critical to the study of cyclic graph decompositions as seen in the following two results from [15] and [8], respectively.

Theorem 1. Let G be a graph with n edges. There exists a cyclic G-decomposition of K_{2n+1} if and only if G has a ρ-labeling.

Theorem 2. Let G be a graph with n edges that has a ρ^+-labeling. Then there exists a cyclic G-decomposition of K_{2nx+1} for all positive integers x.

Let G be a graph with n edges and Eulerian components and let h be a β-labeling of G. It is well-known (see [15]) that we must have $n \equiv 0$ or $3 \pmod 4$. Moreover, if such a G is bipartite, then $n \equiv 0 \pmod 4$. These conditions hold since for such a G, $\sum_{e \in E(G)} \bar{h}(e) = n(n+1)/2$. This sum must in turn be even, since each vertex is incident with an even number of edges and $\bar{h}(e) = |g(u) - g(v)|$, where u and v are the endvertices of e. Thus we must have $4 | n(n+1)$. Clearly, the same will hold if such a G admits a σ-labeling. We shall refer to this restriction as the *parity condition.* There are no such restrictions on $|E(G)|$ if h is a ρ-labeling.

Theorem 3. (Parity Condition) If a graph G with Eulerian components and n edges has a σ-labeling, then $n \equiv 0$ or $3 \pmod 4$. If such a G is bipartite, then $n \equiv 0 \pmod 4$.

The study of graph decompositions is a popular branch of modern combinatorial design theory (see [5] for an overview). In particular, the study of G-decompositions of K_{2n+1} (and of K_{2nx+1}) when G is a graph with n edges (and x is a positive integer) has attracted considerable attention.

The study of graph labelings is also quite popular (see Gallian [10] for a dynamic survey). Theorems 1 and 2 provide powerful links between the two areas. Much of the attention on labelings has been on graceful labelings (i.e., β-labelings). Unfortunately, the parity condition "disqualifies" large classes of graphs from admitting graceful labelings. This difficulty is compounded by the fact that certain classes of graphs with ρ-labelings meet the parity condition, yet fail to be graceful.

In this manuscript, we will focus on labelings of 2-regular graphs (i.e., the vertex-disjoint union of cycles). It is conjectured by El-Zanati and Vanden Eynden (see [3]) that every 2-regular graph G with n edges admits a ρ-labeling (and thus divides K_{2n+1} cyclically). Here, we shall show that this conjecture holds if G is 2-regular with at most two components.

2. Some of the Known Results for 2-regular graphs

The following is known for cycles (see [13], [14] and [8]).

Theorem 4. Let $m \geq 3$ be an integer. Then, C_m admits an α-labeling if $m \equiv 0 \pmod 4$, a ρ-labeling if $m \equiv 1 \pmod 4$, a ρ^{++}-labeling if $m \equiv 2 \pmod 4$, and a β-labeling if $m \equiv 3 \pmod 4$.

For 2-regular graphs with two components, we have the following from Abrham and Kotzig [2].

Theorem 5. Let $m \geq 3$ and $n \geq 3$ be integers. Then the graph $C_m \bigcup C_n$ has a β-labeling if and only if $m+n \equiv 0$ or $3 \pmod 4$. Moreover, $C_m \bigcup C_n$ has an α-labeling if and only if both m and n are even and $m+n \equiv 0 \pmod 4$.

A restricted ρ-labeling (called a γ-labeling) for almost-bipartite graphs was introduced in [4]. A nonbipartite graph G is *almost-bipartite* if G contains an edge whose removal renders the graph bipartite. It was shown in [4] that if such a G with n edges admits a γ-labeling, then G divides K_{2nx+1} cyclically. It was also shown that odd cycles of length exceeding three and 2-regular graphs with one odd and one even component (with the exception of $C_3 \bigcup C_4$) admit γ-labelings.

For 2-regular graphs with more than two components, the following is known. In [12], Kotzig shows that $3C_{4k+1}$ admits a β-labeling for all $k \geq 2$. In [6], it is shown that rC_3 admits a ρ-labeling for all $r \geq 1$. In [9], Eshghi shows that $C_{2m} \bigcup C_{2n} \bigcup C_{2k}$ has an α-labeling for all m, n, and $k \geq 2$ with $m+n+k \equiv 0 \pmod 2$ except when $m = n = k = 2$. In [1], Abrham and Kotzig show that rC_4 has an α-labeling for all positive integers $r \neq 3$. In [7], it is shown that rC_m admits a ρ-labeling for $m \geq 3$ and $r \leq 4$. An additional result follows by combining results from [8] and from [3].

Theorem 6. Let G be a 2-regular bipartite graph of order n. Then G has a σ^{++}-labeling if $n \equiv 0 \pmod 4$ and a ρ^{++}-labeling if $n \equiv 2 \pmod 4$.

