The n-Queens Problem With Diagonal Constraints

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Abstract

For a solution S of the n-queens problem, let $M(S)$ denote the maximum of the absolute values of the diagonal numbers of S, and let $m(S)$ denote the minimum of those absolute values. For $n \geq 4$, let $F(n)$ denote the minimum value of $M(S)$, and let $f(n)$ denote the maximum value of $m(S)$, as S ranges over all solutions of the n-queens problem. Say that a solution S is an *n-champion* if $M(S) = F(n)$ and $m(S) = f(n)$.

Approximately linear bounds are given for $F(n)$ and $f(n)$, along with computational results and several constructions together providing evidence that the bounds are excellent. It is shown that, in the range $4 \leq n \leq 24$, n-champions exist except for $n = 11, 16, 21, 22$.

Keywords: n-queens problem, Parallelogram Law.

1 Introduction

The *n-queens problem* requires placing n queens on an $n \times n$ chessboard so that no two attack each other. This combinatorial problem has been studied from many viewpoints. Ahrens showed [1] by construction that a solution exists for all positive integers n except 2 and 3; some recent work [5] has aimed at determining good bounds for the number of solutions for each $n \geq 4$. Also, solutions of particular forms have been studied [2, 3, 4].

In this paper, we examine the n-queens problem with constraints on the diagonals that the queens can occupy. For a precise statement, some notation is necessary.

We will identify the $n \times n$ chessboard with a square of side length n in the Cartesian plane, having sides parallel to the coordinate axes. We take the origin of the coordinate system to be the center of the board, and refer to board squares by the coordinates of their centers. The square (x,y) is in *column* x and *row* y. The *difference diagonal* (respectively *sum diagonal*) through square (x,y) is the set of all board squares with centers on the line of slope $+1$ (respectively -1) through the point (x,y). The value of $y - x$ is the same for each square (x,y) on a difference diagonal, and we will refer to the diagonal by this value. Similarly, the value of $y + x$ is the same for each square on a sum diagonal, and we associate this value to the diagonal. Note that when n is even, each of the column and row numbers is half an odd integer. However, for any n the diagonal numbers are integers.

Let S be a set of squares of the $n \times n$ board; then $M(S)$ will denote the maximum of the absolute values of the diagonal numbers of S, and $m(S)$ will denote the minimum of those absolute values. (The definitions of the diagonal numbers imply $M(S) = \max\{|y| + |x| : (x,y) \in S\}$ and $m(S) = \min\{||y| - |x|| : (x,y) \in S\}$.)

For each integer $n \geq 4$, let $F(n)$ be the minimum value of $M(S)$, and $f(n)$ the maximum value of $m(S)$, as S ranges over all solutions of the n-queens problem. In this paper, we examine the functions F and f.

One motivation for our work arose from the second author's paper [6] on the *queen domination problem*: finding for each positive integer k the least size of a set D of squares of the $k \times k$ board such that queens on the squares of D attack or occupy every square of the board.

The construction employed in [6] for dominating sets required choosing an n, $n \ll k$, and a solution S of the n-queens problem, applying the function $h(x,y) = (2(y+x), 2(y-x))$ to S, and placing copies of $h(S)$ on the $k \times k$ board. The smaller the sub-board that $h(S)$ fit inside, the better the result was; since h is a clockwise rotation by $\pi/4$ radians (and dilation by $2\sqrt{2}$), h interchanges diagonals with orthogonals. Thus it was desirable for $M(S)$ to be as small as possible.

A second motivation came from consideration of how the number $Q(n)$ of solutions of the n-queens problem grows as n increases. It was recently conjectured [5] that there exists $\beta > 0$ with $\lim_{n \to \infty}[\ln Q(n)/n \ln n] = \beta$. This would imply that for any $\epsilon > 0$, $Q(n) > n^{(\beta - \epsilon)n}$ for sufficiently large n.

