New Families of Cordial Graphs

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Abstract

In this paper we show the cordiality of following families of graphs: (1) Pyramid graphs, (2) One point unions of plys, (3) One point unions of wheel related graphs, (4) Path unions of shells of different sizes, (5) Path unions of flags of different sizes.

By a graph G we always mean a simple loop-less graph with vertex set $V(G)$ and edge set $E(G)$. Let $f : V(G) \longrightarrow \{0,1\}$ be a binary map. By $v_f(0)$ and $v_f(1)$ we mean the number of vertices which are assigned the value 0 and 1 respectively by f. Such a binary function f induces a mapping $f : E(G) \longrightarrow \{0,1\}$ given by $f(e) = | f(u) - f(v) |$ where $e = uv \in E(G)$. By $e_f(0), e_f(1)$ we mean the number of edges which are assigned the values 0 and 1 respectively by f. The map f is called a *cordial labeling* if $| v_f(0) - v_f(1) | \leq 1$ and $| e_f(0) - e_f(1) | \leq 1$. A graph G is said to be *cordial* if it admits a cordial labeling. For a binary labeling f of a graph G, by the *dual* $\hat{f}$ we mean the labeling obtained by interchanging the labels 0 and 1 in f. One notes that $v_{\hat{f}}(0) = v_f(1), v_{\hat{f}}(1) = v_f(0)$ and $e_{\hat{f}}(0) = e_f(0), e_{\hat{f}}(1) = e_f(1)$. For a graph G by the *index of cordiality* $i(G)$, we mean the minimum of $| e_f(0) - e_f(1) |$ where the minimum is taken over all the binary labelings of G which are equitable on the vertices. It has been proved by Cahit that an Eulerian graph with number of edges $\equiv 2(\bmod 4)$ is not cordial.

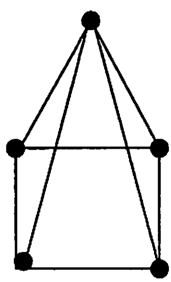
Pyramid Graphs

[1]This work was partially supported by Department of Science and Technology, Government of India.

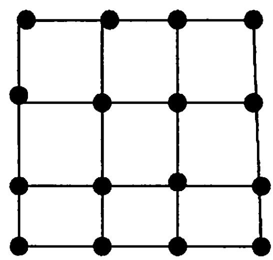
The family $\{PY_n, n = 1, 2, \cdots\}$ of Pyramid Graphs is defined inductively as follows:

Definition: The first member PY_1 is just $C_4 \times K_1$. The vertex of K_1 is denoted by u and the vertices of C_4 are denoted by $\{u^1_{1,1}, u^1_{1,2}, u^1_{2,1}, u^1_{2,2}\}$. The edge set is given by

$$E(PY_1) = \{uu^1_{i,j} \mid 1 \le i, j \le 2\} \bigcup \{u^1_{1,1}u^1_{1,2}, u^1_{1,2}u^1_{2,2}, u^1_{2,2}u^1_{2,1}, u^1_{2,1}u^1_{1,1}\}.$$



PY_1



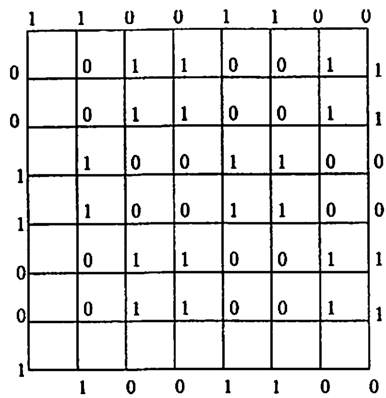
Base of PY_2

If PY_{n-1} is constructed inductively, then to construct PY_n, one takes new vertices $\{u^n_{i,j} \mid 1 \le i, j \le 2^n\}$. This is added to PY_{n-1} as the base for PY_n in the form of the grid $P_{2^n} \times P_{2^n}$. Thus,

$$\begin{aligned}
V(PY_n) &= V(PY_{n-1}) \bigcup \{u^n_{i,j} \mid 1 \le i, j \le 2^n\}, \\
E(PY_n) &= E(PY_{n-1}) \bigcup_{i=1}^{2^n} \{u^n_{i,j}u^n_{i,j+1} \mid 1 \le j \le 2^n - 1\} \\
&\quad \bigcup_{j=1}^{2^n} \{u^n_{i,j}u^n_{i+1,j} \mid 1 \le i \le 2^n - 1\} \\
&\quad \bigcup_{i,j} \{u^{n-1}_{i,j}u^n_{2i-1,2j-1}, u^{n-1}_{i,j}u^n_{2i-1,2j}, u^{n-1}_{i,j}u^n_{2i,2j-1}, u^{n-1}_{i,j}u^n_{2i,2j}\}
\end{aligned}$$

Thus, $\mid V(PY_n) \mid = 1 + 4 + 4^2 + \cdots + 4^n$ and $\mid E(PY_n) \mid = \mid E(PY_{n-1}) \mid + 4^n + 2^{n-1}(2^n - 1)$, that is, PY_n has odd number of vertices and even number of edges.

Theorem 1: The pyramid graph PY_n is cordial for every integer $n \geq 1$.
Proof: We define the labeling as follows: Define $f(u) = 0, f(u^1_{1,1}) = 1 = f(u^1_{1,2})$,
$(u^1_{2,1}) = 0 = f(u^1_{2,2})$. One can see that at this stage three vertices have been labeled 0, two have been labeled 1, four edges have received label 0 and four edges have received label 1, that is the labeling is equitable on the vertices as well as the edges.



Base Labels of PY_3

For the base of PY_n, we define f as follows:

$$\begin{aligned}
f(u^n_{i,j}) &= 1 \quad i \equiv 1,0(\bmod 4) \text{and} j \equiv 1,2(\bmod 4), \\
f(u^n_{i,j}) &= 0 \quad i \equiv 1,0(\bmod 4) \text{and} j \equiv 3,0(\bmod 4), \\
f(u^n_{i,j}) &= 1 \quad i \equiv 2,3(\bmod 4) \text{and} j \equiv 1,2(\bmod 4), \\
f(u^n_{i,j}) &= 0 \quad i \equiv 2,3(\bmod 4) \text{and} j \equiv 3,0(\bmod 4).
\end{aligned}$$

One can easily check that $v_f(0) = 1 + 2 + 2^3 + \cdots + 2^{2n-1}$ and $v_f(1) = +2 + 2^3 + \cdots + 2^{2n-1}$. That is, the labeling is equitable on the set of vertices of PY_n. We have seen that the labeling is equitable on the edges of PY_1. The cordiality of PY_n follows from the observation that the attachment and labeling of each of the successive bases creates equal number of vertices as well as edges with labels 0 and 1. □

One Point Union of Plys

Definition: A $t-$ply $P_t(u,v)$ is a graph with t paths, each of length at least two and such that no two paths have a vertex in common except for the end vertices u and v.

We say that a path P is of type i if $l(P) \equiv i(\text{mod}4)), i = 1,2,3,4$. Denote by t_i, the number of paths of the type $i, i = 1,2,3,4$. Then

$$t = t_1 + t_2 + t_3 + t_4 \cdots\cdots (\text{I})$$

If $e =| E(P_t(u,v)) |$, then

$$e \equiv (t_1 + 2\, t_2 + 3\, t_3)(\text{mod}4) \cdots\cdots (\text{II})$$

Further, let $t_1 = 4\, s_1 + x_1, t_3 = 4\, s_3 + x_3, t_2 = 2\, s_2 + x_2, t_4 = 2\, s_4 + x_4, 0 \leq x_1,\, x_3 \leq 3$ and $0 \leq x_2,\, x_4 \leq 1$. By (II), it follows that

$$e \equiv x_1 + 2\, x_2 + 3\, x_3(\text{mod}4) \cdots\cdots (\text{III})$$

Since x_1, x_3 take 4 values each and x_2, x_4 take 2 values each, there will in all be 64 cases to be considered for cordiality. In the following 8 cases, $P_t(u,v)$ is Eulerian and by (III), $e \equiv 2$ (mod4), and hence $P_t(u,v)$is not cordial.

(1) $x_1 = x_3 = 0; x_2 = x_4 = 1.$
(2) $x_1 = x_2 = x_4 = 0, x_3 = 2.$
(3) $x_1 = x_2 = x_3 = x_4 = 1.$
(4) $x_1 = 1, x_2 = x_4 = 0, x_3 = 3.$
(5) $x_1 = 2, x_2 = x_4 = x_3 = 0.$
(6) $x_1 = 2 = x_3, x_2 = x_4 = 1.$
(7) $x_1 = 3, x_2 = x_4 = 0, x_3 = 1.$
(8) $x_1 = x_3 = 3, x_2 = x_4 = 1.$

Cordiality of single plys have been studied in detail in [5]. Let $P \equiv \{u, v_1, \cdots, v_n, v\}$ be a typical path with end points u and v in the t-ply $P_t(u,v)$. The length $l(P)$ of this path is $n+1$. The type of labelings to be used for such $t-$ply graphs have been given in detail in [5]. The classification of plys is done according to the lengths of paths present in those plys. We give only the relevant details herein in the form of a table, in which the letters 'E' and 'NE' denote an Eulerian, non-Eulerian graph respectively. Further let 'e' be the number of edges in the corresponding $t-$ply graph and let $e \equiv r(\mathrm{mod}4), 0 \leq r \leq 3$.

Property	A_1	A_2	A_3	A_4	A_5	A_6	B	C
E/NE	E	NE	NE	NE	E	E	NE	E
r	0	1	3	2	3	1	0	2
	0000	0030	0010	0021	0011	0031	0001	0020
	0121	0111	0131	0100	0130	0110	0120	0101
	1010	1000	1020	1031	1021	1001	1011	1030
	1131	1121	1101	1110	1100	1120	1130	1111
(x_1, x_2, x_3, x_4)	2020	2010	2030	2001	2031	2011	2021	2000
	2101	2131	2111	2120	2110	2130	2100	2121
	3030	3020	3000	3011	3001	3021	3031	3010
	3111	3101	3121	3130	3120	3100	3110	3131

We next list the labelings that we will require, to label the various types of plys in the one point unions. Throughout, we note that the vertex v in the ply graph will be taken as the central vertex in the one point union. We emphasize that the labelings listed here, are part of the huge list of labelings provided in [5]. As we do not need the exact formulae of these labelings, we take their exitance for granted.

As before,let 'e' be the number of edges in the corresponding $t-$ply graph and let $e \equiv r(\mathrm{mod}4), 0 \leq r \leq 3$. We have to handle those plys which are Eulerian with $e \equiv 2(\mathrm{~mod}4)$ rather carefully because they are not cordial. Following table gives the list of all those labelings which we use.

Type	E/NE	r	f	Label for u	Label for v	Relation of vertex labels	Relation of edge labels
A_1	E	0	α_1	1	0	$v_f(0) = v_f(1)$	$e_f(0) = e_f(1)$
			α_2	1	0	$v_f(0) + 2 = v_f(1)$	$e_f(0) = e_f(1)$
A_2	NE	1	β_1	1	0	$v_f(0) = v_f(1)$	$e_f(0) + 1 = e_f(1)$
			β_2	1	0	$v_f(0) + 2 = v_f(1)$	$e_f(0) + 1 = e_f(1)$
			β_3	0	0	$v_f(0) = v_f(1) + 2$	$e_f(0) = e_f(1) + 1$
			β_4	0	0	$v_f(0) = v_f(1)$	$e_f(0) = e_f(1) + 1$
A_3	NE	3	γ_1	1	0	$v_f(0) = v_f(1)$	$e_f(0) = e_f(1) + 1$
			γ_2	1	0	$v_f(0) + 2 = v_f(1)$	$e_f(0) = e_f(1) + 1$
			γ_3	0	0	$v_f(0) = v_f(1) + 2$	$e_f(0) + 1 = e_f(1)$
			γ_4	0	0	$v_f(0) = v_f(1)$	$e_f(0) + 1 = e_f(1)$
A_4	NE	2	δ_1	1	0	$v_f(0) + 1 = v_f(1)$	$e_f(0) = e_f(1)$
			δ_2	0	0	$v_f(0) = v_f(1) + 1$	$e_f(0) + 2 = e_f(1)$
A_5	E	3	θ_1	1	0	$v_f(0) + 1 = v_f(1)$	$e_f(0) + 1 = e_f(1)$
			θ_2	1	0	$v_f(0) + 1 = v_f(1)$	$e_f(0) = e_f(1) + 3$
A_6	E	1	ϕ_1	1	0	$v_f(0) + 1 = v_f(1)$	$e_f(0) = e_f(1) + 1$
			ϕ_2	1	0	$v_f(0) + 1 = v_f(1)$	$e_f(0) + 3 = e_f(1)$
B	NE	0	μ_1	0	0	$v_f(0) + 1 = v_f(1)$	$e_f(0) = e_f(1)$
			μ_2	0	0	$v_f(0) = v_f(1) + 1$	$e_f(0) = e_f(1)$
			μ_3	1	0	$v_f(0) + 1 = v_f(1)$	$e_f(0) = e_f(1) + 2$
C	E	2	ξ_1	1	0	$v_f(0) = v_f(1)$	$e_f(0) = e_f(1) + 2$
			ξ_2	1	0	$v_f(0) = v_f(1)$	$e_f(0) + 2 = e_f(1)$
			ξ_3	1	0	$v_f(0) + 2 = v_f(1)$	$e_f(0) = e_f(1) + 2$
			ξ_4	1	0	$v_f(0) + 2 = v_f(1)$	$e_f(0) + 2 = e_f(1)$

We firstly observe that for every labeling listed above, for each type of graph, at least one end vertex has the label 0. Moreover, for each type from $\{A_1, \cdots, A_6, B, C\}$, we have a labeling which is not cordial; in some of them the vertices are not equitably labeled and in the remaining the edges are not equitably labeled.

Definition: Let f be a cordial labeling of a single t-ply graph $P_t(u, v,)$.

Let $\tilde{f}$ be a binary labeling of $P_t(u,v)$ such that $\tilde{f}(u) = f(v), \tilde{f}(v) = f(u)$. Also, on any path, $u, v_1, v_2, \cdots, v_n, v$ in $P_t(u,v)$ let $\tilde{f}(v_i) = f(v_{n-i+1})$ i.e. $\tilde{f}(v_1) = f(v_n)\tilde{f}(v_2) = f(v_{n-1})$ and so on. We call $\tilde{f}$ the **inversion** labeling of f. Then, $v_{\tilde{f}}(0) = v_f(0), v_{\tilde{f}}(1) = v_f(1)$; $e_{\tilde{f}}(0) = e_f(0), e_{\tilde{f}}(1) = e_f(1)$.

Definition: Let $P_{t^{(i)}}(u_i, v)$ be single $t^{(i)}$-plys, $1 \leq i \leq n$. A one point union G of these $nt^{(i)}$-plys is the graph obtained by taking v as a common vertex such that any two plys $P_{t^{(i)}}(u_i, v)$ and $P_{t^{(j)}}(u_j, v), i \neq j$ are edge disjoint and do not have any vertex in common except v.

Theorem: The one point union of n single $t^{(i)}$-plys G is cordial if and only if G is not Eulerian with $\mid E(G) \mid \equiv 2(\bmod 4)$.

Proof: Let $n_i, 1 \leq i \leq 6$ be the number of $t_{(i)}$-plys of Type A_i in G. Let n_7, n_8 denote the number of $t_{(i)}$-plys of Types B, C respectively. Then $n_i = 2p_i + r_i, i = 1,2,3,8; 0 \leq r_i \leq 1$ and $n_i = 4p_i + r_i, i = 5,6; 0 \leq r_i \leq 3$. Let f be a binary labeling of G described below: The labeling of G will be done in two stages.Let $v'_f(0), v'_f(1), e'_f(0), e'_f(1)$ denote the number of vertices and edges which receive the label 0, 1 respectively after Stage 1. Let $v''_f(0), v''_f(1), e''_f(0), e''_f(1)$ denote the number of vertices and edges which receive the label 0, 1 respectively at Stage 2.

Stage 1: In this stage we label vertices in all the plys except r_i plys of the respective type, $1 \leq i \leq 8$. Those remaining will be labeled in Stage 2.

In each of $\left\lfloor \frac{n_1}{2} \right\rfloor$ graphs of Type A_1, use the labeling α_1 and for each of the remaining $\left\lfloor \frac{n_1}{2} \right\rfloor$ graphs of Type A_1, use the labeling $\tilde{\alpha}_2$. If n_1 is even, all graphs of Type A_1 are labeled. If n_1 is odd, then one block of Type A_1 remains. This will be labeled after Stage 2.

