NEGATIVE RESULTS ON THE STABILITY PROBLEM WITHIN CO-HEREDITARY CLASSES

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ABSTRACT. Let S be a stable set in a graph G, possibly $S = \emptyset$. The subgraph $G - N[S]$, where $N[S]$ closed neighborhood of S, is called a *co-stable subgraph* of G. We denote by CSub(G) the set of all co-stable subgraphs of G. A class of graphs $\mathcal{P}$ is called *co-hereditary* if $G \in \mathcal{P}$ implies CSub(G) $\subseteq \mathcal{P}$. Our result: If the set of all minimal forbidden co-stable subgraphs for a non-empty co-hereditary class $\mathcal{P}$ is finite, then Stable Set is an NP-complete problem within $\mathcal{P}$. Also, we prove that the decision problem of recognizing whether a graph has a fixed graph H as a co-stable subgraph is NP-complete for each non-trivial graph H.
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1. Co-hereditary classes

We use standard graph-theoretic terminology, see for example Melnikov, Sarvanov, Tyshkevich, Yemelichev, and Zverovich [9]. A complete graph of order n is denoted by K_n. Here we admit the existence of the *null-graph* K_0. The *neighborhood* $N(v) = N_G(v)$ of a vertex v in a graph G is the set of all vertices that are adjacent to v. The *closed neighborhood* of v is $N[v] = \{v\} \cup N(v)$. For a set $X \subseteq V(G)$, $N(X) = \bigcup_{x \in X} N(x)$ and $N[X] = X \cup N(x)$. A set $S \subseteq V(G)$ is *stable* if $N(S) \cap S = \emptyset$. A *maximal* stable set is not contained in any other stable set. A stable set S is *maximum* in a graph G if $|S| \geq |S'|$ for each stable set S' of G.

Definition 1. *Let S be a stable set of G, possibly $S = \emptyset$. The subgraph $G - N[S]$ is called a* co-stable subgraph *of G. We denote by* CSub(G) *the set of all co-stable subgraphs of G* [considered up to isomorphism].

For example, the 5-cycle C_5 has the following co-stable subgraphs: C_5, K_2 and K_0, that is CSub(C_5) $= \{C_5, K_2, K_0\}$. Clearly, each co-stable subgraph is an induced subgraph, but not conversely.

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Definition 2. *A class of graphs $\mathcal{P}$ is called* co-hereditary *if it is closed under taking co-stable subgraphs, i.e., $G \in \mathcal{P}$ implies* $\mathrm{CSub}(G) \subseteq \mathcal{P}$.

A *hereditary class* is a class closed under taking induced subgraphs. Each hereditary class is co-hereditary, but not conversely. Recall that a graph is called *well-covered* if all its maximal stable sets have the same cardinality. Well-covered graphs constitute a well-known example of a co-hereditary class. Indeed, suppose that H is a co-stable subgraph of a well-covered graph G. It means that $H = G - N[S]$ for some stable set S of G. If there are two maximal stable sets S_1 and S_2 in H of different size, then $S \cup S_1$ and $S \cup S_2$ would be two maximal stable sets of G of different size, contradicting to the fact that G is a well-covered graph.

For a set of graphs Z, we put

$$\mathrm{FCS}(Z) = \{G : \mathrm{CSub}(G) \cap Z = \emptyset\}.$$

If $\mathcal{P} = \mathrm{FCS}(Z)$ then Z is called a set of *forbidden co-stable subgraphs* for $\mathcal{P}$ characterizing $\mathcal{P}$. A forbidden co-stable subgraph $H \in Z$ for $\mathcal{P}$ is *minimal* if $\mathrm{CSub}(H) \setminus \{H\} \subseteq \mathcal{P}$. The following observations are straightforward.

Proposition 1. (i) *$\mathcal{P}$ is a co-hereditary class if and only if* $\mathcal{P} = \mathrm{FCS}(Z)$ *for some set Z.*

(ii) *The inclusion-wise minimal* [with respect to the partial order "to be a co-stable subgraph"] *set Z that satisfies* (i) *is uniquely defined. Moreover, it coincides with the set of all minimal forbidden co-stable subgraphs for $\mathcal{P}$.*

Proposition 1 shows that each co-hereditary class can be characterized in terms of forbidden co-stable subgraphs. Such a characterization for the class $\mathcal{WELL}$ of all well-covered graphs was found by Zverovich [15]. No other characterizations of co-hereditary classes are known.

2. Stable Set Problem

The *stability number* of a graph G, denoted by $\alpha(G)$, is the maximum cardinality of a stable set in G. The problem of calculating $\alpha(G)$ is polynomial-time equivalent to the following decision problem.

Decision Problem 1 (Stable Set).