Such rapid growth invites the imposition of further constraints. We examine the effect of requiring $M(S) = F(n)$ and $m(S) = f(n)$ on the number of solutions S, for $4 \leq n \leq 24$; the results are in Table 2. For example, there are 3860 such solutions for $n = 20$, while $Q(20) = 39029188884$ is reported in [5].

2 Bounds and values for $F(n)$ and $f(n)$

We first observe some simple relations between $M(S)$ and $m(S)$ for a solution S of the n-queens problem, and their consequences.

Proposition 1 *For any n and solution S of the n-queens problem, $M(S) - m(S) \geq (n/2) - 1$ and $M(S) + m(S) \leq n - 1$. Therefore $0 \leq m(S) \leq n/4$, so $f(n) \leq n/4$, and $(n/2) - 1 \leq M(S) \leq n - 1$, so $F(n) \geq (n/2) - 1$.*

Proof. By the definitions of $M(S)$ and $m(S)$, the difference diagonals available for squares of S are numbered $-M(S), \ldots, -m(S)$ and $m(S), \ldots, M(S)$; there are $2(M(S) - m(S) + 1)$ of these if $m(S) > 0$, and $2M(S) + 1$ if $m(S) = 0$. The first inequality of the proposition then follows from the fact that the n squares of S lie on different difference diagonals.

By the definition of $m(S)$, S contains no square on a sum or difference diagonal with number of absolute value less than $m(S)$. These squares include all those on sum or difference diagonals whose numbers have absolute value exceeding $n - 1 - m(S)$, so $M(S) \leq n - 1 - m(S)$. This gives the second inequality of the proposition, and the two inequalities imply the stated bounds on $m(S)$ and $M(S)$, and on $f(n)$ and $F(n)$. ∎

With $S = \{(x_i, y_i) : 1 \leq i \leq n\}$ a solution of the n-queens problem, let $d_i = y_i - x_i$ and $s_i = y_i + x_i$ for each i, $1 \leq i \leq n$. Since the board has n columns and n rows, and no two squares of S share a column or row,

$$\{x_i : 1 \leq i \leq n\} = \{y_i : 1 \leq i \leq n\} = \left\{\frac{1-n}{2}, \frac{3-n}{2}, \ldots, \frac{n-3}{2}, \frac{n-1}{2}\right\}. \tag{1}$$

This implies

$$\sum_{i=1}^{n} d_i = \sum_{i=1}^{n} s_i = \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i = 0. \tag{2}$$

For each i, the Parallelogram Law gives $2x_i^2 + 2y_i^2 = (y_i - x_i)^2 + (y_i + x_i)^2 = d_i^2 + s_i^2$. Then summing from 1 to n, and using (1) to evaluate the left side of the resulting equation, we obtain

$$(n^3 - n)/3 = \sum_{i=1}^{n} d_i^2 + \sum_{i=1}^{n} s_i^2. \tag{3}$$

Using (2) and (3), we can now considerably improve on the bounds for F and f given in Proposition 1.

Theorem 2 *For each $n \geq 4$,*

$$F(n) \geq \begin{cases} \left\lceil \left(\sqrt{\frac{7n^2-4}{3}} + n - 2\right)/4 \right\rceil & \text{if } n \text{ is even,} \\ \left\lceil \left(\sqrt{\frac{7n^2+10n+3}{3}} + n - 3\right)/4 \right\rceil & \text{if } n \text{ is odd,} \end{cases}$$

and

$$f(n) \leq \begin{cases} \left\lfloor \left(\sqrt{\frac{7(n^2+4)}{3}} - n + 2\right)/4 \right\rfloor & \text{if } n \text{ is even,} \\ \left\lfloor \left(\sqrt{\frac{7n^2+2n+3}{3}} - n - 1\right)/4 \right\rfloor & \text{if } n \text{ is odd.} \end{cases}$$

Proof. We will use similar methods to establish each of the four stated bounds; we consider the maximum (for F) and the minimum (for f) possible values of the right side of (3). For this we need the following definitions.