Out of $2p_2$ of the graphs of Type A_2, use the labeling β_1 for each of p_2 of these and the labeling β_3 for each of the the other p_2 graphs. If n_2 is even, all the plys of Type A_1 are labeled. If n_2 is odd the remaining ply will be labeled in Stage 2.

Of $2p_3$ of the graphs of Type A_3, use the labeling γ_1 for each of p_3 of these and the labeling γ_3 for each of the other p_3. If at all one ply of Type A_3 is remaining it will be labeled in Stage 2.

In each of the n_4 graphs of Type A_4, use the labeling $\bar{\bar{\delta}}_1$.

Out of $4p_5$ of the graphs of Type A_5, for each of $3p_5$ of these use the labeling $\bar{\bar{\theta}}_1$ and for each of the other p_5 graphs of the same type, use the labeling $\bar{\bar{\theta}}_2$.

Out of $4p_6$ of the graphs of Type A_6, for each of $3p_6$ of these, use the labeling $\bar{\bar{\phi}}_1$ and for each of the other p_6 graphs of the same type, use the labeling $\bar{\bar{\phi}}_2$.

In each of the n_7 graphs of Type B use the labeling μ_2.

Out of $2p_8$ of the graphs of Type C, for each of p_8 of these, use the labeling ξ_1 and for each of the other p_8 graphs of the same type, use the labeling $\bar{\bar{\xi}}_4$. Then

$v'_f(0) = \left\lfloor \frac{n_1}{2} \right\rfloor (v_{\alpha_1}(0)-1) + \left\lfloor \frac{n_1}{2} \right\rfloor (v_{\bar{\bar{\alpha}}_2}(0)-1) + p_2(v_{\beta_1}(0)-1) + p_2(v_{\beta_3}(0)-1)$
$+p_3(v_{\gamma_1}(0)-1) + p_3(v_{\gamma_3}(0)-1) + n_4(v_{\bar{\bar{\delta}}_1}(0)-1) + 3p_5(v_{\bar{\bar{\theta}}_1}(0)-1) + p_5(v_{\bar{\bar{\theta}}_2}(0)-$
$1)+$
$p_6(v_{\bar{\bar{\phi}}_1}(0)-1) + p_6(v_{\bar{\bar{\phi}}_2}(0)-1) + n_7(v_{\mu_2}(0)-1) + p_8(v_{\xi_1}(0)-1) + p_8(v_{\bar{\bar{\xi}}_4}(0)-$
$1)+1,$
$= \left\lfloor \frac{n_1}{2} \right\rfloor (v_{\alpha_1}(1)-1) + \left\lfloor \frac{n_1}{2} \right\rfloor (v_{\bar{\bar{\alpha}}_2}(1)+1) + p_2(v_{\beta_1}(1)-1) + p_2(v_{\beta_3}(1)+1)$
$+p_3(v_{\gamma_1}(1)-1) + p_3(v_{\gamma_3}(1)+1) + n_4 v_{\bar{\bar{\delta}}_1}(1) + 3p_5 v_{\bar{\bar{\theta}}_1}(1) + p_5 v_{\bar{\bar{\theta}}_2}(1) + 3p_6(v_{\bar{\bar{\phi}}_1}(1)$
$+ p_6(v_{\bar{\bar{\phi}}_2}(1) + n_7 v_{\mu_2}(1) + p_8(v_{\xi_1}(1)-1) + p_8(v_{\bar{\bar{\xi}}_4}(1)+1) + 1$
$= \left\lfloor \frac{n_1}{2} \right\rfloor [v_{\alpha_1}(1) + v_{\bar{\bar{\alpha}}_2}(1)] + p_2[(v_{\beta_1}(1) + v_{\beta_3}(1)] + p_3[v_{\gamma_1}(1) + v_{\gamma_3}(1)] + n_4 v_{\bar{\bar{\delta}}_1}(1)$
$+p_5[3v_{\bar{\bar{\theta}}_1}(1) + v_{\bar{\bar{\theta}}_2}(1)] + p_6[3v_{\bar{\bar{\phi}}_1}(1) + v_{\bar{\bar{\phi}}_2}(1)] + n_7 v_{\mu_2}(1) + p_8[v_{\xi_1}(1) + v_{\bar{\bar{\xi}}_4}(1)] + 1.$

But

$v'_f(1) = \left\lfloor \frac{n_1}{2} \right\rfloor [v_{\alpha_1}(1) + v_{\bar{\bar{\alpha}}_2}(1)] + p_2[(v_{\beta_1}(1) + v_{\beta_3}(1)] + p_3[v_{\gamma_1}(1)$
$+v_{\gamma_3}(1)] + n_4 v_{\bar{\bar{\delta}}_1}(1) + p_5[3v_{\bar{\bar{\theta}}_1}(1) + v_{\bar{\bar{\theta}}_2}(1)] + p_6[3v_{\bar{\bar{\phi}}_1}(1) + v_{\bar{\bar{\phi}}_2}(1)] + n_7 v_{\mu_2}(1) +$
$p_8[v_{\xi_1}(1) + v_{\bar{\bar{\xi}}_4}(1)].$

Thus $v'_f(0) = v'_f(1) + 1$.

Also,

$e'_f(0) = \left\lfloor \frac{n_1}{2} \right\rfloor [e_{\alpha_1}(0) + e_{\bar{\bar{\alpha}}_2}(0)] + p_2[e_{\beta_1}(0) + e_{\beta_3}(0)] + p_3[e_{\gamma_1}(0) + e_{\gamma_3}(0)] +$
$n_4 e_{\bar{\bar{\delta}}_1}(0)$
$+p_5[3e_{\bar{\bar{\theta}}_1}(0) + e_{\bar{\bar{\theta}}_2}(0)] + p_6[3e_{\bar{\bar{\phi}}_1}(0) + e_{\bar{\bar{\phi}}_2}(0)] + n_7 e_{\mu_2}(0) + p_8[e_{\xi_1}(0) + e_{\bar{\bar{\xi}}_4}(0)]$
$= \left\lfloor \frac{n_1}{2} \right\rfloor [e_{\alpha_1}(1) + e_{\bar{\bar{\alpha}}_2}(1)] + p_2[e_{\beta_1}(1) - 1 + e_{\beta_3}(1) + 1] + p_3[e_{\gamma_1}(1) + 1 + e_{\gamma_3}(1) - 1]$
$+ n_4 e_{\bar{\bar{\delta}}_1}(1) + p_5[3e_{\bar{\bar{\theta}}_1}(1) - 1 + e_{\bar{\bar{\theta}}_2}(1) + 3] + p_6[3e_{\bar{\bar{\phi}}_1}(1) + 1 + e_{\bar{\bar{\phi}}_2}(1) - 3]$

$+ n_7 e_{\mu_2}(1) + p_8[e_{\xi_1}(1) + 2 + e_{\bar{\xi}_4}(1) - 2]$.

But we have

$e'_f(1) = \left\lfloor \frac{n_1}{2} \right\rfloor [e_{\alpha_1}(1) + e_{\bar{\alpha}_2}(1)] + p_2[e_{\beta_1}(1) + e_{\beta_3}(1)] + p_3[e_{\gamma_1}(1) + e_{\gamma_3}(1)] + n_4 e_{\bar{\delta}_1}(1)$

$+ p_5[3e_{\bar{\theta}_1}(1) + e_{\bar{\theta}_2}(1)] + p_6[3e_{\bar{\bar{\phi}}_1}(1) + e_{\bar{\phi}_2}(1)] + n_7 e_{\mu_2}(1) + p_8[e_{\xi_1}(1) + e_{\bar{\xi}_4}(1)]$.

Consequently, we have $e'_f(0) = e'_f(1)$.

We now list in tabular form, the 128 cases that arise due to the various values of r_i. The choice of the appropriate labeling made in each case is indicated alongside, in brackets. Where $r_i > 1$, we have more than one block of the same type and the labeling used for each block is also mentioned. Thus for instance an entry of the form $2(\bar{\bar{\phi}}_1, \bar{\bar{\phi}}_2)$ means that for one of the two blocks of that type we use the labeling $\bar{\bar{\phi}}_1$ and for the other block, we use $\bar{\bar{\phi}}_2$. Cases which are problematic are marked with an asterix (*) and will be dealt with separately. The table is broken according to the value of r_2, r_3, r_5, r_6 and r_8. If $r_1 = 1$ we deal with it later.

r_2	r_3	r_5	r_6	r_8	Vertex condition	Edge Condition
0	0	0	0	0	0	0
0	0	0	0	1	*	*
0	0	0	$1(\bar{\bar{\phi}}_1)$	0	$v''_f(0) = v''_f(1)$	$e''_f(0) = e''_f(1) + 1$
0	0	0	$1(\bar{\bar{\phi}}_1)$	$1(\xi_2)$	$v''_f(0) + 1 = v''_f(1)$	$e''_f(0) + 1 = e''_f(1)$
0	0	0	2	0	*	*
0	0	0	$2(\bar{\bar{\phi}}_1, \bar{\bar{\phi}}_1)$	$1(\xi_2)$	$v''_f(0) = v''_f(1) + 1$	$e''_f(0) = e''_f(1)$
0	0	0	$3(\bar{\bar{\phi}}_1, \bar{\bar{\phi}}_1, \bar{\bar{\phi}}_2)$	0	$v''_f(0) = v''_f(1)$	$e''_f(0) + 1 = e''_f(1)$
0	0	0	$3(\bar{\bar{\phi}}_1, \bar{\bar{\phi}}_1, \bar{\bar{\phi}}_1)$	$1(\xi_2)$	$v''_f(0) + 1 = v''_f(1)$	$e''_f(0) = e''_f(1) + 1$
0	0	$1(\bar{\bar{\theta}}_1)$	0	0	$v''_f(0) = v''_f(1)$	$e''_f(0) + 1 = e''_f(1)$
0	0	$1(\bar{\bar{\theta}}_1)$	0	$1(\xi_1)$	$v''_f(0) + 1 = v''_f(1)$	$e''_f(0) = e''_f(1) + 1$
0	0	$1(\bar{\bar{\theta}}_1)$	$1(\bar{\bar{\phi}}_1)$	0	$v''_f(0) = v''_f(1)$	$e''_f(0) = e''_f(1)$
0	0	1	1	1	*	*
0	0	$1(\bar{\bar{\theta}}_1)$	$2(\bar{\bar{\phi}}_1, \bar{\bar{\phi}}_1)$	0	$v''_f(0) = v''_f(1)$	$e''_f(0) = e''_f(1) + 1$
0	0	$1(\bar{\bar{\theta}}_1)$	$2(\bar{\bar{\phi}}_1, \bar{\bar{\phi}}_1)$	$1(\xi_2)$	$v''_f(0) + 1 = v''_f(1)$	$e''_f(0) + 1 = e''_f(1)$
0	0	1	3	0	*	*
0	0	$1(\bar{\bar{\theta}}_1)$	$3(\bar{\bar{\phi}}_1, \bar{\bar{\phi}}_1, \bar{\bar{\phi}}_1)$	$1(\xi_2)$	$v''_f(0) + 1 = v''_f(1)$	$e''_f(0) = e''_f(1)$
0	0	2	0	0	*	*

r_2	r_3	r_5	r_6	r_8	Vertex conditon	Edge Condition
0	0	$2(\hat{\theta}_1,\hat{\theta}_1)$	0	$1(\xi_1)$	$v''_f(0)+1=v''_f(1)$	$e''_f(0)=e''_f(1)$
0	0	$2(\bar{\hat{\theta}}_1,\bar{\hat{\theta}}_1)$	$1(\bar{\hat{\phi}}_1)$	0	$v''_f(0)=v''_f(1)$	$e''_f(0)+1=e''_f(1)$
0	0	$2(\bar{\hat{\theta}}_1,\bar{\hat{\theta}}_1)$	$1(\bar{\hat{\phi}}_1)$	$1(\xi_1)$	$v''_f(0)+1=v''_f(1)$	$e''_f(0)=e''_f(1)+1$
0	0	$2(\bar{\hat{\theta}}_1,\bar{\hat{\theta}}_1)$	$2(\bar{\hat{\phi}}_1,\bar{\hat{\phi}}_1)$	0	$v''_f(0)=v''_f(1)$	$e''_f(0)=e''_f(1)$
0	0	2	2	1	*	*
0	0	$2(\bar{\hat{\theta}}_1,\bar{\hat{\theta}}_1)$	$3(\bar{\hat{\phi}}_1,\bar{\hat{\phi}}_1,\bar{\hat{\phi}}_1)$	0	$v''_f(0)=v''_f(1)$	$e''_f(0)=e''_f(1)+1$
0	0	$2(\bar{\hat{\theta}}_1,\bar{\hat{\theta}}_1)$	$3(\bar{\hat{\phi}}_1,\bar{\hat{\phi}}_1,\bar{\hat{\phi}}_1)$	$1(\xi_2)$	$v''_f(0)+1=v''_f(1)$	$e''_f(0)+1=e''_f(1)$
0	0	$3(\bar{\hat{\theta}}_1,\bar{\hat{\theta}}_1,\bar{\hat{\theta}}_2)$	0	0	$v''_f(0)=v''_f(1)$	$e''_f(0)=e''_f(1)+1$
0	0	$3(\bar{\hat{\theta}}_1,\bar{\hat{\theta}}_1,\bar{\hat{\theta}}_1)$	0	$1(\xi_1)$	$v''_f(0)+1=v''_f(1)$	$e''_f(0)+1=e''_f(1)$
0	0	3	1	0	*	*
0	0	$3(\bar{\hat{\theta}}_1,\bar{\hat{\theta}}_1,\bar{\hat{\theta}}_1)$	$1(\bar{\hat{\phi}}_1)$	$1(\xi_1)$	$v''_f(0)+1=v''_f(1)$	$e''_f(0)=e''_f(1)$
0	0	$3(\bar{\hat{\theta}}_1,\bar{\hat{\theta}}_1,\bar{\hat{\theta}}_1)$	$2(\bar{\hat{\phi}}_1,\bar{\hat{\phi}}_1)$	0	$v''_f(0)=v''_f(1)$	$e''_f(0)+1=e''_f(1)$
0	0	$3(\bar{\hat{\theta}}_1,\bar{\hat{\theta}}_1,\bar{\hat{\theta}}_1)$	$2(\bar{\hat{\phi}}_1,\bar{\hat{\phi}}_1)$	$1(\xi_1)$	$v''_f(0)+1=v''_f(1)$	$e''_f(0)=e''_f(1)+1$
0	0	$3(\bar{\hat{\theta}}_1,\bar{\hat{\theta}}_1,\bar{\hat{\theta}}_1)$	$3(\bar{\hat{\phi}}_1,\bar{\hat{\phi}}_1,\bar{\hat{\phi}}_1)$	0	$v''_f(0)=v''_f(1)$	$e''_f(0)=e''_f(1)$
0	0	3	3	1	*	*
0	$1(\gamma_1)$	0	0	0	$v''_f(0)+1=v''_f(1)$	$e''_f(0)=e''_f(1)+1$
0	$1(\bar{\hat{\gamma}}_2)$	0	0	$1(\xi_2)$	$v''_f(0)=v''_f(1)$	$e''_f(0)+1=e''_f(1)$
0	$1(\gamma_4)$	0	$1(\bar{\hat{\phi}}_1)$	0	$v''_f(0)+1=v''_f(1)$	$e''_f(0)=e''_f(1)$
0	$1(\bar{\hat{\gamma}}_2)$	0	$1(\hat{\phi}_1)$	$1(\xi_2)$	$v''_f(0)=v''_f(1)$	$e''_f(0)=e''_f(1)$
0	$1(\gamma_4)$	0	$2(\bar{\hat{\phi}}_1,\bar{\hat{\phi}}_1)$	0	$v''_f(0)+1=v''_f(1)$	$e''_f(0)=e''_f(1)+1$
0	$1(\bar{\hat{\gamma}}_2)$	0	$2(\bar{\hat{\phi}}_1,\bar{\hat{\phi}}_1)$	$1(\xi_2)$	$v''_f(0)=v''_f(1)$	$e''_f(0)=e''_f(1)+1$
0	$1(\gamma_1)$	0	$3(\bar{\hat{\phi}}_1,\bar{\hat{\phi}}_1,\bar{\hat{\phi}}_2)$	0	$v''_f(0)+1=v''_f(1)$	$e''_f(0)=e''_f(1)$
0	$1(\gamma_3)$	0	$3(\bar{\hat{\phi}}_1,\bar{\hat{\phi}}_1,\phi_1)$	$1(\xi_2)$	$v''_f(0)=v''_f(1)$	$e''_f(0)=e''_f(1)$
0	$1(\gamma_1)$	$1(\bar{\hat{\theta}}_1)$	0	0	$v''_f(0)+1=v''_f(1)$	$e''_f(0)=e''_f(1)$
0	$1(\gamma_3)$	$1(\theta_1)$	0	$1(\xi_1$	$v''_f(0)=v''_f(1)$	$e''_f(0)=e''_f(1)$
0	$1(\gamma_1)$	$1(\bar{\hat{\theta}}_1)$	$1(\bar{\hat{\phi}}_1)$	0	$v''_f(0)+1=v''_f(1)$	$e''_f(0)=e''_f(1)+1$
0	$1(\bar{\hat{\gamma}}_2)$	$1(\bar{\hat{\theta}}_1)$	$1(\bar{\hat{\phi}}_1)$	$1(\xi_2)$	$v''_f(0)=v''_f(1)$	$e''_f(0)+1=e''_f(1)$
0	$1(\gamma_4)$	$1(\bar{\hat{\theta}}_1)$	$2(\bar{\hat{\phi}}_1,\bar{\hat{\phi}}_1)$	0	$v''_f(0)+1=v''_f(1)$	$e''_f(0)=e''_f(1)$
0	$1(\bar{\hat{\gamma}}_2)$	$1(\bar{\hat{\theta}}_1)$	$2(\bar{\hat{\phi}}_1,\bar{\hat{\phi}}_1)$	$1(\xi_2)$	$v''_f(0)=v''_f(1)$	$e''_f(0)=e''_f(1)$
0	$1(\gamma_4)$	$1(\bar{\hat{\theta}}_1)$	$3(\bar{\hat{\phi}}_1,\bar{\hat{\phi}}_1,\bar{\hat{\phi}}_1)$	0	$v''_f(0)+1=v''_f(1)$	$e''_f(0)=e''_f(1)+1$
0	$1(\bar{\hat{\gamma}}_4)$	$1(\bar{\hat{\theta}}_1)$	$3(\bar{\hat{\phi}}_1,\bar{\hat{\phi}}_1,\bar{\hat{\phi}}_1)$	$1(\xi_2)$	$v''_f(0)=v''_f(1)$	$e''_f(0)=e''_f(1)+1$