Instance: *A graph G and an integer k.*

Question: *Is there a stable set S of G with $|S| \geq k$?*

Let us consider the corresponding constructive variant of this problem.

Constructive Problem 1 (Stable Set).

Instance: *A graph G and an integer k.*

Output: *A stable set S of G with $|S| \geq k$ or "No"* [there is no such a stable set in G].

Let $\mathcal{P}$ be a class of graphs. Imagine that we have an oracle O that, given a graph $G \in \mathcal{P}$, returns $\alpha(G)$ in polynomial time. Can we construct a maximum stable set of each graph in $\mathcal{P}$ in polynomial time using O only? The answer is "yes" for co-hereditary classes.

Proposition 2. *If $\mathcal{P}$ is a co-hereditary class then* Constructive Problem Stable Set *and* Decision Problem Stable Set *are polynomial-time equivalent.*

Proof. An output of the constructive problem can be easily transformed to an answer to the question of the decision problem. Conversely, let Alg be an algorithm for solving Decision Problem Stable Set. Given an instance (G, k) to Constructive Problem, we proceed as follows.

Algorithm 1.

Input: *A pair (G, k).*

Step 1. *Set $S = \emptyset$.*

Step 2. *If $V(G) = \emptyset$ then return S and* Stop.

Step 3. *For each vertex $u \in V(G)$, apply* Alg *to the instance $(G - N[u], k - 1)$.* Note that the co-stable subgraph $G - N[u]$ is in $\mathcal{P}$ by co-hereditariness.

Step 4. *If the answer is* No *for each vertex, then return* No *and* Stop.
If the answer is Yes *for a vertex u then set $G = G - N[u]$ and $k = k - 1$, include u into S and go to* Step 2.

The algorithm either constructs a stable set S of G with $|S| \geq k$, or returns "No" [there is no such a set]. □

In view of Proposition 2, the model of co-hereditary classes is very natural when we deal with Decision Problem Stable Set. However, almost all known results on the problem were obtained for hereditary classes. Among them perfect graphs (Grötschel, Lovász, and Schrijver [5]), claw-free graphs (Minty [10], see also corrections in Nakamura and Tamura [11], and the unweighted version in Sbihi [14]), chair-free graphs (Alekseev [1]), $(P_5, \overline{P}_5)$-free graphs and their extensions (Giakoumakis and Rusu [4], and Zverovich and Zverovich [20]), AH-free graphs (Hertz and de Werra [6]), pentagraphs (Zverovich and Zverovich [21]), superbipartite graphs (Zverovich and Zverovich [19]), a class of Mahadev and Reed and its extensions (Mahadev and Reed [8], Rautenbach [12] and Zverovich [17]), etc. Some general methods for Stable Set Problem in hereditary classes were developed by Alekseev [2], Rautenbach, Zverovich, and Zverovich [13], Zverovich [16], Zverovich [18], and Zverovich and Zverovich [22].

The class of well-covered graphs is the only known co-hereditary non-hereditary class of graphs for which Stable Set Problem can be solved in polynomial-time. Why do not people consider other such classes? We provide some reasons for that.

3. Negative results

A class of graphs $\mathcal{P}$ is called *α-polynomial* if there exists a polynomial-time algorithm for Stable Set Problem within $\mathcal{P}$. A class of graphs $\mathcal{P}$ is *α-complete* if Decision Problem Stable Set is NP-complete within $\mathcal{P}$.

Definition 3. *For an integer $n \geq 1$, we define an n-*replication *of a graph G as a graph* $\mathrm{Rep}(G,n)$ *defined as follows:*

- *introduce a copy V_i of the complete graph K_n for each vertex v_i of G, and*
- *when $i \neq j$, a vertex $x \in V_i$ is adjacent to a vertex $y \in V_j$ if and only if the vertices v_i and v_j are adjacent in G.*

In other words, $\mathrm{Rep}(G,n)$ is obtained from G by inflating each vertex of G with $n-1$ new vertices. Clearly, $G = \mathrm{Rep}(G,1)$.

Theorem 1. *Let Z be the set of all minimal forbidden co-stable subgraphs for a co-hereditary class $\mathcal{P}$. If Z is finite and $Z \neq \{K_0\}$, then $\mathcal{P}$ is α-complete.*

Proof. Let N be the maximum order of a graph in Z. Clearly, Z cannot contain an $(N+1)$-replication of a graph $G \neq K_0$.

Claim 1. *For each graph G, the graph $H = \mathrm{Rep}(G, N+1)$ belongs to $\mathcal{P}$.*

Proof. We consider an arbitrary co-stable subgraph H' of H. By definition $H' = H - N_H[S]$ for a stable set S of H. Without loss of generality we may assume that $S \subseteq V(G)$. It follows that $H' = \mathrm{Rep}(G', N+1)$, where $G' = G - N_G[S]$. We show that $H' \notin Z$.