In [11], Kotzig shows that if $r > 1$, then rC_3 does not admit a β-labeling. Similarly, he shows that rC_5 does not admit a β-labeling for any r. These 2-regular graphs (rC_3 for $r > 1$ and rC_5 for $r \geq 1$) fail to admit β-labelings even when the parity condition is satisfied. It is thus reasonable to consider labelings that are less restrictive than β-labelings when studying 2-regular graphs.

3. Main results

We shall show that if $m \geq 3$ and $n \geq 3$ are integers, then $G = C_m \bigcup C_n$ admits a ρ-labeling. By Theorem 5, if $m + n \equiv 0$ or $3 \pmod 4$, then G admits a β-labeling (and thus a ρ-labeling). By Theorem 6, if both m and n are even and if $m + n \equiv 2 \pmod 4$, then G admits a ρ^{++}-labeling. By the results from [4], if $m + n \equiv 1 \pmod 4$, then G admits a γ-labeling. Thus, it suffices to show G admits a ρ-labeling when $m + n \equiv 2 \pmod 4$, and both m and n are odd. Some additional definitions and notational conventions are necessary.

Denote the path with consecutive vertices $a_1, a_2, \ldots, a_k$ by $(a_1, \ldots, a_k)$. By $(a_1, a_2, \ldots, a_k) + (b_1, b_2, \ldots, b_j)$, where $a_k = b_1$, we mean the path $(a_1, \ldots, a_k, b_2, \ldots, b_j)$.

To simplify our consideration of various labelings, we will sometimes consider graphs whose vertices are named by distinct nonnegative integers, which are also their labels.

Let a, b, and h be integers with $0 \leq a \leq b$ and $h > 0$. Set $d = b - a$. We define the path

$$P(a, h, b) = (a, a + h + 2d - 1, a + 1, a + h + 2d - 2, a + 2, \ldots, b - 1, b + h, b).$$

It is easily checked that $P(a, h, b)$ is simple and

$$V(P(a, h, b)) = [a, b] \bigcup [b + h, b + h + d - 1].$$

Furthermore, the edge labels of $P(a, h, b)$ are distinct and

$$\bar{E}(P(a, h, b)) = [h, h + 2d - 1].$$

These formulas will be used extensively in the proofs that follow.

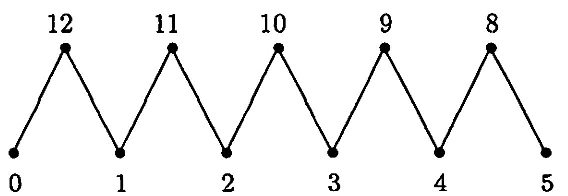

FIGURE 1. The path $P(0, 3, 5)$.

Theorem 7. Let x and y be positive integers and let $G = C_{4x+1} \bigcup C_{4y+1}$. Then G has a ρ-labeling.

Proof. Let G_1 and G_2 be the two cycles in G, defined as follows:

$$\begin{aligned} G_1 &= P(0, 2x+4y+4, x-1) + P(x-1, 4y+3, 2x-1) \\ &+ (2x-1, 2x, 4x+4y+3, 0), \\ G_2 &= P(4x+4y+4, 2y+2, 4x+5y+4) + P(4x+5y+4, 3, 4x+6y+3) \\ &+ (4x+6y+3, 4x+6y+5, 4x+8y+6, 4x+4y+4). \end{aligned}$$

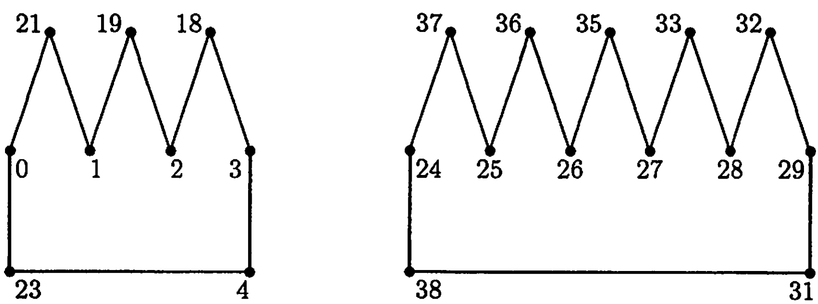

FIGURE 2. A ρ-labeling of $C_9 \bigcup C_{13}$.