r_2	r_3	r_5	r_6	r_8	Vertex conditon	Edge Condition
0	$1(\gamma_1)$	$2(\theta_1,\theta_1)$	0	0	$v_f''(0)+1=v_f''(1)$	$e_f''(0)+1=e_f''(1)$
0	$1(\gamma_2)$	$2(\theta_1,\theta_1)$	0	$1(\xi_1)$	$v_f''(0)=v_f''(1)$	$e_f''(0)=e_f''(1)+1$
0	$1(\gamma_1)$	$2(\theta_1,\theta_1)$	$1(\phi_1)$	0	$v_f''(0)+1=v_f''(1)$	$e_f''(0)=e_f''(1)$
0	$1(\gamma_4)$	$2(\theta_1,\theta_1)$	$1(\phi_1)$	$1(\xi_1)$	$v_f''(0)=v_f''(1)$	$e_f''(0)=e_f''(1)$
0	$1(\gamma_1)$	$2(\theta_1,\theta_1)$	$2(\phi_1,\phi_1)$	0	$v_f''(0)+1=v_f''(1)$	$e_f''(0)=e_f''(1)+1$
0	$1(\gamma_2)$	$2(\theta_1,\theta_1)$	$2(\phi_1,\phi_1)$	$1(\xi_2)$	$v_f''(0)=v_f''(1)$	$e_f''(0)+1=e_f''(1)$
0	$1(\gamma_4)$	$2(\theta_1,\theta_1)$	$3(\phi_1,\phi_1,\phi_1)$	0	$v_f''(0)+1=v_f''(1)$	$e_f''(0)=e_f''(1)$
0	$1(\gamma_2)$	$2(\theta_1,\theta_1)$	$3(\varphi_1,\varphi_1,\phi_1)$	$1(\xi_1)$	$v_f''(0)=v_f''(1)$	$e_f''(0)=e_f''(1)$
0	$1(\gamma_3)$	$3(\theta_1,\theta_1,\theta_2)$	0	0	$v_f''(0)+1=v_f''(1)$	$e_f''(0)=e_f''(1)$
0	$1(\gamma_2)$	$3(\theta_1,\theta_1,\theta_1)$	0	$1(\xi_1)$	$v_f''(0)=v_f''(1)$	$e_f''(0)=e_f''(1)$
0	$1(\gamma_1)$	$3(\theta_1,\theta_1,\theta_1)$	$1(\phi_1)$	0	$v_f''(0)+1=v_f''(1)$	$e_f''(0)+1=e_f''(1)$
0	$1(\gamma_2)$	$3(\theta_1,\theta_1,\theta_1)$	$1(\phi_1)$	$1(\xi_1)$	$v_f''(0)=v_f''(1)$	$e_f''(0)=e_f''(1)+1$
0	$1(\gamma_1)$	$3(\theta_1,\theta_1,\theta_1)$	$2(\phi_1,\varphi_1)$	0	$v_f''(0)+1=v_f''(1)$	$e_f''(0)=e_f''(1)$
0	$1(\gamma_3)$	$3(\theta_1,\theta_1,\theta_2)$	$2(\phi_1,\phi_1)$	$1(\xi_2)$	$v_f''(0)=v_f''(1)$	$e_f''(0)=e_f''(1)$
0	$1(\gamma_4)$	$3(\theta_1,\theta_1,\theta_1)$	$3(\phi_1,\varphi_1,\phi_1)$	0	$v_f''(0)+1=v_f''(1)$	$e_f''(0)=e_f''(1)+1$
0	$1(\gamma_3)$	$3(\theta_1,\theta_1,\theta_1)$	$3(\phi_1,\phi_1,\phi_1)$	$1(\xi_2)$	$v_f''(0)=v_f''(1)$	$e_f''(0)=e_f''(1)+1$
$1(\beta_1)$	0	0	0	0	$v_f''(0)+1=v_f''(1)$	$e_f''(0)+1=e_f''(1)$
$1(\beta_2)$	0	0	0	$1(\xi_1)$	$v_f''(0)=v_f''(1)$	$e_f''(0)=e_f''(1)$
$1(\beta_1)$	0	0	$1(\phi_1)$	0	$v_f''(0)+1=v_f''(1)$	$e_f''(0)=e_f''(1)$
$1(\beta_3)$	0	0	$1(\phi_1)$	$1(\xi_2)$	$v_f''(0)=v_f''(1)$	$e_f''(0)=e_f''(1)$
$1(\beta_1)$	0	0	$2(\phi_1,\phi_1)$	0	$v_f''(0)+1=v_f''(1)$	$e_f''(0)=e_f''(1)+1$
$1(\beta_2)$	0	0	$2(\phi_1,\phi_1)$	$1(\xi_2)$	$v_f''(0)=v_f''(1)$	$e_f''(0)+1=e_f''(1)$
$1(\beta_4)$	0	0	$3(\phi_1,\phi_1,\phi_2)$	0	$v_f''(0)+1=v_f''(1)$	$e_f''(0)=e_f''(1)$
$1(beta_2)$	0	0	$3(\phi_1,\phi_1,\phi_1)$	$1(\xi_2)$	$v_f''(0)=v_f''(1)$	$e_f''(0)=e_f''(1)$
$1(\beta_4)$	0	$1(\theta_1)$	0	0	$v_f''(0)+1=v_f''(1)$	$e_f''(0)=e_f''(1)$
$1(\beta_2)$	0	$1(\theta_1)$	0	$1(\xi_1)$	$v_f''(0)=v_f''(1)$	$e_f''(0)=e_f''(1)$
$1(\beta_1)$	0	$1(\theta_1)$	$1(\phi_1)$	0	$v_f''(0)+1=v_f''(1)$	$e_f''(0)+1=e_f''(1)$

r_2	r_3	r_5	r_6	r_8	Vertex conditon	Edge Condition
$1(\bar{\bar{\beta}}_2)$	0	$1(\hat{\theta}_1)$	$1(\hat{\varphi}_1)$	$1(\xi_1)$	$v''_f(0) = v''_f(1)$	$e''_f(0) = e''_f(1) + 1$
$1(\beta_1)$	0	$1(\dot{\bar{\theta}}_1)$	$2(\bar{\bar{\phi}}_1, \dot{\varphi}_1)$	0	$v''_f(0) + 1 = v''_f(1)$	$e''_f(0) = e''_f(1)$
$1(\beta_3)$	0	$1(\dot{\bar{\theta}}_1)$	$2(\bar{\bar{\phi}}_1, \dot{\varphi}_1)$	$1(\xi_2)$	$v''_f(0) = v''_f(1)$	$e''_f(0) = e''_f(1)$
$1(\beta_4)$	0	$1(\dot{\bar{\theta}}_1)$	$3(\bar{\bar{\phi}}_1, \dot{\varphi}_1, \dot{\varphi}_1)$	0	$v''_f(0) + 1 = v''_f(1)$	$e''_f(0) = e''_f(1) + 1$
$1(\bar{\bar{\beta}}_2)$	0	$1(\dot{\bar{\theta}}_1)$	$3(\bar{\bar{\phi}}_1, \dot{\varphi}_1, \dot{\varphi}_1)$	$1(\xi_2)$	$v''_f(0) = v''_f(1)$	$e''_f(0) = e''_f(1) + 1$
$1(\beta_4)$	0	$2(\dot{\bar{\theta}}_1, \dot{\theta}_1)$	0	0	$v''_f(0) + 1 = v''_f(1)$	$e''_f(0) + 1 = e''_f(1)*$
$1(\beta_3)$	0	$2(\dot{\bar{\theta}}_1, \dot{\theta}_1)$	0	$1(\xi_1)$	$v''_f(0) = v''_f(1)$	$e''_f(0) + 1 = e''_f(1)$
$1(\beta_4)$	0	$2(\dot{\theta}_1, \bar{\bar{\theta}}_1)$	$1(\dot{\varphi}_1)$	0	$v''_f(0) + 1 = v''_f(1)$	$e''_f(0) = e''_f(1)$
$1(\bar{\bar{\beta}}_2)$	0	$2(\dot{\bar{\theta}}_1, \dot{\bar{\theta}}_1)$	$1(\dot{\varphi}_1)$	$1(\xi_1)$	$v''_f(0) = v''_f(1)$	$e''_f(0) = e''_f(1)$
$1(\beta_1)$	0	$2(\dot{\bar{\theta}}_1, \bar{\bar{\theta}}_1)$	$2(\bar{\bar{\phi}}_1, \dot{\phi}_1)$	0	$v''_f(0) + 1 = v''_f(1)$	$e''_f(0) + 1 = e''_f(1)$
$1(\bar{\bar{\beta}}_2)$	0	$2(\hat{\theta}_1, \hat{\theta}_1)$	$2(\bar{\bar{\phi}}_1, \hat{\phi}_1)$	$1(\xi_1)$	$v''_f(0) = v''_f(1)$	$v''_f(0) = v''_f(1) + 1$
$1(\beta_1)$	0	$2(\dot{\bar{\theta}}_1, \bar{\bar{\theta}}_1)$	$3(\bar{\bar{\phi}}_1, \dot{\phi}_1, \dot{\phi}_1)$	0	$v''_f(0) + 1 = v''_f(1)$	$e''_f(0) = e''_f(1)$
$1(beta_3)$	0	$2(\dot{\bar{\theta}}_1, \bar{\bar{\theta}}_1)$	$3(\bar{\bar{\phi}}_1, \dot{\phi}_1, \dot{\varphi}_2)$	$1(\xi_1)$	$v''_f(0) = v''_f(1)$	$e''_f(0) + 1 = e''_f(1)$
$1(\bar{\bar{\beta}}_2)$	0	$3(\dot{\bar{\theta}}_1, \dot{\bar{\theta}}_1, \dot{\bar{\theta}}_2)$	0	0	$v''_f(0) + 1 = v''_f(1)$	$e''_f(0) = e''_f(1)$
$1(\beta_3)$	0	$3(\dot{\bar{\theta}}_1, \dot{\bar{\theta}}_1, \dot{\theta}_1)$	0	$1(\xi_1)$	$v''_f(0) = v''_f(1)$	$e''_f(0) = e''_f(1)$
$1(\beta_4)$	0	$3(\dot{\theta}_1, \dot{\bar{\theta}}_1, \dot{\bar{\theta}}_1)$	$1(\dot{\varphi}_1)$	0	$v''_f(0) + 1 = v''_f(1)$	$e''_f(0) + 1 = e''_f(1)$
$1(\beta_2)$	0	$3(\dot{\theta}_1, \dot{\theta}_1, \dot{\theta}_1)$	$1(\dot{\varphi}_1)$	$1(\xi_1)$	$v''_f(0) = v''_f(1)$	$e''_f(0) = e''_f(1) + 1$
$1(\beta_4)$	0	$3(\dot{\bar{\theta}}_1, \dot{\bar{\theta}}_1, \dot{\bar{\theta}}_1)$	$2(\bar{\bar{\phi}}_1, \dot{\varphi}_1)$	0	$v''_f(0) + 1 = v''_f(1)$	$e''_f(0) = e''_f(1)$
$1(\beta_3)$	0	$3(\dot{\bar{\theta}}_1, \dot{\bar{\theta}}_1, \dot{\bar{\theta}}_1)$	$2(\bar{\bar{\phi}}_1, \dot{\varphi}_1)$	$1(\xi_2)$	$v''_f(0) = v''_f(1)$	$e''_f(0) = e''_f(1)$
$1(\beta_1)$	0	$3(\dot{\bar{\theta}}_1, \dot{\bar{\theta}}_1, \dot{\bar{\theta}}_1)$	$3(\bar{\bar{\phi}}_1, \dot{\phi}_1, \dot{\phi}_1)$	0	$v''_f(0) + 1 = v''_f(1)$	$e''_f(0) + 1 = e''_f(1)$
$1(\beta_1)$	0	$3(\dot{\bar{\theta}}_1, \dot{\bar{\theta}}_1, \dot{\bar{\theta}}_1)$	$3(\bar{\bar{\phi}}_1, \dot{\phi}_1, \dot{\phi}_1)$	$1(\xi_1)$	$v''_f(0) = v''_f(1)$	$e''_f(0) = e''_f(1) + 1$
$1(\bar{\bar{\beta}}_2)$	$1(\gamma_1)$	0	0	0	$v''_f(0) = v''_f(1)$	$e''_f(0) = e''_f(1) + 1$
$1(\beta_1)$	$1(\gamma_4)$	0	0	$1(\xi_1)$	$v''_f(0) + 1 = v''_f(1)$	$e''_f(0) = e''_f(1)$
$1(\beta_1)$	$1(\bar{\bar{\gamma}}_2)$	0	$1(\dot{\phi}_1)$	0	$v''_f(0) = v''_f(1)$	$e''_f(0) = e''_f(1) + 1$
$1(\beta_1)$	$1(\bar{\bar{\gamma}}_2)$	0	$1(\dot{\varphi}_1)$	$1(\xi_2)$	$v''_f(0) + 1 = v''_f(1)$	$e''_f(0) + 1 = e''_f(1)$
$1(\beta_1)$	$1(\gamma_3)$	0	$2(\bar{\bar{\phi}}_1, \dot{\varphi}_1)$	0	$v''_f(0) = v''_f(1)$	$e''_f(0) = e''_f(1)$
$1(\beta_1)$	$1(\bar{\bar{\gamma}}_2)$	0	$2(\bar{\bar{\phi}}_1, \dot{\varphi}_1)$	$1(\xi_2)$	$v''_f(0) + 1 = v''_f(1)$	$e''_f(0) = e''_f(1)$