Each graph contain K_0 as a co-stable subgraph. Since all forbidden co-stable subgraphs in Z are minimal and $Z \neq \{K_0\}$, we have $K_0 \notin Z$. Hence if $G' = K_0$ then $H' = K_0 \notin Z$. Also, if $G' \neq K_0$ then $|V(H')| \geq N + 1$. The definition of N implies that $H' \notin Z$. Thus, no co-stable subgraph of H belongs to Z. It follows that $H \in \mathcal{P}$. □

Claim 2. *For each graph G and each integer $n \geq 1$, $\alpha(G) = \alpha(H)$, where $H = \text{Rep}(G, n)$.*

Proof. It follows directly from Definition 3. □

Claim 1 and Claim 2 give a polynomial-time reduction from the general Stable Set Problem to the same problem within $\mathcal{P}$, namely $G \to \text{Rep}(G, N+1)$. Since Stable Set is known to be NP-complete in general (Karp [7]), the result follows. □

An *n-replication* is the graph $\text{Rep}(G, n)$ for some graph G.

Corollary 1. *Let $\mathcal{P}$ be an α-polynomial co-hereditary class of graphs defined by a set Z of forbidden co-stable subgraphs. If $P \neq NP$ then Z contains an n-replication for each $n \geq 1$.*

Proof. If Z does not contains an n-replication for some $n \geq 1$, then we may use the reduction $G \to \text{Rep}(G, n)$ to show that $\mathcal{P}$ is α-complete [as in the proof of Theorem 1]. The assumption $P \neq NP$ produces a contradiction, since α-polynomial class cannot be α-complete. □

For example, the trivial class $\mathcal{O} = \{K_0\}$ is α-polynomial. As it follows from Proposition 3 below, $\mathcal{O}$ can be defined by the set $Z_1 = \{K_n : n \geq 1\}$ of forbidden co-stable subgraphs. Clearly, Z_1 contains an n-replication for each $n \geq 1$. A graph is *non-trivial* if it has at least one vertex.

Proposition 3. *Each non-trivial graph G contains a non-trivial complete graph as a co-stable subgraph.*

Proof. Let H be the smallest non-trivial co-stable subgraph of G. Suppose that H is not complete, i.e., H contains distinct non-adjacent vertices u and v. We delete $N_H[u]$ from H, and obtain a co-stable subgraph H' of G. The subgraph H' is non-trivial, since $v \in V(H')$. We arrive to a contradiction: $|V(H')| < |V(H)|$. Thus, H is a complete graph. □

Unfortunately, the converse of Corollary 1 does not hold in general, i.e., if Z contains an n-replication for each $n \geq 1$, then $\mathrm{FCS}(Z)$ is not necessarily an α-polynomial class [assuming $P \neq NP$]. We consider the set $Z = \{K_n : n = 2, 3, 4, \ldots\}$ that contains an n-replication for each $n \geq 1$.

Proposition 4. *The class* $\mathcal{P} = \mathrm{FCS}(K_n : n = 2, 3, 4, \ldots)$ *is* α*-complete.*

Proof. Given a graph G, we construct a graph H by subdividing each edge of G with two new vertices. It is well-known and easy to see that $\alpha(H) = \alpha(G) + |E(G)|$. Since graph H is triangle-free, it does not contain K_n as a co-stable subgraph for all $n \geq 3$. Further, we define a 2-*duplication of* H as the following graph $\mathrm{Dup}(H, 2)$:

- introduce a set V_i of two non-adjacent vertices for each vertex v_i of H, and
- a vertex $x \in V_i$ is adjacent to a vertex $y \in V_j$ if and only if $i \neq j$ and the vertices v_i and v_j are adjacent in G.

It is easy to see that each co-stable subgraph in $\mathrm{Dup}(H, 2)$ consists of a 2-duplication of a co-stable subgraph of H and, possibly, some isolated vertices. Therefore $\mathrm{Dup}(H, 2)$ does not contain K_n as a co-stable subgraph for all $n \geq 2$. In other words, $\mathrm{Dup}(H, 2)$ is in $\mathcal{P}$. Also, $\alpha(\mathrm{Dup}(H, 2)) = 2\alpha(H) = 2\alpha(G) + 2|E(G)|$. Thus, we have constructed a polynomial-time reduction from the general Stable Set Problem to the same problem within $\mathcal{P}$. □

Open Problem 1. *Find necessary and sufficient conditions for* Z *under which* $\mathrm{FCS}(Z)$ *is an* α*-polynomial class, assuming* $P \neq NP$.

4. Complexity

Here we consider the following problem, where H is a fixed graph.