Now we compute

$$\begin{aligned} V(G_1) &= [0, 2x-1] \bigcup [3x+4y+3, 4x+4y+1] \bigcup [2x+4y+2, 3x+4y+1] \\ &\bigcup \{2x, 4x+4y+3\} \\ V(G_2) &= [4x+4y+4, 4x+6y+3] \bigcup [4x+7y+6, 4x+8y+5] \\ &\bigcup [4x+6y+6, 4x+7y+4] \bigcup \{4x+6y+5, 4x+8y+6\}. \end{aligned}$$

We can order these as

$$[0, 2x-1], 2x, [2x+4y+2, 3x+4y+1], [3x+4y+3, 4x+4y+1], 4x+4y+3$$

from G_1, and

$$\begin{aligned} &[4x+4y+4, 4x+6y+3], 4x+6y+5, [4x+6y+6, 4x+7y+4], \\ &[4x+7y+6, 4x+8y+5], 4x+8y+6 \end{aligned}$$

from G_2. We see that the vertices of the two cycles are distinct and contained in $[0, 8x+8y+4]$. Note that if $y = 1$ then the set $[4x+6y+6, 4x+7y+4]$ is empty. Likewise if $x = 1$ then $[3x+4y+3, 4x+4y+1]$ will also be empty. These conditions however do not change the proof.

Likewise we compute

$$\begin{aligned}\bar{E}(G_1) &= [2x+4y+4, 4x+4y+1] \\ &\bigcup [4y+3, 2x+4y+2] \bigcup \{1, 2x+4y+3, 4x+4y+3\}, \\ \bar{E}(G_2) &= [2y+2, 4y+1] \bigcup [3, 2y] \bigcup \{2, 2y+1, 4y+2\}.\end{aligned}$$

We can order these as the edge label 1 from G_1,

$$2, [3, 2y], 2y+1, [2y+2, 4y+1], 4y+2$$

from G_2, and

$$[4y+3, 2x+4y+2], 2x+4y+3, [2x+4y+4, 4x+4y+1], 4x+4y+3$$

from G_1. Thus $\bar{E}(G) = [1, 4x+4y+1] \bigcup \{4x+4y+3\}$. Since $2(4x+4y+2)+1-(4x+4y+3) = 4x+4y+2$, we have a ρ-labeling. As with the vertex labels, if $y=1$ the set $[3, 2y]$ will be empty. Likewise if $x=1$, then $[2x+4y+4, 4x+4y+1]$ will also be empty. Neither condition would change the proof. □

Theorem 8. Let x and y be nonnegative integers and let $G = C_{4x+3} \bigcup C_{4y+3}$. Then G has a ρ-labeling.

Proof. The two cycles will be defined as follows:

$$\begin{aligned}G_1 &= P(0, 2x+4y+6, x) + P(x, 4y+5, 2x) + (2x, 2x+2, 4x+4y+7, 0), \\ G_2 &= P(4x+4y+8, 2y+4, 4x+5y+8) + P(4x+5y+8, 3, 4x+6y+8) \\ &+ (4x+6y+8, 4x+6y+9, 4x+8y+12, 4x+4y+8).\end{aligned}$$

Now we compute

$$\begin{aligned}V(G_1) &= [0, 2x] \bigcup [3x+4y+6, 4x+4y+5] \bigcup [2x+4y+5, 3x+4y+4] \\ &\bigcup \{2x+2, 4x+4y+7\} \\ V(G_2) &= [4x+4y+8, 4x+6y+8] \bigcup [4x+7y+12, 4x+8y+11] \\ &\bigcup [4x+6y+11, 4x+7y+10] \bigcup \{4x+6y+9, 4x+8y+12\}.\end{aligned}$$

We can order these as

$$[0, 2x], 2x+2, [2x+4y+5, 3x+4y+4], [3x+4y+6, 4x+4y+5], 4x+4y+7$$

from G_1, and

$$\begin{aligned}&[4x+4y+8, 4x+6y+8], 4x+6y+9, [4x+6y+11, 4x+7y+10], \\ &[4x+7y+12, 4x+8y+11], 4x+8y+12\end{aligned}$$

from G_2. We see that the vertices of the two cycles are distinct and contained in $[0, 2(4x+4y+6)] = [0, 8x+8y+12]$. Note that if $y=0$, then the sets $[4x+6y+11, 4x+7y+10]$ and $[4x+7y+12, 4x+8y+11]$ are empty. Likewise if $x=0$, then the sets $[2x+4y+5, 3x+4y+4]$ and $[3x+4y+6, 4x+4y+5]$ will also be empty. This however does not change the proof.

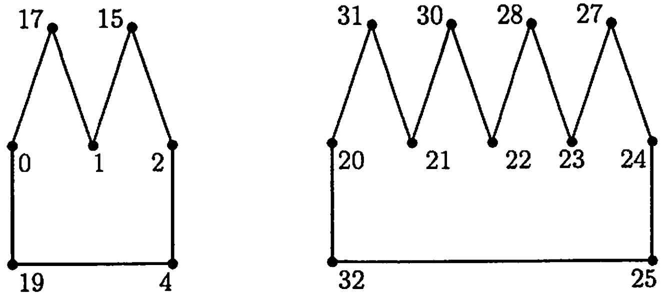

FIGURE 3. A ρ-labeling of $C_7 \bigcup C_{11}$.