r_2	r_3	r_5	r_6	r_8	Vertex conditon	Edge Condition
$1(\beta_1)$	$1(\bar{\hat{\gamma}}_2)$	0	$3(\hat{\phi}_1,\hat{\phi}_1,\hat{\phi}_2)$	0	$v''_f(0) = v''_f(1)$	$e''_f(0)+1 = e''_f(1)$
$1(\beta_1)$	$1(\bar{\hat{\gamma}}_2)$	0	$3(\bar{\hat{\phi}}_1,\bar{\hat{\phi}}_1,\varphi_1)$	$1(\xi_2)$	$v''_f(0)+1 = v''_f(1)$	$e''_f(0) = e''_f(1)+1$
$1(\beta_1)$	$1(\bar{\hat{\gamma}}_2)$	$1(\bar{\hat{\theta}}_1)$	0	0	$v''_f(0) = v''_f(1)$	$e''_f(0)+1 = e''_f(1)$
$1(\beta_1)$	$1(\bar{\hat{\gamma}}_2)$	$1(\bar{\hat{\theta}}_1)$	0	$1(\xi_1)$	$v''_f(0)+1 = v''_f(1)$	$e''_f(0) = e''_f(1)+1$
$1(\beta_1)$	$1(\bar{\hat{\gamma}}_2)$	$1(\bar{\hat{\theta}}_1)$	$1(\bar{\hat{\phi}}_1)$	0	$v''_f(0) = v''_f(1)$	$e''_f(0) = e''_f(1)$
$1(\beta_1)$	$1(\gamma_3)$	$1(\bar{\hat{\theta}}_1)$	$1(\bar{\hat{\phi}}_1)$	$1(\xi_1)$	$v''_f(0)+1 = v''_f(1)$	$e''_f(0) = e''_f(1)$
$1(\beta_1)$	$1(\bar{\hat{\gamma}}_2)$	$1(\bar{\hat{\theta}}_1)$	$2(\bar{\hat{\phi}}_1,\dot{\varphi}_1)$	0	$v''_f(0) = v''_f(1)$	$e''_f(0) = e''_f(1)+1$
$1(\beta_1)$	$1(\hat{\gamma}_2)$	$1(\bar{\hat{\theta}}_1)$	$2(\bar{\hat{\phi}}_1,\bar{\hat{\varphi}}_1)$	$1(\xi_2)$	$v''_f(0)+1 = v''_f(1)$	$e''_f(0)+1 = e''_f(1)$
$1(\beta_1)$	$1(\gamma_3)$	$1(\bar{\hat{\theta}}_1)$	$3(\bar{\hat{\phi}}_1,\bar{\hat{\phi}}_1,\bar{\hat{\varphi}}_1)$	0	$v''_f(0) = v''_f(1)$	$e''_f(0) = e''_f(1)$
$1(\beta_1)$	$1(\bar{\hat{\gamma}}_2)$	$1(\bar{\hat{\theta}}_1)$	$3(\bar{\hat{\phi}}_1,\bar{\hat{\phi}}_1,\bar{\hat{\varphi}}_1)$	$1(\xi_2)$	$v''_f(0)+1 = v''_f(1)$	$e''_f(0) = e''_f(1)+1$
$1(\beta_3)$	$1(\gamma_1)$	$2(\bar{\hat{\theta}}_1,\bar{\hat{\theta}}_1)$	0	0	$v''_f(0) = v''_f(1)$	$e''_f(0) = e''_f(1)$
$1(\beta_1)$	$1(\bar{\hat{\gamma}}_2)$	$2(\bar{\hat{\theta}}_1,\bar{\hat{\theta}}_1)$	0	$1(\xi_1)$	$v''_f(0)+1 = v''_f(1)$	$e''_f(0) = e''_f(1)$
$1(\beta_1)$	$1(\bar{\hat{\gamma}}_2)$	$2(\bar{\hat{\theta}}_1,\bar{\hat{\theta}}_1)$	$1(\bar{\hat{\phi}}_1)$	0	$v''_f(0) = v''_f(1)$	$e''_f(0)+1 = e''_f(1)$
$1(\beta_1)$	$1(\bar{\hat{\gamma}}_2)$	$2(\bar{\hat{\theta}}_1,\bar{\hat{\theta}}_1)$	$1(\bar{\hat{\phi}}_1)$	$1(\xi)$	$v''_f(0)+1 = v''_f(1)$	$e''_f(0) = e''_f(1)+1$
$1(\beta_1)$	$1(\bar{\hat{\gamma}}_2)$	$2(\bar{\hat{\theta}}_1,\bar{\hat{\theta}}_1)$	$2(\bar{\hat{\phi}}_1,\dot{\varphi}_1)$	0	$v''_f(0) = v''_f(1)$	$e''_f(0) = e''_f(1)+1$
$1(\beta_1)$	$1(\gamma_3)$	$2(\bar{\hat{\theta}}_1,\bar{\hat{\theta}}_1)$	$2(\bar{\hat{\phi}}_1,\hat{\varphi}_1)$	$1(\xi_1)$	$v''_f(0)+1 = v''_f(1)$	$e''_f(0) = e''_f(1)$
$1(\beta_1)$	$1(\bar{\hat{\gamma}}_2)$	$2(\bar{\hat{\theta}}_1,\bar{\hat{\theta}}_1)$	$3(\bar{\hat{\phi}}_1,\bar{\hat{\phi}}_1.\dot{\varphi}_1)$	0	$v''_f(0) = v''_f(1)$	$e''_f(0) = e''_f(1)+1$
$1(\beta_1)$	$1(\bar{\hat{\gamma}}_2)$	$2(\bar{\hat{\theta}}_1,\bar{\hat{\theta}}_1)$	$3(\bar{\hat{\phi}}_1,\bar{\hat{\phi}}_1,\varphi_1)$	$1(\xi_2)$	$v''_f(0)+1 = v''_f(1)$	$e''_f(0)+1 = e''_f(1)$
$1(\beta_1)$	$1(\bar{\hat{\gamma}}_2)$	$3(\bar{\hat{\theta}}_1,\bar{\hat{\theta}}_1,\bar{\hat{\theta}}_2)$	0	0	$v''_f(0) = v''_f(1)$	$e''_f(0) = e''_f(1)+1$
$1(\beta_1)$	$1(\bar{\hat{\gamma}}_2)$	$3(\bar{\hat{\theta}}_1,\bar{\hat{\theta}}_1,\bar{\hat{\theta}}_1)$	0	$1(\xi_1)$	$v''_f(0)+1 = v''_f(1)$	$e''_f(0)+1 = e''_f(1)$
$1(\beta_4)$	$1(\bar{\hat{\gamma}}_2)$	$3(\bar{\hat{\theta}}_1,\bar{\hat{\theta}}_1,\bar{\hat{\theta}}_1)$	$1(\bar{\hat{\phi}}_1)$	0	$v''_f(0) = v''_f(1)$	$e''_f(0) = e''_f(1)$
$1(\beta_1)$	$1(\bar{\hat{\gamma}}_2)$	$3(\bar{\hat{\theta}}_1,\bar{\hat{\theta}}_1,\bar{\hat{\theta}}_1)$	$1(\bar{\hat{\phi}}_1)$	$1(\xi_1)$	$v''_f(0)+1 = v''_f(1)$	$e''_f(0) = e''_f(1)$
$1(\beta_1)$	$1(\bar{\hat{\gamma}}_2)$	$3(\bar{\hat{\theta}}_1,\bar{\hat{\theta}}_1,\bar{\hat{\theta}}_1)$	$2(\bar{\hat{\phi}}_1,\bar{\hat{\phi}}_1)$	0	$v''_f(0) = v''_f(1)$	$e''_f(0)+1 = e''_f(1)$
$1(\beta_1)$	$1(\bar{\hat{\gamma}}_2)$	$3(\bar{\hat{\theta}}_1,\bar{\hat{\theta}}_1,\bar{\hat{\theta}}_2)$	$2(\bar{\hat{\phi}}_1,\bar{\varphi}_1$	$1(\xi_1)$	$v''_f(0)+1 = v''_f(1)$	$e''_f(0) = e''_f(1)+1$
$1(\beta_1)$	$1(\bar{\hat{\gamma}}_2)$	$3(\bar{\hat{\theta}}_1.\bar{\hat{\theta}}_1,\bar{\hat{\theta}}_1)$	$3(\bar{\hat{\phi}}_1,\bar{\hat{\phi}}_1.\bar{\hat{\varphi}}_1)$	0	$v''_f(0) = v''_f(1)$	$e''_f(0) = e''_f(1)$
$1(\beta_1)$	$1(\gamma_3)$	$3(\bar{\hat{\theta}}_1,\bar{\hat{\theta}}_1,\bar{\hat{\theta}}_1)$	$3(\bar{\hat{\phi}}_1,\bar{\hat{\phi}}_1.\varphi_1)$	$1(\xi_1)$	$v''_f(0)+1 = v''_f(1)$	$e''_f(0) = e''_f(1)$

Excepting in the cases marked with an asterix(*), the above table shows that $v'_f(0)+v''_f(0) = v'_f(1)+v''_f(1)$ or $v'_f(0)+v''_f(0) = v'_f(1)+v''_f(1)+1$. If there remains a block of Type A_1, and if $v'_f(0)+v''_f(0) = v'_f(1)+v''_f(1)$, then choosing the labeling $\bar{\alpha}_2$ for this block, we will obtain a labeling f for which $v_f(0) = v_f(1)+1$. If there remains a block of Type A_1 and if $v'_f(0)+v''_f(0) = v'_f(1)+v''_f(1)+1$ then choosing the labeling α_1 for this block, we obtain $v_f(0) = v_f(1)$. Hence, in all cases except possibly the starred ones, G is cordial.

We now take a look at the starred cases. For easy reference, we list them in the table below. Note that we also give along with each case, two alternate labeling functions f one in which $e_f(0) = e_f(1) + 2$ and in the other $e_f(0) + 2 = e_f(1)$. In both, $| v_f(0) - v_f(1) | \leq 1$.

r_2	r_3	r_5	r_6	r_8	Vertex conditon	Edge Condition
0	0	0	0	$1(\xi_1)$	$v''_f(0) + 1 = v''_f(1)$	$e''_f(0) = e''_f(1) + 2$
				$1\xi_2$	$v''_f(0) + 1 = v''_f(1)$	$e''_f(0) + 2 = e''_f(1)$
0	0		$2(\dot{\bar{\phi}}_1, \dot{\bar{\phi}}_1)$	0	$v''_f(0) = v''_f(1)$	$e''_f(0) = e''_f(1) + 2$
			$2(\dot{\bar{\phi}}_1, \dot{\bar{\phi}}_1)$	0	$v''_f(0) = v''_f(1)$	$e''_f(0) + 2 = e''_f(1)$
0	0	$1(\dot{\bar{\theta}}_1)$	$1(\dot{\bar{\phi}}_1)$	$1(\xi_1)$	$v''_f(0) + 1 = v''_f(1)$	$e''_f(0) = e''_f(1) + 2$
				$1(\xi_2)$	$v''_f(0) + 1 = v''_f(1)$	$e''_f(0) + 2 = e''_f(1)$
0	0	$1(\dot{\bar{\theta}}_1)$	$3(\dot{\bar{\phi}}_1, \dot{\bar{\phi}}_1, \dot{\bar{\phi}}_1)$	0	$v''_f(0) = v''_f(1)$	$e''_f(0) = e''_f(1) + 2$
			$3(\dot{\bar{\phi}}_1; \dot{\bar{\phi}}_1, \dot{\bar{\phi}}_2)$		$v''_f(0) = v''_f(1)$	$e''_f(0) + 2 = e''_f(1)$
0	0	$2(\dot{\bar{\theta}}_1, \dot{\bar{\theta}}_1)$	0	0	$v''_f(0) = v''_f(1)$	$e''_f(0) + 2 = e''_f(1)$
		$2(\dot{\bar{\theta}}_1, \dot{\bar{\theta}}_2)$			$v''_f(0) = v''_f(1)$	$e''_f(0) = e''_f(1) + 2$
0	0	$2(\dot{\bar{\theta}}_1, \dot{\bar{\theta}}_1)$	$2(\dot{\bar{\phi}}_1; \dot{\bar{\phi}}_1)$	$1(\xi_1)$	$v''_f(0) = v''_f(1)$	$e''_f(0) = e''_f(1) + 2$
				$1(\xi_2)$	$v''_f(0) = v''_f(1)$	$e''_f(0) + 2 = e''_f(1)$
0	0	$3(\dot{\bar{\theta}}_1, \dot{\bar{\theta}}_1, \dot{\bar{\theta}}_1)$	$1(\dot{\bar{\phi}}_1)$	0	$v''_f(0) = v''_f(1)$	$e''_f(0) + 2 = e''_f(1)$
		$3(\dot{\bar{\theta}}_1, \dot{\bar{\theta}}_1, \dot{\bar{\theta}}_1)$			$v''_f(0) = v''_f(1)$	$e''_f(0) = e''_f(1) + 2$
0	0	$3(\dot{\bar{\theta}}_1, \dot{\bar{\theta}}_1, \dot{\bar{\theta}}_1)$	$3((\dot{\bar{\phi}}_1, \dot{\bar{\phi}}_1, \dot{\bar{\phi}}_1)$	$1(\xi_1)$	$v''_f(0) + 1 = v''_f(1)$	$e''_f(0) = e''_f(1) + 2$
				$1(\xi_2)$	$v''_f(0) + 1 = v''_f(1)$	$e''_f(0) + 2 = e''_f(1)$

We now take a look at the starred cases. For easy reference, we list them in the table above. Note that we also give along with each case, two alternate labeling functions f one in which $e_f(0) = e_f(1) + 2$ and in the other $e_f(0) + 2 = e_f(1)$. In both, $| v_f(0) - v_f(1) | \leq 1$.

We note that the number of edges in a graph of

(i)Type A_5 is congruent to 3(mod4),

(ii)Type A_6 is congruent to 1(mod4),

(iii)Type C is congruent to 2(mod4).

Thus in the eight cases given in the table, $| E(G) | \equiv 2(\text{mod}4)$. Further, graphs of Types A_1, A_5, A_6, C are Eulerian whereas, those of Types A_2, A_3, A_4, B are non Eulerian.Hence, if $n_2 = 0, n_3 = 0, n_4 = 0, n_7 = 0$ then in the one point union G, there are no graphs of Type A_2, A_3, A_4, B, hence G is Eulerian. Therefore,in the above cases, if $n_2 = 0, n_3 = 0, n_4 = 0, n_7 = 0$ then G is Eulerian with $| E(G) | \equiv 2(\text{ mod } 4)$. By Theorem 1.2 , G cannot be cordial. However as indicated by the table, we will have proved that the index of cordiality of G, in each case, is 2.

Next suppose that at least one of n_2, n_3, n_4, n_7 is non-zero. Then G is non-Eulerian. In this case we disturb the labeling in Stage 1 so as to obtain a cordial labeling of G.

Firstly, suppose that $n_2 \neq 0$. Then $n_2 = 2p_2$ that is there are at least two graphs of Type A_2. In Stage 1, we had used the labeling β_1 for p_2 of these and the labeling β_3 for the remaining p_2. Now, in the altered labeling, we use β_1 for p_2 of these; the labeling β_3 for $(p_2 - 1)$ of these and the labeling $\bar{\bar{\beta}}_2$ for the remaining one of this type of ply. Then at Stage 1, we obtain $e'_f(0) + 2 = e'_f(1)$. Now, in Stage 2, we choose the labeling from the table in which $e''_f(0) = e''_f(1) + 2$. This will ensure that $e_f(0) = e_f(1)$.

In case, we have $n_2 = 0, n_3 \neq 0$. Then $n_3 = 2p_3$ that is there are at least two graphs of Type A_3. In Stage 1, we had used the labeling γ_1 for p_3 of these and the labeling γ_3 for the remaining p_3. Now, in the altered labeling, we use γ_1 for p_3 of these; the labeling γ_3 for $(p_3 - 1)$ of these and the labeling $\bar{\bar{\gamma}}_2$ for the remaining one of this type of ply.Then at Stage 1, we obtain $e'_f(0) = e'_f(1) + 2$. Now, in Stage 2, we choose the labeling from the table in which $e''_f(0) + 2 = e''_f(1)$.

Next suppose that $n_2 = 0, n_3 = 0, n_4 \neq 0$. Then in one of the graphs of Type A_4, instead of using the labeling $delta_1$, at Stage 1, use the labeling δ_2. Then at Stage 1, we obtain $e'_f(0) + 2 = e'_f(1)$. Now, in Stage 2, we choose the labeling in which $e''_f(0) = e''_f(1) + 2$.

Finally, suppose that $n_2 = 0, n_3 = 0, n_4 = 0, n_7 \neq 0$. Then in one of the graphs of Type A_4, instead of using the labeling μ_1 at Stage 1, use the labeling μ_3. Then at Stage 1, we obtain $e'_f(0) = e'_f(1) + 2$. Now, in Stage 2, we choose the labeling from the table in which $e''_f(0) + 2 = e''_f(1)$.