Decision Problem 2 (Co-Stable Subgraph H).

Instance: *A graph* G.

Question: *Is* H *isomorphic to a co-stable subgraph of* G*?*

Co-Stable Subgraph K_0 is a trivial problem, since each graph contains K_0 as a co-stable subgraph.

Theorem 2. Co-Stable Subgraph H *is an NP-complete problem for each graph* $H \neq K_0$.

Proof. The problem is in NP. Indeed, if we guess a stable set S of G, we can check in polynomial time that $H \cong G - N[S]$. To show NP-hardness, we use a polynomial-time reduction from Satisfiability Problem, see Garey and Johnson [3]. Let $C = \{c_1, c_2, \ldots, c_m\}$ be a set of clauses over a set of literals $L = \{x_1, \overline{x}_1, x_2, \overline{x}_2, \ldots, x_n, \overline{x}_n\}$. We define a graph G^* as follows:

- $V(G^*) = U \cup C^* \cup L^*$, where the sets U, C^* and L^* are pairwise disjoint,
- U induces H,
- $G^*(C^*)$ is a disjoin union of complete subgraphs $C^1, C^2, \ldots, C^m$, each having exactly $k+1$ vertices, where $k = |V(H)|$,
- L^* consists of $2n$ pairwise disjoint sets $X_1, \overline{X}_1, X_2, \overline{X}_2, \ldots, X_n, \overline{X}_n$, where $|X_i| = |\overline{X}_i| = k+1$ for every $i = 1, 2, \ldots, n$,
- $G^*(L^*)$ is a union of complete graphs induced by $X_i \cup \overline{X}_i$, $i = 1, 2, \ldots, n$,
- every vertex of U is adjacent to every vertex of C^*,
- there are no edges between U and L^*, and
- vertices $x \in L^*$ and $c \in C^*$ are adjacent if and only if either
  - $x \in X_i$, $c \in C^j$ and the clause c_j contains the variable x_i, or
  - $x \in \overline{X}_i$, $c \in C^j$ and the clause c_j contains the literal $\overline{x}_i$.

Claim 3. *A set U induces a co-stable subgraph of G^* if and only if there exists a truth assignment for L that satisfies C.*

Proof. Suppose that $H = G^*(U)$ is a co-stable subgraph of G. It means that there exists a stable set $S \subseteq V(G^*)$ such that $H = G^* - N[S]$. Since every vertex of U is adjacent to every vertex of C^*, we have $S \subseteq L^*$. Clearly, every vertex of C^* is adjacent to some vertex of S, that is S *dominates* C^*. Since S is a stable set, S contains at most one vertex from each complete graph K^i induced by $X_i \cup \overline{X}_i$. Since $S \subseteq L^*$ and $H = G^* - N[S]$, S contains exactly one vertex from each K^i. Thus, we can define a truth assignment ϕ setting $x_i = 1$ if and only if $S \cap X_i \neq \emptyset$. Since S dominates C^*, ϕ is a satisfying truth assignment for C.

Conversely, if there is a truth assignment ϕ that satisfies C, then we define a stable set $S \subseteq L^*$ as follows: S contains a vertex from X_i (respectively, $\overline{X}_i$) if and only if $\phi(x_i) = 1$ (respectively, $\phi(x_i) = 0$). Since exactly one of x_i, $\overline{x}_i$ is true, every vertex of L^* is in $N[S]$. The assignment ϕ satisfies C, so S dominates C^*, and therefore $C^* \subseteq N[S]$. We have $L^* \cup C^* \subseteq N[S]$. The construction of G implies $U \cap N[S] = \emptyset$. Thus, $N[S] = L^* \cup C^*$, or $H = G - N[S] \in \mathrm{CSub}(G)$, and the result follows. □

Claim 4. If $W \subseteq V(G^*)$ induces a co-stable subgraph of G^* isomorphic to H, then $W = U$.

Proof. Suppose that W contains a vertex $w \notin U$. By the construction, w belongs to a complete subgraph K of order $k+1$ such that $N[a] = N[b]$ for all $a, b \in V(K)$.

Further, W induces a co-stable subgraph, i.e., $W = V(G) \setminus N[S]$ for some stable set S of G. Since $w \in W$, we have $w \notin N[S]$. It follows that $S \cap N[w] = \emptyset$. Since $N[w] = N[z]$ for each vertex $z \in V(K)$, we obtain that $V(K) \subseteq W$, a contradiction to the fact that $|V(K)| > k = |V(H)| = |W|$. The contradiction implies that $W \subseteq U$. Since $|W| = |U|$, we have $W = U$. □

□

Theorem 2 shows that there is no an obvious recognition algorithm for finitely defined co-hereditary classes of graphs in opposition to finitely defined hereditary classes.

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