Likewise we compute

$$\begin{aligned}\bar{E}(G_1) &= [2x+4y+6, 4x+4y+5]\bigcup[4y+5, 2x+4y+4] \\ &\bigcup \{2, 2x+4y+5, 4x+4y+7\}, \\ \bar{E}(G_2) &= [2y+4, 4y+3]\bigcup[3, 2y+2]\bigcup\{1, 2y+3, 4y+4\}.\end{aligned}$$

We can order these as the edge label 1 from G_2, 2 from G_1,

$$[3, 2y+2], 2y+3, [2y+4, 4y+3], 4y+4$$

from G_2, and

$$[4y+5, 2x+4y+4], 2x+4y+5, [2x+4y+6, 4x+4y+5], 4x+4y+7$$

from G_1. Thus $\bar{E}(G) = [1, 4x+4y+5]\bigcup\{4x+4y+7\}$. Since $2(4x+4y+6)+1-(4x+4y+7) = 4x+4y+6$, we have a ρ-labeling. Again, if $y=0$, then the sets $[3, 2y+2]$ and $[2y+4, 4y+3]$ are empty. Likewise if $x=0$, then $[4y+5, 2x+4y+4]$ and $[2x+4y+6, 4x+4y+5]$ will also be empty. These cases will not however change the proof. □

4. Concluding Remarks

We summarize the known results for labelings of $C_m \bigcup C_n$ in the table below.

$m \pmod 4$	$n \pmod 4$	Labeling of $C_m \bigcup C_n$	Reference
0	0	α	[2]
0	1	γ	[4]
0	2	ρ^{++}	[3]
0	3	β γ if $(m,n) \neq (4,3)$	[2] [4]
1	1	ρ	this paper
1	2	β γ	[2] [4]
1	3	β	[2]
2	2	α	[2]
2	3	γ	[4]
3	3	ρ	this paper

Table 1. Labelings of $C_m \bigcup C_n$.

This work was done while the first author was an undergraduate student at Illinois State University. A group of undergraduates at Illinois State is currently investigating the various labelings of 2-regular graphs with three components.

References

[1] J. Abrham and A. Kotzig, All 2-regular graphs consisting of 4-cycles are graceful, *Discrete Math.* **135** (1994), 1–14.

[2] J. Abrham and A. Kotzig, Graceful valuations of 2-regular graphs with two components, *Discrete Math.* **150** (1996), 3–15.

[3] A. Blinco and S.I. El-Zanati, A note on the cyclic decomposition of complete graphs into bipartite graphs, *Bull. Inst. Combin. Appl.*, 40 (2004), 77–82.

[4] A. Blinco, S.I. El-Zanati and C. Vanden Eynden, On the cyclic decomposition of complete graphs into almost-bipartite graphs, *Discrete Math.*, to appear.

[5] J. Bosák, *Decompositions of Graphs*, Kluwer Academic Publishers Group, Dordrecht, 1990.

[6] J.H. Dinitz and P. Rodney, Disjoint difference families with block size 3, *Util. Math.* **52** (1997), 153–160.

[7] D. Donovan, S. I. El-Zanati, C. Vanden Eynden and S. Sutinuntopas, Labelings of unions of up to four uniform cycles, *Australas. J. Combin.*, **29** (2004), 323–336.

[8] S.I. El-Zanati, C. Vanden Eynden and N. Punnim, On the cyclic decomposition of complete graphs into bipartite graphs, *Australas. J. Combin.* **24** (2001), 209–219.

[9] K. Eshghi, The existence and construction of α-valuations of 2-regular graphs with 3 components, Ph.D. Thesis, Industrial Engineering Dept., University of Toronto, 1997.

[10] J.A. Gallian, A dynamic survey of graph labeling, *Electron. J. Combin.*, Dynamic Survey 6, 106 pp.

[11] A. Kotzig, β-valuations of quadratic graphs with isomorphic components, *Util. Math.*, 7 (1975), 263–279.

[12] A. Kotzig, Recent results and open problems in graceful graphs, *Congress. Numer.* 44 (1984), 197–219.

[13] A. Rosa, On the cyclic decompositions of the complete graph into polygons with odd number of edges, *Časopis Pěst. Mat.* 91 (1966), 53–63.

[14] A. Rosa, On the cyclic decomposition of the complete graph into $(4m+2)$-gons, *Mat.-Fyz. Časopis Sloven. Akad. Vied* 16 (1966), 349–352.

[15] A. Rosa, On certain valuations of the vertices of a graph, in: Théorie des graphes, journées internationales d'études, Rome 1966 (Dunod, Paris, 1967), 349–355.