In this case too, if one ply of Type A_1 remains, then the appropriate labeling can be chosen for it as explained earlier. Hence in all cases except when G is Eulerian with $|E(G)| \equiv 2(\bmod 4)$, we have obtained a cordial labeling for G. This finishes the proof. □

One Point Union of Wheel Related Graphs.

It is known that a large class of wheel related graphs are cordial [1]. The one point unions of wheels, fans and flags of same size have been proved to be cordial by Shee and Ho[7]. We now consider the one point union of the following graphs.

(i) Helms (ii) Closed Helms (iii) Flower graphs (iv) Gear graphs (v) Sunflower graphs

A *wheel* W_n is the Cartesian product $K_1 \times C_n$. Thus if $V(C_n) = \{v_1, \cdots, v_n\}$ then $V(W_n) = \{u, v_1, \cdots, v_n\}$ where $v_1, \cdots, v_n$ are the vertices of C_n and u is called the central vertex. A *Helm* H_n is obtained from the wheel by attaching a pendant vertex to each of the vertices on C_n in W_n. The vertex set of H_n is then $V = \{u, v_1, \cdots, v_n, w_1, \cdots, w_n\}$ where $w_1, \cdots, w_n$ are the pendant vertices. *closed helm* CH_n is obtained by taking a helm H_n and by adding edges $\{w_n w_1, w_i w_{i+1} \mid 1 \leq i \leq n-1\}$. *flower* FL_n is a graph obtained from the helm H_n by attaching each of its pendant vertices to its central vertex to the edge set $E(H_n)$. A *gear* graph G_n is obtained from a wheel W_n by inserting a vertex on each of the cyclic edges of C_n in W_n. A *sunflower graph* SF_n has vertex set and edge set as follows:

$V(SF_n) = \{u, v_i \mid 1 \le i \le n\} \bigcup \{w_i \mid 1 \le i \le n\}$,
$E(SF_n) = \{uv_i, v_iv_{i+1}, w_iv_i, w_iv_{i+1} \mid 1 \le i \le n\}$, where the value of $i+1$ is taken modulo n.

It was proved by Andar, Boxwala and Limaye [1], that the helm H_n, the closed helm CH_n, the flower graph FL_n, the gear graph G_n, the sunflower graph SF_n and various other families of wheel related graphs are cordial. We use the labelings given in [1].

Let g_1 be the binary labeling of a helm H_n given by $g_1(u) = 0$, and

$$g_1(v_i) = \begin{cases} 1, & i \equiv 1, 2 \bmod 4 \\ 0, & i \equiv 0, 3 \bmod 4 \end{cases}$$

$$g_1(w_i) = \begin{cases} 0, & i \equiv 1 \bmod 2 \\ 1, & i \equiv 0 \bmod 2 \end{cases}$$

Theorem: The one point union of helms is cordial.

Proof: We first give additional binary labelings of a helm H_n as follows:

(i)A binary labeling g_2 is defined as $g_2(u) = 0 = g_2(v_n)$
$g_2(v_i) = 1, i \equiv 1, 2(\bmod 4), i \ne n$, $g_2(v_i) = 0, i \equiv 0, 3(\bmod 4)$. $g_2(w_i) = 1, i \equiv 1(\bmod 2), i \ne n$, $g_2(w_i) = 0, i \equiv 0(\bmod 2)$, and $g_2(w_n) = n$.

(ii) A binary labeling g_3 is defined as $g_3(u) = 0 = g_3(v_{n-1}), g_3(w_{n-1}) = 1, g_3(w_n) = 0$,
$g_3(v_i) = 1, i \equiv 1, 2(\bmod 4), i \ne n-1, g_3(v_i) = 0, i \equiv 0, 3(\bmod 4), g_3(w_i) = 0, i \equiv 1(\bmod 2), i \ne n-1, g_3(w_i) = 1, i \equiv 0(\bmod 2), i \ne n$.

(iii) A binary labeling g_4 is defined as $g_4(u) = 0 = g_4(v_{n-2}), g_4(w_{n-2}) = 1$,
$g_4(v_i) = 1, i \equiv 1, 2(\text{mod}4), i \ne n-2, g_4(v_i) = 0, i \equiv 0, 3(\text{mod}4), g_4(w_i) = 0, i \equiv 1(\text{mod}2), i \ne n-2, g_4(w_i) = 1, i \equiv 0(\text{mod}2)$.

If $n = 4q + r, 0 \le 3,$. depending on the value of r, we use the labeling as described in the table below.

r	Labeling	Vertex condition	Edge condition
0	g_1	$v_{g_1}(0) = v_{g_1}(1) + 1$	$e_{g_1}(0) = e_{g_1}(1)$
1	g_1	$v_{g_1}(0) = v_{g_1}(1) + 1$	$e_{g_1}(0) + 1 = e_{g_1}(1)$
	g_2	$v_{g_2}(0) = v_{g_2}(1) + 1$	$e_{g_2}(0) = e_{g_2}(1) + 1$
2	g_3	$v_{g_3}(0) = v_{g_3}(1) + 1$	$e_{g_3}(0) = e_{g_3}(1)$
3	g_1	$v_{g_1}(0) = v_{g_1}(1) + 1$	$e_{g_1}(0) + 1 = e_{g_1}(1)$
	g_4	$v_{g_4}(0) = v_{g_4}(1) + 1$	$e_{g_4}(0) = e_{g_1}(1) + 1$

Let $H(n_1, \cdots, n_k)$ denote a one point union of k helms $H_{n_i}, 1 \leq i \leq k$. Let p_1 be the number of helms in the one point union for which $n_i \equiv 1(\text{mod}4)$ and p_2 the number of helms for which $n_i \equiv 3(\text{mod}4)$. Then $p_1 = 2q_1 + s_1; p_2 = 2q_2 + s_2; 0 \leq s_1, s_2 \leq 1$. In each helm H_{n_i} with $n_i \equiv 0(\text{mod}4)$, use the labeling g_1 and in each helm with $n_i \equiv 2(\text{mod}4)$, use g_3. In q_1 of the helms with $n_i \equiv 1(\text{mod}4)$, use g_1 and in the q_1 of such helms use g_2. Of $2q_2$ helms with $n_i \equiv 3(\text{mod}4)$, for q_2 of them use g_1 and for the other q_2 use g_4.

There now remain s_1 helms H_{n_i} with $n_i \equiv 1(\text{mod}4)$ and s_2 helms H_{n_i} with $n_i \equiv 3(\text{mod}4)$. If $s_1 = 0, s_2 \neq 0$ or $s_1 \neq 0, s_2 = 0$ then for the remaining helm use g_1.
If $s_1 = 1, s_2 = 1$; then use g_1 for the helm for which $n_i \equiv 1(\text{mod}4)$ and use g_4 for the helm for which $n_i \equiv 3(\text{mod}4)$.

Let f be a binary labeling of $H(n_1, \cdots, n_k)$ whose restriction to each of the H_{n_i} is the labeling chosen above. It is clear that
(i)If $\underline{s_1 = 0, s_2 = 0 \text{ or } s_1 = 1, s_2 = 1}$, we get $v_f(0) = v_f(1) + 1, e_f(0) = e_f(1)$.
(ii)If $\underline{s_1 = 0, s_2 = 1 \text{ or } s_1 = 1, s_2 = 0}$, we get $v_f(0) = v_f(1) + 1, e_f(0) + 1 = e_f(1)$.
Hence f is a cordial labeling. □

Remark: The closed helm CH_n, the flower graph FL_n and the sunflower graph SF_n are all cordial [1]. The cordial labeling g given [1] satisfies $v_g(0) = v_g(1) + 1, e_g(0) = e_g(1)$. When we take one point unions of many copies of these graphs with arbitrary sizes and the central vertex as the common vertex, all we have to do is repeat the labelings given in [1]. That gives a cordial labelings of the corresponding one point unions. Hence we have following:

(1) The one point union $CH(n_1, \cdots, n_k)$ of closed helms of sizes $n_1, n_2, \cdots, n_k$ is cordial.

(2) The one point union $FL(n_1, \cdots, n_k)$ of flower graphs of sizes $n_1, n_2, \cdots, n_k$ is cordial.

(3) The one point union $SF(n_1, \cdots, n_k)$ of sunflower graphs of sizes $n_1, n_2, \cdots, n_k$ is cordial. □

We now consider the one point union of gear graphs, with the central vertex as the common vertex. Let G_n denote a gear graph with central vertex u,

$V(G_n) = \{u, v_1, v_2, \cdots, v_n; w_1, w_2, \cdots, w_n\}$,

$E(G_n) = \{uv_i, v_iw_i, w_iv_{i+1} | 1 \leq i \leq n-1\} \bigcup \{v_nw_n, w_nv_1\}$.

We call a gear graph G_n even or odd according as n is even or odd. Let g_1 be the cordial labeling given by: $g(u) = 0, g(v_i) = 1 = g(w_i), i \equiv 1(\text{mod}2)$ and $g(v_i) = g(w_i) = 0, i \equiv 0(\text{mod}2)$. In addition to this, we require the following binary labelings of G_n.

(i) The labeling g_2 is as follows: $g_2(u) = 0, g_2(v_i) = 1,\ \ i \equiv 1(\bmod 2), i \neq 1, g_2(v_i) = 0,\ \ i \equiv 0(\bmod 2), i = 1$

$g_2(w_i) = 1,\ \ i \equiv 0(\bmod 2), g_2(w_i) = 0,\ \ i \equiv 0(\text{mod}2)$.

(ii) The labeling g_3 is as follows: $g_3(u) = 0, g_3(v_i) = 1,\ \ i \equiv 1(\bmod 2), g_3(v_i) = 0,\ \ i \equiv 0(\bmod 2)$,

$g_3(w_i) = 1,\ \ i \equiv 0(\bmod 2), i \neq 1, g_3(w_1) = 0,\ \ i \equiv 0(\text{mod}2)$.

If n is even we note that, $v_{g_1}(0) = n+1, v_{g_1}(1) = n; e_{g_1}(0) = e_{g_1}(1) = \frac{3n}{2}$; that is $e_{g_1}(0) = e_{g_1}(1), v_{g_1}(0) = v_{g_1}(1) + 1$.

When n is odd, $v_{g_2}(0) = n+1, v_{g_2}(1) = n; e_{g_2}(0) = \frac{3n-1}{2}, e_{g_1}(1) = \frac{3n+1}{2}$; that is $e_{g_2}(0) + 1 = e_{g_2}(1), v_{g_2}(0) = v_{g_2}(1) + 1$.

When n is odd, $v_{g_3}(0) = n+1, v_{g_3}(1) = n; e_{g_3}(0) = \frac{3n+1}{2}, e_{g_1}(1) = \frac{3n-1}{2}$; that is $e_{g_3}(0) = e_{g_3}(1) + 1, v_{g_3}(0) = v_{g_3}(1) + 1$.

Theorem: The one point union of gear graphs is cordial

Proof: Let $G(n_1, n_2, \cdots, n_k)$ be the one point union of k gear graphs $G_{n_i}, 1 \leq i \leq k$. Amongst these k gear graphs, let t of them be even gear graphs. Then the remaining $k-t$ of them are odd gear graphs. Let f_i be a binary labeling assigned to each gear graph G_{n_i} as follows:
For n_i even, let $f_i(v) = g_1(v), v \in V(G_{n_i})$. In $\left\lfloor \frac{k-t}{2} \right\rfloor$ of the odd gear graphs, let $f_i(v) = g_2(v), v \in V(G_{n_i})$. In $\left\lceil \frac{k-t}{2} \right\rceil$ of the odd gear graphs, let $f_i(v) = g_3(v), v \in V(G_{n_i})$. Let f be a binary labeling of $G(n_1, n_2, \cdots, n_k)$ defined as
$f(v) = f_i(v), v \in V(G_{n_i}), 1 \leq i \leq k$. Evidently, $v_f(0) = v_f(1) + 1$.

In every gear graph, with even number of points, the edges are equitably labeled. Further in each of the $\left\lfloor \frac{k-t}{2} \right\rfloor$ of the odd gear graphs, to which the labeling g_2 has been assigned, the number of edges with the label 0 is one less than the number of edges with the label 1; while in each of the $\left\lceil \frac{k-t}{2} \right\rceil$ of the odd gear graphs to which the labeling g_3 has been assigned, the number of edges with the label 0 is one greater than the number of edges with the label 1. Hence if $k-t$ is even, then $e_f(0) = e_f(1)$ and if $k-t$ is odd, then $e_f(0) = e_f(1) + 1$. Hence $G(n_1, n_2, \cdots, n_k)$ is cordial. □

Path unions of shell related graphs

In 1993, Shee and Ho called a graph obtained by taking graphs $G_1, \cdots, G_n$ and adding an edge from one vertex of G_i to one vertex of $G_{i+1}, 1 \leq i \leq n-1$, a path union of $G_1, \cdots, G_n$. In the same paper they proved that the path union of k shells $k \geq 2$, of the same size is cordial.

Definition: While taking the path union of shells, we join the apex of S_{n_i} to the apex of $S_{n_{i+1}}, 1 \leq i \leq k-1$. This path union is denoted by $PS(n_1, n_2, \cdots, n_k)$. If the edge joining one vertex of S_{n_i} to a vertex of $S_{n_{i+1}}$ does not necessarily join the two apex points, then the resulting path union is denoted by $GPS(n_1, n_2, \cdots, n_k)$ and is called the *generalised path union*.

In a shell S_n, we have $| V(S_n) |= n$ and $| E(S_n) |= 2n-3$. It clearly follows that $| V(PS(n_1, n_2, \cdots, n_k)) |= (n_1 + n_2 + \cdots + n_k)$ and
$| E(PS(n_1, n_2, \cdots, n_k)) |= 2(n_1 + n_2 + \cdots + n_k) - (2k+1)$.

Let $g_1, g_2, g_3, g_4, g_5, g_6, g_7$ be the binary labelings of shell S_n defined as follows:

(1) $g_1(u) = 0, g_1(v_i) = 1$, if $i \equiv 1, 2(\text{mod } 4)$ and $g_1(v_i) = 0$, if $i \equiv 0, 3(\text{mod } 4)$.

(2) $g_2(u) = 0, g_2(v_i) = 0$, if $i \equiv 1, 2(\text{mod } 4)$ and $g_2(v_i) = 1$, if $i \equiv 0, 3(\text{mod } 4)$.

(3) $g_3(u) = 0, g_3(v_i) = 1$, if $i \equiv 1, 2(\text{mod } 4), i \neq n-3$, and $g_3(v_i) = 0$, if $i \equiv 0, 3(\text{ mod } 4), i \neq n-1$. Let $g_3(v_{n-3}) = 0$ and $g_3(v_{n-1}) = 1$.

(4) $g_4(u) = 0, g_4(v_i) = 1$, if $i \equiv 1, 2(\text{mod } 4), i \neq n-2$, and $g_4(v_i) = 0$, if $i \equiv 0, 3(\text{mod } 4)$. Let $g_4(v_{n-2}) = 0$.

(5) $g_5(u) = 0, g_5(v_i) = 1$, if $i \equiv 1, 2(\text{mod } 4), i \neq n-1$, and $g_5(v_i) = 0$, if $i \equiv 0, 3(\text{mod } 4)$. Let $g_5(v_{n-1}) = 0$.

(6) $g_6(u) = 0 = g_6(n_{n-1}), g_6(v_{n-3}) = 1, g_6(v_i) = 1,\ i \equiv 1, 2\ (\text{ mod } 4), i \neq n-1, g_6(v_i) = 0,\ i \equiv 0, 3\ (\text{ mod } 4), i \neq (n-3)$.

(7) $g_7(u) = 0, g_7(v_{n-2}) = 1, g_7(v_i) = 1,\ i \equiv 1, 2\ (\text{ mod } 4), g_7(v_i) = 0,\ i \equiv 0, 3\ (\text{ mod } 4), i \neq (n-2)$.

If $n = 4q+r, 0 \leq r \leq 3$, depending on the value of r, we use the following vertex and edge label conditions for the labelings mentioned above.

r	Labeling	Vertex condition	Edge condition
0	g_1	$v_{g_1}(0) = v_{g_1}(1)$	$e_{g_1}(0) + 1 = e_{g_1}(1)$
	g_3	$v_{g_3}(0) = v_{g_3}(1)$	$e_{g_3}(0) = e_{g_3}(1) + 1$
1	g_1	$v_{g_1}(0) = v_{g_1}(1) + 1$	$e_{g_1}(0) = e_{g_1}(1) + 1$
	g_3	$v_{g_3}(0) = v_{g_3}(1) + 1$	$e_{g_3}(0) + 1 = e_{g_3}(1)$
	g_7	$v_{g_7}(0) + 1 = v_{g_7}(1)$	$e_{g_7}(0) + 1 = e_{g_7}(1)$
2	g_1	$v_{g_1}(0) = v_{g_1}(1)$	$e_{g_1}(0) + 1 = e_{g_1}(1)$
	g_6	$v_{g_6}(0) = v_{g_6}(1)$	$e_{g_6}(0) = e_{g_6}(1) + 1$
3	g_1	$v_{g_1}(0) + 1 = v_{g_1}(1)$	$e_{g_1}(0) + 1 = e_{g_1}(1)$
	g_4	$v_{g_4}(0) = v_{g_4}(1) + 1$	$e_{g_4}(0) = e_{g_4}(1) + 1$
	g_5	$v_{g_5}(0) = v_{g_5}(1) + 1$	$e_{g_5}(0) + 1 = e_{g_5}(1)$

Theorem: The path union $PS(n_1, n_2, \cdots, n_k)$ is cordial for each finite sequence $n_1, n_2, \cdots, n_k$.

Proof: Consider the path union $PS(n_1, n_2, \cdots, n_k)$ of k shells $S_{n_1}, S_{n_2}, \cdots S_{n_k}$. Let f_i be the labeling chosen for S_{n_i} and let f be the labeling for $PS(n_1, n_2, \cdots, n_k)$ defined by $f(v) = f_i(v), v \in V(S_{n_i})$. We will choose f_i carefully. Let there be t shells of even width. Then the number of shells of odd width is $k - t$. In every shell of even width, let $f_i = g_1$. For the shells of odd width, do as follows: In the first $\left\lfloor \frac{k-t}{2} \right\rfloor$ such shells use

$$\begin{aligned} f_i &= g_7, \quad n_i \equiv 1(\text{mod}4) \\ &= g_1, \quad n_i \equiv 3(\text{mod}4). \end{aligned}$$

In the last $\left\lceil \frac{k-t}{2} \right\rceil$ such shells use

$$\begin{aligned} f_i &= g_1, \quad n_i \equiv 1(\text{mod}4) \\ &= g_5, \quad n_i \equiv 3(\text{mod}4). \end{aligned}$$

The way we have defined the labeling f, in each of the shells of even width, the vertices are equitably labeled. In each of the first $\left\lfloor \frac{k-t}{2} \right\rfloor$ shells of odd width, the number of vertices with the label 0 is one less than the number of vertices with the label 1. Finally,in each of the last $\left\lceil \frac{k-t}{2} \right\rceil$ shells of odd width, the number of vertices with the label 0 is one more than the number of vertices with the label 1.

Hence, $v_f(0) = v_f(1)$, if $k - t$ is even and $v_f(0) = v_f(1) + 1$, if $k - t$ is odd.

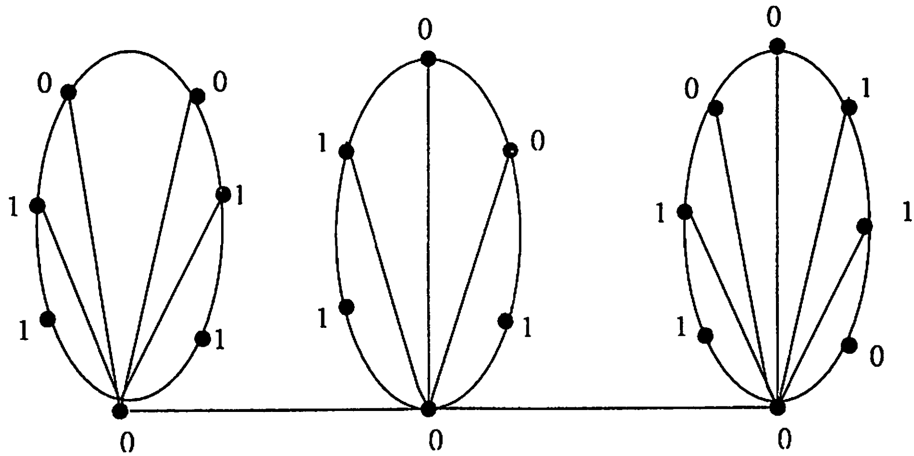
Further each of the $k - 1$ connecting edges receives the label 0 and in each of the k shells, the number of edges with the label 0 is one less than the number of edges with the label 1.

Hence, $e_f(0) + 1 = e_f(1)$.

The proof can also be given inductively as follows. In fact we prove that we can always obtain a labeling f for the above path union which will satisfy $v_f(0) - v_f(1) = 0$ or 1 and $e_f(0) + 1 = e_f(1)$.

For $k = 2$, consider the path union $PS(n_1, n_2)$ of two shells S_{n_1} and S_{n_2}. Let f be a binary labeling of $PS(n_1, n_2)$ defined as $f(v) = f_i(v), v \in$

$V(S_{n_i}), i = 1, 2$. Depending on the values of n_1, n_2 we choose the labelings f_1, f_2 for S_{n_1}, S_{n_2} respectively as shown in the table below:



labeling of $PS(7, 6, 8)$.

Let $n_1 = 4q_1 + r_1, n_2 = 4q_2 + r_2; 0 \leq r_1, r_2 \leq 3$.

$r_1(f_1)$	$r_2(f_2)$	Vertex condition for f	Edge condition for f
$0(g_1)$	$0(g_1)$	$v_f(0) = v_f(1)$	$e_f(0) + 1 = e_f(1)$
$0(g_1)$	$1(g_3)$	$v_f(0) = v_f(1) + 1$	$e_f(0) + 1 = e_f(1)$
$0(g_1)$	$2(g_1)$	$v_f(0) = v_f(1)$	$e_f(0) + 1 = e_f(1)$
$0(g_1)$	$3(g_5)$	$v_f(0) = v_f(1) + 1$	$e_f(0) + 1 = e_f(1)$
$1(g_3)$	$1(g_7)$	$v_f(0) = v_f(1)$	$e_f(0) + 1 = e_f(1)$
$1(g_3)$	$2(g_1)$	$v_f(0) = v_f(1) + 1$	$e_f(0) + 1 = e_f(1)$
$1(g_3)$	$3(g_1)$	$v_f(0) = v_f(1)$	$e_f(0) + 1 = e_f(1)$
$2(g_1)$	$2(g_1)$	$v_f(0) = v_f(1)$	$e_f(0) + 1 = e_f(1)$
$2(g_1)$	$3(g_5)$	$v_f(0) = v_f(1) + 1$	$e_f(0) + 1 = e_f(1)$
$3(g_1)$	$3(g_5)$	$v_f(0) = v_f(1)$	$e_f(0) + 1 = e_f(1)$

¿From the above table, it is evident that such a labeling can be given. Next assume that the result is true for some natural number m; that is for a path union of m shells, we have a labeling f for which $v_f(0) - v_f(1) = 0$ or 1 and $e_f(0) + 1 = e_f(1)$. Now consider a path union of $m + 1$ shells

$PS(n_1, \cdots, n_{m+1})$. Let f_1 be the cordial labeling of $PS(n_1, \cdots, n_m)$ and let f_2 be a labeling chosen for $S_{n_{m+1}}$ as explained below. Let f be the binary labeling of $PS(n_1, \cdots, n_{m+1})$ defined as
$f(v) = f_1(v), v \in V(PS(n_1, \cdots, n_m))$ and $f(v) = f_2(v), v \in V(S_{n_{m+1}})$. Let $n_{m+1} \equiv r_{m+1} (\text{mod} 4)$.

Case 1: $v_{f_1}(0) = v_{f_1}(1), e_{f_1}(0) + 1 = e_{f_1}(1)$.

$r_{m+1}(f_2)$	Vertex condition for f	Edge condition for f
$0(g_1)$	$v_f(0) = v_f(1)$	$e_f(0) + 1 = e_f(1)$
$1(g_3)$	$v_f(0) = v_f(1) + 1$	$e_f(0) + 1 = e_f(1)$
$2(g_1)$	$v_f(0) = v_f(1)$	$e_f(0) + 1 = e_f(1)$
$3(g_5)$	$v_f(0) = v_f(1) + 1$	$e_f(0) + 1 = e_f(1)$

Case 2: $v_{f_1}(0) = v_{f_1}(1) + 1, e_{f_1}(0) + 1 = e_{f_1}(1)$.

$r_{m+1}(f_2)$	Vertex condition for f	Edge condition for f
$0(g_1)$	$v_f(0) = v_f(1) + 1$	$e_f(0) + 1 = e_f(1)$
$1(g_7)$	$v_f(0) = v_f(1)$	$e_f(0) + 1 = e_f(1)$
$2(g_1)$	$v_f(0) = v_f(1) + 1$	$e_f(0) + 1 = e_f(1)$
$3(g_1)$	$v_f(0) = v_f(1)$	$e_f(0) + 1 = e_f(1)$

That f is a cordial labeling and that it satisfies the aforementioned conditions is evident from the above tables. □

We next show that while taking the path union one does not necessarily have to join the apexes to obtain cordiality.

Theorem: The generalized path union $GPS(n_1, n_2, \cdots, n_k)$, is cordial for each finite sequence $n_1, n_2, \cdots, n_k$.

Proof: The proof can be given inductively as follows.

For $k = 2$, consider the generalized path union $GPS(n_1, n_2)$ of two shells S_{n_1} and S_{n_2}. Let f be a binary labeling of $GPS(n_1, n_2)$ defined as $f(v) = f_i(v), v \in V(S_{n_i}), i = 1, 2$. Depending on the values of n_1, n_2 we choose the labelings f_1, f_2 for S_{n_1}, S_{n_2} respectively as shown in the table below:

Let $n_1 = 4q_1 + r_1, n_2 = 4q_2 + r_2; 0 \leq r_1, r_2 \leq 3$.

$r_1(f_1)$	$r_2(f_2)$	Vertex condition for f	Edge condition for f
$0(g_1)$	$0(g_3)$	$v_f(0) = v_f(1)$	$e_f(0) = e_f(1)$
$0(g_1)$	$1(g_3)$	$v_f(0) = v_f(1) + 1$	$e_f(0) = e_f(1)$
$0(g_1)$	$2(g_6)$	$v_f(0) = v_f(1)$	$e_f(0) = e_f(1)$
$0(g_1)$	$3(g_4)$	$v_f(0) = v_f(1) + 1$	$e_f(0) = e_f(1)$
$1(g_1)$	$1(g_7)$	$v_f(0) = v_f(1)$	$e_f(0) = e_f(1)$
$1(g_1)$	$2(g_1)$	$v_f(0) = v_f(1) + 1$	$e_f(0) = e_f(1)$
$1(g_1)$	$3(g_1)$	$v_f(0) = v_f(1)$	$e_f(0) = e_f(1)$
$2(g_1)$	$2(g_6)$	$v_f(0) = v_f(1)$	$e_f(0) = e_f(1)$
$2(g_1)$	$3(g_4)$	$v_f(0) = v_f(1) + 1$	$e_f(0) = e_f(1)$
$3(g_1)$	$3(g_4)$	$v_f(0) = v_f(1)$	$e_f(0) = e_f(1)$

We note that while writing down the edge label conditions for f, we have taken into account only the edges of S_{n_1} and S_{n_2}. However we have not considered the label of the edge connecting the two shells. This edge can either have the label 0 or the label 1 depending on the labels of its end vertices. ¿From the above table, it is evident then that either $e_f(0) = e_f(1)+1$ or $e_f(0) + 1 = e_f(1)$, thus guaranteeing cordiality of $GPS(n_1, n_2)$.

Next assume that the result is true for some natural number m; that is, that a generalized path union of m shells is cordial. Now consider a generalized path union of $m+1$ shells $GPS(n_1, \cdots, n_{m+1})$. Since $GPS(n_1, \cdots, n_m)$ is a generalized path union of m shells, by induction hypothesis, it is cordial. Let therefore f_1 be a cordial labeling of $GPS(n_1, \cdots, n_m)$ and let f_2 be a labeling chosen for $S_{n_{m+1}}$ as explained below. Let f be the binary labeling of $GPS(n_1, \cdots, n_{m+1})$ defined as

$f(v) = f_1(v), v \in V(GPS(n_1, \cdots, n_m))$ and $f(v) = f_2(v), v \in V(S_{n_{m+1}})$. Let $n_{m+1} \equiv r_{m+1}(\text{mod} 4)$. Without loss of generality, we can assume that $v_{f_1}(0) \geq v_{f_1}(1)$. Let w_m be the vertex of S_{n_m} connected to a vertex w_{m+1} in $S_{n_{m+1}}$. Let $e'_f(0), e'_f(1)$ be the number of edges with the labels 0 and 1 respectively in $GPS(n_1, \cdots, n_{m+1}) \backslash \{w_m w_{m+1}\}$. Then

Case 1: $v_{f_1}(0) = v_{f_1}(1), e_{f_1}(0) = e_{f_1}(1) + 1$.

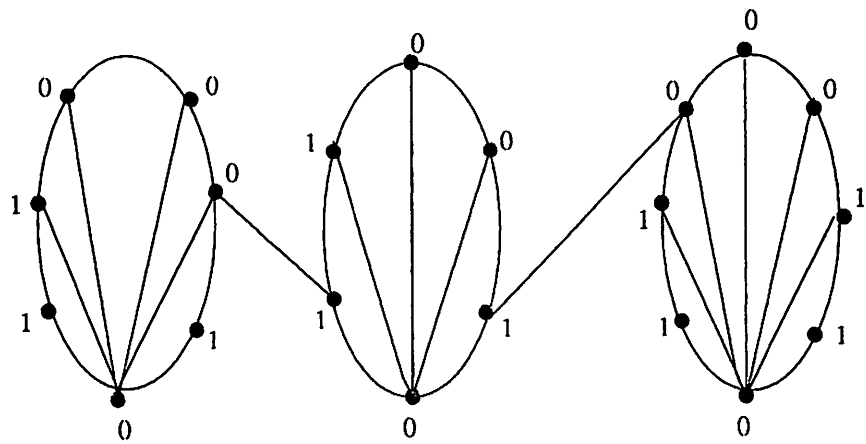
$r_{m+1}(f_2)$	Vertex condition for f	Edge condition for f
$0(g_1)$	$v_f(0) = v_f(1)$	$e'_f(0) = e'_f(1)$
$1(g_3)$	$v_f(0) = v_f(1) + 1$	$e'_f(0) = e'_f(1)$
$2(g_1)$	$v_f(0) = v_f(1)$	$e'_f(0) = e'_f(1)$
$3(g_1)$	$v_f(0) + 1 = v_f(1)$	$e'_f(0) = e'_f(1)$

Case 2: $v_{f_1}(0) = v_{f_1}(1) + 1, e_{f_1}(0) = e_{f_1}(1) + 1$.

$r_{m+1}(f_2)$	Vertex condition for f	Edge condition for f
$0(g_1)$	$v_f(0) = v_f(1) + 1$	$e'_f(0) = e'_f(1)$
$1(g_7)$	$v_f(0) = v_f(1)$	$e'_f(0) = e'_f(1)$
$2(g_1)$	$v_f(0) = v_f(1) + 1$	$e'_f(0) = e'_f(1)$
$3(g_1)$	$v_f(0) = v_f(1)$	$e'_f(0) = e'_f(1)$

Case 3: $v_{f_1}(0) = v_{f_1}(1), e_{f_1}(0) + 1 = e_{f_1}(1)$.

$r_{m+1}(f_2)$	Vertex condition for f	Edge condition for f
$0(g_3)$	$v_f(0) = v_f(1)$	$e'_f(0) = e'_f(1)$
$1(g_1)$	$v_f(0) = v_f(1) + 1$	$e'_f(0) = e'_f(1)$
$2(g_6)$	$v_f(0) = v_f(1)$	$e'_f(0) = e'_f(1)$
$3(g_4)$	$v_f(0) = v_f(1) + 1$	$e'_f(0) = e'_f(1)$



Labeling of $GPS(7,6,8)$

Case 4: $v_{f_1}(0) = v_{f_1}(1) + 1, e_{f_1}(0) + 1 = e_{f_1}(1)$.

$r_{m+1}(f_2)$	Vertex condition for f	Edge condition for f
$0(g_3)$	$v_f(0) = v_f(1) + 1$	$e'_f(0) = e'_f(1)$
$1(\hat{g}_1)$	$v_f(0) = v_f(1)$	$e'_f(0) = e'_f(1)$
$2(g_6)$	$v_f(0) = v_f(1) + 1$	$e'_f(0) = e'_f(1)$
$3(\hat{g}_4)$	$v_f(0) = v_f(1)$	$e'_f(0) = e'_f(1)$

In all cases observe that $e'_f(0) = e'_f(1)$. Hence $e_f(0) = e_f(1) + 1$ or $e_f(0) + 1 = e_f(1)$ according as the induced label for the edge $w_m w_{m+1}$ is 0 or 1. Thus f is a cordial labeling. □

Next we investigate the cordiality of a *Stemmed shell* which we now define:

Definition: A Stemmed shell ST_n is obtained by attaching a pendant vertex w to the apex u of a shell S_n of width n. The vertex w will be called the root of ST_n.

Observe that $|V(ST_n)| = n + 1; |E(ST_n)| = (2n - 2)$. A stemmed shell will be said to be of even (odd) width if n is even(odd).

In what follows, we prove that the path union of stemmed shells is cordial. We use the labelings $g_1, g_2, \cdots, g_7$, as defined earlier, in what follows. We first prove the following result:

Theorem: All stemmed shells are cordial.

Proof: Let ST_n be a stemmed shell with apex u and root w. Then $V(ST_n) = V(S_n)\bigcup\{w\}, E(ST_n) = E(S_n)\bigcup\{uw\}$. We define a cordial labeling of ST_n as follows:

Case 1: $n \equiv 0(\text{mod}4)$

Let $f_1(v) = g_1(v), v \in V(S_n), f_1(w) = 0$. Then since $v_{g_1}(0) = v_{g_1}(1), e_{g_1}(0)+1 = e_{g_1}(1)$, it follows that $v_{f_1}(0) = v_{f_1}(1) + 1, e_{f_1}(0) = e_{f_1}(1)$.

Case 2: $n \equiv 1(\text{mod}4)$

Let $f_2(v) = g_1(v), v \in V(S_n), f_1(w) = 1$. Then since $v_{g_1}(0) = v_{g_1}(1) + 1, e_{g_1}(0) = e_{g_1}(1) + 1$, it follows that $v_{f_2}(0) = v_{f_2}(1), e_{f_2}(0) = e_{f_2}(1)$.

Case 3: $n \equiv 2(\text{mod}4)$

Let $f_3(v) = g_1(v), v \in V(S_n), f_3(w) = 0$. Then since $v_{g_1}(0) = v_{g_1}(1), e_{g_1}(0)+1 = e_{g_1}(1)$, it follows that $v_{f_3}(0) = v_{f_3}(1) + 1, e_{f_3}(0) = e_{f_3}(1)$.

Case 4: $n \equiv 3(\text{mod}4)$

Let $f_4(v) = g_1(v), v \in V(S_n), f_4(w) = 0$. Then since $v_{g_1}(0)+1 = v_{g_1}(1), e_{g_1}(0)+1 = e_{g_1}(1)$, it follows that $v_{f_4}(0) = v_{f_4}(1), e_{f_4}(0) = e_{f_4}(1)$.

In each case ST_n is cordial. Hence the result. □

Remarks: (a) For $n \equiv 0(\text{mod}4)$, if we define $f_5(v) = g_2(v), v \in V(S_n), f_5(w) = 1$, then $v_{f_5}(0) = v_{f_5}(1) + 1, e_{f_5}(0) = e_{f_5}(1)$. Notice that $f_1(w) = 0, f_5(w) = 1$. Also the duals of f_1, f_5 that is $\hat{f}_1, \hat{f}_5$ satisfy

$v_{\hat{f}_1}(0) + 1 = v_{\hat{f}_1}(1), e_{\hat{f}_1}(0) = e_{\hat{f}_1}(1), \hat{f}_1(w) = 1$; $v_{\hat{f}_5}(0) + 1 = v_{\hat{f}_5}(1), e_{\hat{f}_5}(0) = e_{\hat{f}_5}(1), \hat{f}_5(w) = 0$;

(b) For $n \equiv 1(\text{mod}4)$, we have $v_{\hat{f}_2}(0) = v_{\hat{f}_2}(1), e_{\hat{f}_2}(0) = e_{\hat{f}_2}(1), \hat{f}_2(w) = 0$;

(c) For $n \equiv 2(\text{mod}4)$, if we define $f_6(v) = g_2(v), v \in V(S_n), f_6(w) = 1$, then $v_{f_6}(0) = v_{f_6}(1) + 1, e_{f_6}(0) = e_{f_6}(1)$. Notice that $f_3(w) = 0, f_6(w) = 1$. Also the duals of f_3, f_6 that is $\hat{f}_3, \hat{f}_6$, satisfy

$v_{\hat{f}_3}(0) + 1 = v_{\hat{f}_3}(1), e_{\hat{f}_3}(0) = e_{\hat{f}_3}(1), \hat{f}_3(w) = 1$; $v_{\hat{f}_6}(0) + 1 = v_{\hat{f}_6}(1), e_{\hat{f}_6}(0) = e_{\hat{f}_6}(1), \hat{f}_6(w) = 0$;

(d) For $n \equiv 3(\text{mod}4)$, we have $v_{\hat{f}_4}(0) = v_{\hat{f}_1}(1), e_{\hat{f}_1}(0) = e_{\hat{f}_4}(1), \hat{f}_4(w) = 1$.

We now discuss the cordiality of Path unions of stemmed shells $ST_{n_1}, \cdots, ST_{n_k}$, with $w_1, \cdots, w_k$, as their respective roots. We note that for the path union, the edges connecting ST_{n_i} to $ST_{n_{i+1}}$ join the vertices w_i to w_{i+1}.

Theorem: All path unions of stemmed shells are cordial.

Proof: Let $PST(n_1, \cdots, n_k)$ be a path union of k stemmed shells $ST_{n_1}, \cdots, ST_{n_k}$, where $n_1, \cdots, n_k$ are not necessarily the same. For convenience, we denote this path union by PST. Define a binary labeling g of PST in two steps as follows: In step 1 we first label the vertices $w_1, \cdots, w_k$. In the next step we label each stemmed shell with the scheme described below. For each stemmed shell ST_{n_i}, the restriction of g to ST_{n_i} will be the labeling chosen for that particular stemmed shell.

Let $g(w_i) = 1, i \equiv 1, 2(\text{mod}4)$ $\quad g(w_i) = 0, i \equiv 0, 3(\text{mod}4)$.

Firstly note that in every stemmed shell of odd width, the vertices and edges are equitably labeled. Thus in the above path union, for a stemmed shell of odd width, we either use the labeling g_1 or its dual according as the label for its root is 0 or 1.

Let t be the number of stemmed shells of even width. In $\left\lfloor \frac{t}{2} \right\rfloor$ of these stemmed shells, use the labeling for which the number of vertices with the label 0 is one less than the number of vertices with the label 1 and in $\left\lceil \frac{t}{2} \right\rceil$ of these use the labeling for which the number of vertices with the label 0 is one more than the number of vertices with the label 1. Since in any stemmed shell of even width also, the edges are equitably labeled, we need to look only at the labels of the connecting edges. If k is even then amongst the connecting edges, $\frac{k}{2}$ receive the label 0 and $\frac{(k-2)}{2}$ edges receive the label 1. Hence in this case,

$e_g(0) = e_g(1) + 1$. However if k is odd, then the connecting edges are also equitably labeled. In this case $e_g(0) = e_g(1)$. Further, if t is even, then $v_g(0) = v_g(1)$ and if t is odd, then $v_g(0) = v_g(1) + 1$. Hence g is a cordial labeling. □

One point unions and Path unions of Flags

We first investigate the cordiality of the one point union of graphs called Flags and later the cordiality of their path unions. All the techniques in this section are similar to those used in the previous results. Because of this, we give only the final result in tabular form.

Definition: A flag FG_n is the graph obtained by joining exactly one vertex of the cycle C_n to an extra vertex called the root.

Let $V(C_n) = \{u_1, u_2, \cdots, u_n\}$ and let v be the pendant vertex attached to the vertex u_1. Then $V(FG_n) = V(C_n)\bigcup\{v\}$ and $E(FG_n) = E(C_n)\bigcup\{u_1v\}$.

We define the following binary labeling for FG_n which we will use in our constructions.

(i)Let f_1 be defined by:
$f_1(u_i) = 0, i \equiv 1, 2(\mod 4), f_1(u_i) = 1, i \equiv 0, 3(\mod 4), f_1(v) = 0.$
(ii)Let f_2 be defined by:
$f_2(u_i) = 0, i \equiv 1, 2(\mod 4), i \neq 2, f_2(u_i) = 1, i \equiv 0, 3(\mod 4), f_2(u_2) = 1, f_2(v) = 0.$
(iii)Let f_3 be defined by:
$f_3(u_i) = 1, i \equiv 1, 2(\mod 4), f_3(u_i) = 0, i \equiv 0, 3(\mod 4), f_3(v) = 0.$
(iv)Let f_4 be defined by:
$f_4(u_i) = 1, i \equiv 1, 2(\mod 4), i \neq n-1, f_4(u_i) = 0, i \equiv 0, 3(\mod 4),$
$f_4(u_{n-1}) = 0, f_4(v) = 0.$
(v)Let f_5 be defined by:
$f_5(u_i) = 1, i \equiv 2, 3(\mod 4), f_5(u_i) = 0, i \equiv 0, 1(\mod 4), f_5(v) = 0.$
(vi)Let f_6 be defined by:
$f_6(u_i) = 1, i \equiv 1, 2(\mod 4), i \neq n, f_6(u_i) = 0, i \equiv 0, 3(\mod 4), f_6(v) = 0 = f_6(u_n).$

We use these labelings in what follows:

Consider a flag FG_n. Let $n = 4q + r, 0 \leq r \leq 3$. Depending on the value of r, we use the labelings for FG_n as shown in the table. In each case, for the labeling used, we note that the vertex v is labeled 0. From this table it is clear that flags are cordial.

r	Labeling f	Vertex condition for f	Edge condition for f
0	f_1	$v_{f_1}(0) = v_{f_1}(1) + 1$	$e_{f_1}(0) = e_{f_1}(1) + 1$
	f_3	$v_{f_3}(0) = v_{f_3}(1) + 1$	$e_{f_3}(0) + 1 = e_{f_3}(1)$
1	f_3	$v_{f_3}(0) = v_{f_1}(1)$	$e_{f_3}(0) = e_{f_3}(1)$
	f_6	$v_{f_6}(0) = v_{f_6}(1) + 2$	$e_{f_6}(0) = e_{f_6}(1)$
2	f_4	$v_{f_4}(0) = v_{f_4}(1) + 1$	$e_{f_4}(0) = e_{f_4}(1) + 1$
	f_5	$v_{f_5}(0) = v_{f_5}(1) + 1$	$e_{f_5}(0) + 1 = e_{f_5}(1)$
3	f_1	$v_{f_1}(0) = v_{f_1}(1) + 2$	$e_{f_1}(0) = e_{f_1}(1)$
	f_2	$v_{f_2}(0) = v_{f_2}(1)$	$e_{f_2}(0) = e_{f_2}(1)$

We now make use of these labelings to show that the one point union of flags is cordial. We remark at the outset that S.C.Shee and Y.S.Ho have proved in [7] that the one point union of n copies of a flag F_n are cordial. Our result is more general, in the sense that we consider the one point union of flags $FG_{n_1}, FG_{n_2}, \cdots, FG_{n_k}$ which are not necessarily of the same size. Thus

Theorem: All one point unions of flags are cordial.

Proof: Let $FG(n_1, n_2, \cdots, n_k)$ denote the one point union of k flags FG_{n_1}, $FG_{n_2}, \cdots, FG_{n_k}$ not necessarily of the same size. Let f be a binary labeling of
$FG(n_1, n_2, \cdots, n_k)$ to be defined in two stages. Let $v'_f(0), v'_f(1)$ denote the number of vertices which receive the labels 0, 1 respectively at the end of Stage 1. Let $e'_f(0), e'_f(1)$ denote the number of edges which receive the labels 0, 1 respectively at the end of Stage 1. Let $v''_f(0), v''_f(1)$ denote the number of vertices which receive the labels 0, 1 respectively at the end of Stage 2 and $e''_f(0), e''_f(1)$ the number of edges, which receive the labels 0, 1 respectively at the end of Stage 2.

<u>Stage 1:</u>

Let p_1 be the number of flags FG_{n_i} for which $n_i \equiv 0(\bmod 4)$. Let

$p_1 = 2q_1 + s_1, s_1 = 0, 1$. Out of $2q_1$ of these flags FG_{n_i} with $n_i \equiv 0(\bmod 4)$, in each of q_1 flags, use the labeling f_1 and in each of the other q_1 flags of this type, use the labeling f_3.

Let p_2 be the number of flags FG_{n_i} for which $n_i \equiv 1(\bmod 4)$. Let $p_2 = 2q_2 + s_2, s_2 = 0, 1$. Out of $2q_2$ of these flags FG_{n_i} with $n_i \equiv 1(\bmod 4)$, in each of q_2 of these, use the labeling f_3 and in each of the other q_2 flags of this type, use the labeling f_6.

Let p_3 be the number of flags FG_{n_i} for which $n_i \equiv 2(\bmod 4)$. Let $p_3 = 2q_3 + s_3, s_3 = 0, 1$. Out of $2q_3$ of these flags FG_{n_i} with $n_i \equiv 2(\bmod 4)$, in each of q_3 of these, use the labeling f_4 and in each of the other q_3 flags of this type, use the labeling f_5.

Finally, let p_4 be the number of flags FG_{n_i} for which $n_i \equiv 3(\bmod 4)$. Let $p_4 = 2q_4 + s_4, s_4 = 0, 1$. Out of $2q_4$ of these flags FG_{n_i} with $n_i \equiv 3(\bmod 4)$, in each of q_4 of these, use the labeling f_1 and in each of the other q_4 flags of this type, use the labeling f_2.

At this stage, we have $v'_f(0) = v'_f(1) + 1, e'_f(0) = e'_f(1)$.

Stage 2:

There now remain to be labeled, s_1 flags for which $n_i \equiv 0(\bmod 4), s_2$ flags for which $n_i \equiv 1(\bmod 4), s_3$ flags for which $n_i \equiv 2(\bmod 4)$ and s_4 flags for which $n_i \equiv 3(\bmod 4)$. We now label these by the binary labeling as shown in the following table. Depending on the values of s_1, s_2, s_3, s_4 we indicate the appropriate labeling function chosen, in parentheses, at the side. The vertex and edge label conditions then obtained in each case, are also mentioned.

s_1	s_2	s_3	s_4	Vertex condition	Edge condition
0	0	0	0	$v''_f(0) = v''_f(1)$	$e''_f(0) = e''_f(1)$
0	0	0	$1(f_2)$	$v''_f(0) + 1 = v''_f(1)$	$e''_f(0) = e''_f(1)$
0	0	$1(f_4)$	0	$v''_f(0) = v''_f(1)$	$e''_f(0) = e''_f(1) + 1$
0	0	$1(f_4)$	$1(f_2)$	$v''_f(0) + 1 = v''_f(1)$	$e''_f(0) = e''_f(1) + 1$
0	$1(f_3)$	0	0	$v''_f(0) + 1 = v''_f(1)$	$e''_f(0) = e''_f(1)$
0	$1(f_3)$	0	$1(f_1)$	$v''_f(0) = v''_f(1)$	$e''_f(0) = e''_f(1)$
0	$1(f_3)$	$1(f_4)$	0	$v''_f(0) + 1 = v''_f(1)$	$e''_f(0) = e''_f(1) + 1$
0	$1(f_3)$	$1(f_4)$	$1(f_1)$	$v''_f(0) = v''_f(1)$	$e''_f(0) = e''_f(1) + 1$
$1(f_1)$	0	0	0	$v''_f(0) = v''_f(1)$	$e''_f(0) = e''_f(1) + 1$
$1(f_1)$	0	0	$1(f_2)$	$v''_f(0) + 1 = v''_f(1)$	$e''_f(0) = e''_f(1) + 1$
$1(f_1)$	0	$1(f_5)$	0	$v''_f(0) = v''_f(1)$	$e''_f(0) = e''_f(1)$
$1(f_1)$	0	$1(f_5)$	$1(f_2)$	$v''_f(0) + 1 = v''_f(1)$	$e''_f(0) = e''_f(1)$
$1(f_1)$	$1(f_3)$	0	0	$v''_f(0) + 1 = v''_f(1)$	$e''_f(0) = e''_f(1) + 1$
$1(f_1)$	$1(f_3)$	0	$1(f_1)$	$v''_f(0) = v''_f(1)$	$e''_f(0) = e''_f(1) + 1$
$1(f_1)$	$1(f_3)$	$1(f_5)$	0	$v''_f(0) + 1 = v''_f(1)$	$e''_f(0) = e''_f(1)$
$1(f_1)$	$1(f_3)$	$1(f_5)$	$1(f_1)$	$v''_f(0) = v''_f(1)$	$e''_f(0) = e''_f(1)$

As can be easily seen from the table, at the second stage, either $v''_f(0) + 1 = v''_f(1)$ or $v''_f(0) = v''_f(1)$, whereas for the edge condition, $e''_f(0) = e''_f(1)$ or $e''_f(0) = e''_f(1) + 1$. Hence the labeling f defined in these two stages, satisfies the conditions $| v_f(0) - v_f(1) | \leq 1$ and $| e_f(0) - e_f(1) | \leq 1$.
It follows therefore that f is a cordial labeling for the one point union $FG(n_1, n_2, \cdots, n_k)$ of k flags. Hence the result. □

We now consider the path union of flags $FG_{n_1}, \cdots, FG_{n_k}$, with $w_1, \cdots, w_k$, as their respective roots. We note that for the path union, the edges connecting FG_{n_i} to $FG_{n_{i+1}}$ join the vertices w_i to w_{i+1}. We use the labelings $f_1, f_2, \cdots, f_5$ for a flag FG_n as mentioned earlier. Depending on the value of n, only some of these labeling functions will be required. These labeling functions, along with the corresponding vertex and edge label conditions, are listed below:

For <u>$n \equiv 0(\bmod 4)$</u> we have from the above,
$v_{f_1}(0) = v_{f_1}(1) + 1$: $e_{f_1}(0) = e_{f_1}(1) + 1$: $f_1(v) = 0$.

and $v_{f_2}(0) + 1 = v_{f_2}(1); e_{f_2}(0) = e_{f_2}(1) + 1; f_2(v) = 0$.
Now $v_{\hat{f}_1}(0) + 1 = v_{\hat{f}_1}(1); e_{\hat{f}_1}(0) = e_{\hat{f}_1}(1) + 1; \hat{f}_1(v) = 1$ and
$v_{\hat{f}_2}(0) = v_{\hat{f}_2}(1) + 1; e_{\hat{f}_2}(0) = e_{\hat{f}_2}(1) + 1; \hat{f}_2(v) = 1$.
For $n \equiv 1 (\mod 4)$ we have
$v_{f_3}(0) = v_{f_3}(1); e_{f_3}(0) = e_{f_3}(1); f_3(v) = 0$.
Its dual gives $v_{\hat{f}_3}(0) = v_{\hat{f}_3}(1); e_{\hat{f}_3}(0) = e_{\hat{f}_3}(1); \hat{f}_3(v) = 1$.
For $n \equiv 2 (\mod 4)$ we have
$v_{f_3}(0) + 1 = v_{f_3}(1); e_{f_3}(0) = e_{f_3}(1) + 1; f_3(v) = 0$.
Its dual gives $v_{\hat{f}_3}(0) = v_{\hat{f}_3}(1) + 1; e_{\hat{f}_3}(0) = e_{\hat{f}_3}(1) + 1; \hat{f}_3(v) = 1$.
Also $v_{f_4}(0) = v_{f_4}(1) + 1; e_{f_4}(0) = e_{f_4}(1) + 1; f_4(v) = 0$.
Its dual gives $v_{\hat{f}_4}(0) + 1 = v_{\hat{f}_4}(1); e_{\hat{f}_4}(0) = e_{\hat{f}_4}(1) + 1; \hat{f}_4(v) = 1$.
For $n \equiv 3 (\mod 4)$ we have
$v_{f_5}(0) = v_{f_5}(1); e_{f_5}(0) = e_{f_5}(1); f_5(v) = 0$.
Its dual gives $v_{\hat{f}_5}(0) = v_{\hat{f}_5}(1); e_{\hat{f}_5}(0) = e_{\hat{f}_5}(1); \hat{f}_5(v) = 1$.

Theorem: All path unions of flags are cordial.

Proof: Let $PFG(n_1, n_2, \cdots, n_k)$ denote the path union of k flags FG_{n_1}, $FG_{n_2}, \cdots . FG_{n_k}$. We use induction on k. In fact we will be obtaining a cordial labeling g for which $v_g(0) \geq v_g(1), e_g(0) \geq e_g(1)$. We first prove the result for $k = 2$.

Let g_1, g_2 be the binary labelings chosen for FG_{n_1}, FG_{n_2} respectively. Let g be a binary labeling of $PFG(n_1, n_2)$ described as follows:

$$
\begin{aligned}
g(w) &= g_1(w), w \in V(FG_{n_1}) \\
&= g_2(w), w \in V(FG_{n_2}).
\end{aligned}
$$

Let $n_1 = 4q_1 + r_1, n_2 = 4q_2 + r_2; 0 \leq r \leq 3$. Depending on the values of r_1, r_2 we choose the labeling function accordingly. The choice made, in each case, is indicated by writing the chosen labeling function alongside in parentheses.The vertex conditions and edge conditions obtained for g are also given.

$n_1(g_1)$	$n_2(g_2)$	Vertex condition for g	Edge condition for g
$0(\hat{f}_1)$	$0(f_1)$	$v_g(0) = v_g(1)$	$e_g(0) = e_g(1) + 1$
$0(f_1)$	$1(\hat{f}_3)$	$v_g(0) = v_g(1) + 1$	$e_g(0) = e_g(1)$
$0(\hat{f}_1)$	$2(f_4)$	$v_g(0) = v_g(1)$	$e_g(0) = e_g(1) + 1$
$0(f_1)$	$3(\hat{f}_5)$	$v_g(0) = v_g(1) + 1$	$e_g(0) = e_g(1)$
$1(f_3)$	$1(f_3)$	$v_g(0) = v_g(1)$	$e_g(0) = e_g(1) + 1$
$1(f_3)$	$2(\hat{f}_3)$	$v_g(0) = v_g(1) + 1$	$e_g(0) = e_g(1)$
$1(f_3)$	$3(f_5)$	$v_g(0) = v_g(1)$	$e_g(0) = e_g(1) + 1$
$2(f_3)$	$2(\hat{f}_3)$	$v_g(0) = v_g(1)$	$e_g(0) = e_g(1) + 1$
$2(\hat{f}_3)$	$3(f_5)$	$v_g(0) = v_g(1) + 1$	$e_g(0) = e_g(1)$
$3(f_5)$	$3(f_5)$	$v_g(0) = v_g(1)$	$e_g(0) = e_g(1) + 1$

Hence g is a cordial labeling satisfying the aforementioned conditions.

Assume that the result is true for some $m \in N$. Then there is a cordial labeling g_1 for $PFG(n_1, n_2, \cdots, n_m)$ such that $v_{g_1}(0) \geq v_{g_1}(1)$ and $e_{g_1}(0) \geq e_{g_1}(1)$. Let g be the labeling for $PFG(n_1, n_2, \cdots, n_m, n_{m+1})$ defined as $g(w) = g_1(w)$, if $w \in V(PFG(n_1, n_2, \cdots, n_m))$ and $g(w) = g_2(w)$, if $w \in V(FG(n_{m+1}))$, where the choice for g_2 is as explained in each case below.

<u>Case 1:</u> $v_{g_1}(0) = v_{g_1}(1); e_{g_1}(0) = e_{g_1}(1)$.

Case 1(a): $g_1(v_m) = 0$.

(i) <u>$n_{m+1} \equiv 0(\text{mod}4)$.</u>

Take g_2 as $\hat{f}_2$. Then as $v_{g_2}(0) = v_{g_2}(1)+1; e_{g_2}(0) = e_{g_2}(1)+1; g_2(v_{m+1}) = 1$; moreover the edge $v_m v_{m+1}$ that is the edge connecting $PFG(n_1, n_2, \cdots, n_m)$ to $FG(n_{m+1})$ receives the label 1. Hence $v_g(0) = v_g(1) + 1$ and $e_g(0) = e_g(1)$.

(ii) <u>$n_{m+1} \equiv 1(\text{mod}4)$.</u>

Take g_2 as f_3. Then as $v_{g_2}(0) = v_{g_2}(1); e_{g_2}(0) = e_{g_2}(1); g_2(v_{m+1}) = 0$; moreover the edge $v_m v_{m+1}$ that is the edge connecting $PFG(n_1, n_2, \cdots, n_m)$ to $FG(n_{m+1})$ receives the label 0. Hence $v_g(0) = v_g(1)$ and $e_g(0) = e_g(1) + 1$.

(iii) <u>$n_{m+1} \equiv 2(\text{mod}4)$.</u>

Take g_2 as $\hat{f}_3$. Then as $v_{g_2}(0) = v_{g_2}(1)+1; e_{g_2}(0) = e_{g_2}(1)+1; g_2(v_{m+1}) = 1$; moreover the edge $v_m v_{m+1}$ that is the edge connecting $PFG(n_1, n_2, \cdots, n_m)$ to $FG(n_{m+1})$ receives the label 1. Hence $v_g(0) = v_g(1) + 1$ and $e_g(0) = e_g(1)$.

(iv) $n_{m+1} \equiv 3(\mathrm{mod} 4)$.

Take g_2 as f_5. Then as $v_{g_2}(0) = v_{g_2}(1)$; $e_{g_2}(0) = e_{g_2}(1)$; $g_2(v_{m+1}) = 0$; moreover the edge $v_m v_{m+1}$ that is the edge connecting $PFG(n_1, n_2, \cdots, n_m)$ to $FG(n_{m+1})$ receives the label 0. Hence $v_g(0) = v_g(1)$ and $e_g(0) = e_g(1) + 1$.

Case 1(b): $g_1(v_m) = 1$.

(i) $n_{m+1} \equiv 0(\mathrm{mod} 4)$.

Take g_2 as f_1. Then arguing as above it follows that $v_g(0) = v_g(1) + 1$ and $e_g(0) = e_g(1)$.

(ii) $n_{m+1} \equiv 1(\mathrm{mod} 4)$.

Take g_2 as $\hat{f}_3$. Then arguing as above it follows that $v_g(0) = v_g(1)$ and $e_g(0) = e_g(1) + 1$.

(iii) $n_{m+1} \equiv 2(\mathrm{mod} 4)$.

Take g_2 as f_4. Then arguing as above it follows that $v_g(0) = v_g(1) + 1$ and $e_g(0) = e_g(1)$.

(iv) $n_{m+1} \equiv 3(\mathrm{mod} 4)$.

Take g_2 as $\hat{f}_5$. Then arguing as above it follows that $v_g(0) = v_g(1)$ and $e_g(0) = e_g(1) + 1$.

Case 2: $v_{g_1}(0) = v_{g_1}(1) + 1$; $e_{g_1}(0) = e_{g_1}(1)$.

Case 2(a): $g_1(v_m) = 0$.

(i) $n_{m+1} \equiv 0(\mathrm{mod} 4)$.

Take g_2 as $\hat{f}_1$. Then $v_g(0) = v_g(1)$ and $e_g(0) = e_g(1)$.

(ii) $n_{m+1} \equiv 1(\mathrm{mod} 4)$.

Take g_2 as f_3. Then $v_g(0) = v_g(1) + 1$ and $e_g(0) = e_g(1) + 1$.

(iii) $n_{m+1} \equiv 2(\mathrm{mod} 4)$.

Take g_2 as $\hat{f}_4$. Then $v_g(0) = v_g(1)$ and $e_g(0) = e_g(1)$.

(iv) $n_{m+1} \equiv 3(\mathrm{mod} 4)$.

Take g_2 as f_5. Then $v_g(0) = v_g(1) + 1$ and $e_g(0) = e_g(1) + 1$.

Case 2(b): $g_1(v_m) = 1$.

(i) $n_{m+1} \equiv 0(\mathrm{mod} 4)$.

Take g_2 as f_2. Then it follows that $v_g(0) = v_g(1)$ and $e_g(0) = e_g(1)$.

(ii) $n_{m+1} \equiv 1(\mathrm{mod} 4)$.

Take g_2 as $\hat{f}_3$. Then it follows that $v_g(0) = v_g(1) + 1$ and $e_g(0) = e_g(1) + 1$.

(iii) $n_{m+1} \equiv 2(\mathrm{mod} 4)$.

Take g_2 as f_3. Then we have $v_g(0) = v_g(1)$ and $e_g(0) = e_g(1)$.

(iv) $n_{m+1} \equiv 3(\text{mod}4)$.

Take g_2 as $\hat{f}_5$. Then we have $v_g(0) = v_g(1) + 1$ and $e_g(0) = e_g(1) + 1$.

Case 3: $v_{g_1}(0) = v_{g_1}(1); e_{g_1}(0) = e_{g_1}(1) + 1$.

Case 3(a): $g_1(v_m) = 0$.

(i) $n_{m+1} \equiv 0(\text{mod}4)$.

Take g_2 as $\hat{f}_2$. Then $v_g(0) = v_g(1) + 1$ and $e_g(0) = e_g(1) + 1$.

(ii) $n_{m+1} \equiv 1(\text{mod}4)$.

Take g_2 as $\hat{f}_3$. Then $v_g(0) = v_g(1)$ and $e_g(0) = e_g(1)$.

(iii) $n_{m+1} \equiv 2(\text{mod}4)$.

Take g_2 as $\hat{f}_3$. Then $v_g(0) = v_g(1) + 1$ and $e_g(0) = e_g(1) + 1$.

(iv) $n_{m+1} \equiv 3(\text{mod}4)$.

Take g_2 as $\hat{f}_5$. Then $v_g(0) = v_g(1)$ and $e_g(0) = e_g(1)$.

Case 3(b): $g_1(v_m) = 1$.

(i) $n_{m+1} \equiv 0(\text{mod}4)$.

Take g_2 as f_1. Then it follows that $v_g(0) = v_g(1) + 1$ and $e_g(0) = e_g(1) + 1$.

(ii) $n_{m+1} \equiv 1(\text{mod}4)$.

Take g_2 as f_3. Then it follows that $v_g(0) = v_g(1)$ and $e_g(0) = e_g(1)$.

(iii) $n_{m+1} \equiv 2(\text{mod}4)$.

Take g_2 as f_4. Then we have $v_g(0) = v_g(1) + 1$ and $e_g(0) = e_g(1) + 1$.

(iv) $n_{m+1} \equiv 3(\text{mod}4)$.

Take g_2 as f_5. Then we have $v_g(0) = v_g(1)$ and $e_g(0) = e_g(1)$.

Case 4: $v_{g_1}(0) = v_{g_1}(1) + 1; e_{g_1}(0) = e_{g_1}(1) + 1$.

Case 4(a): $g_1(v_m) = 0$.

(i) $n_{m+1} \equiv 0(\text{mod}4)$.

Take g_2 as $\hat{f}_1$. Then $v_g(0) = v_g(1)$ and $e_g(0) = e_g(1) + 1$.

(ii) $n_{m+1} \equiv 1(\text{mod}4)$.

Take g_2 as $\hat{f}_3$. Then $v_g(0) = v_g(1) + 1$ and $e_g(0) = e_g(1)$.

(iii) $n_{m+1} \equiv 2(\text{mod}4)$.

Take g_2 as $\hat{f}_4$. Then $v_g(0) = v_g(1)$ and $e_g(0) = e_g(1) + 1$.

(iv) $n_{m+1} \equiv 3(\text{mod}4)$.

Take g_2 as $\hat{f}_5$. Then $v_g(0) = v_g(1) + 1$ and $e_g(0) = e_g(1)$.

Case 4(b): $g_1(v_m) = 1$.

(i) $n_{m+1} \equiv 0 (\mathrm{mod} 4)$.

Take g_2 as f_2. Then it follows that $v_g(0) = v_g(1)$ and $e_g(0) = e_g(1) + 1$.

(ii) $n_{m+1} \equiv 1 (\mathrm{mod} 4)$.

Take g_2 as f_3. Then it follows that $v_g(0) = v_g(1) + 1$ and $e_g(0) = e_g(1)$.

(iii) $n_{m+1} \equiv 2 (\mathrm{mod} 4)$.

Take g_2 as f_3. Then we have $v_g(0) = v_g(1)$ and $e_g(0) = e_g(1) + 1$.

(iv) $n_{m+1} \equiv 3 (\mathrm{mod} 4)$.

Take g_2 as f_5. Then we have $v_g(0) = v_g(1) + 1$ and $e_g(0) = e_g(1)$.

Hence in all cases, we see that the labeling g for $PFG(n_1, n_2, \cdots, n_m, n_{m+1})$ is cordial. Therefore,by induction, it follows that the path union of flags $PFG(n_1, n_2, \cdots, n_k)$ is cordial. □

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