Definitions. Let n, ℓ be integers, with $n \geq 1$ and $\ell \geq 0$, and let $k_1, \ldots, k_n$ be distinct integers satisfying $\sum_{i=1}^{n} k_i = 0$. If $|k_i| \leq \ell$ for each i, then $(k_i)_{i=1}^{n}$ is an *(n, ℓ, max)-sequence.* If $|k_i| \geq \ell$ for each i, then $(k_i)_{i=1}^{n}$ is an *(n, ℓ, min)-sequence.*

For $n \geq 4$, let S be any solution of the n-queens problem and let $M = M(S)$ and $m = m(S)$. Then each of the sequences $(d_i)_{i=1}^{n}$ and $(s_i)_{i=1}^{n}$ of diagonal numbers of S is both an (n, M, max)-sequence and an (n, m, min)-sequence, by (2). So if B is the maximum value of $\sum_{i=1}^{n} k_i^2$ among all (n, M, max)-sequences, and b is the minimum value of $\sum_{i=1}^{n} k_i^2$ among all (n, m, min)-sequences, then (3) implies $2b \leq (n^3 - n)/3 \leq 2B$.

To bound $F(n)$, we consider an (n, M, max)-sequence $K = (k_i)_{i=1}^{n}$. If integers $h < j$ occur in K and $h - 1$ and $j + 1$ do not, and $-M \leq h - 1$ and $j + 1 \leq M$, then by replacing h and j with $h - 1$ and $j + 1$ in K, we obtain an (n, M, max)-sequence with larger square sum. Using this fact and now assuming that K has maximum square sum among (n, M, max)-sequences, we see that for some positive integers s and u, K consists of a run $-M, 1 - M, \ldots, -s$, another run $u, u + 1, \ldots, M$, and possibly a single integer t intermediate between the two runs. Since negating each term in an (n, M, max)-sequence gives an (n, M, max)-sequence with the same square sum, we may assume $s \geq u$.

If there is no intermediate term, then $\sum_{i=1}^{n} k_i = 0$ implies $s = u$, so n is even. If there is an intermediate term t, then $\sum_{i=1}^{n} k_i = 0$ implies $t = \sum_{i=u+1}^{s} i$. Since $-s < t < u$, and therefore $|t| < s$, it follows that $s = u$, $t = 0$, and n is odd. For even n, (3) implies $(n^3 - n)/3 \leq 4 \sum_{i=M-(n/2)+1}^{M} i^2$, which simplifies to $(7n^2 - 4)/3 \leq (4M + 2 - n)^2$. For odd n, (3) gives

$(n^3-n)/3 \le 4\sum_{i=M-((n-3)/2)}^{M} i^2$, which simplifies to $(7n^2+10n+3)/3 \le (4M+3-n)^2$. These inequalities give the bound stated for $F(n)$.

To bound $f(n)$, consider an $(n, m, \min)$-sequence $K = (k_i)_{i=1}^{n}$ having minimum square sum among such sequences. Since $f(n) \ge 0$ by definition, we may assume that $m \ge 1$.

For any integers $h < j$ that occur in K, it must not be possible to replace them in K with $h+1$ and $j-1$ and get another $(n, m, \min)$-sequence, as this sequence would have smaller square sum. This implies two facts: first, that either the positive terms of K consist of a run with least member m, or the negative terms of K consist of a run with greatest member $-m$; without loss of generality, we can assume the latter. Second, that then the positive terms consist of a run with at most one term omitted. That is, there are integers $s, w \ge 1$ and $u \ge 0$ such that the terms of K in increasing order are $-m-s+1, \ldots, -m$ and then $m+u, m+u+1, \ldots, m+u+n-s$, except that some term w is omitted from the second run. (Here $m+u+1 \le w \le m+u+n-s$, with $w = m+u+n-s$ being the case where the positive terms are an unbroken run.) Say that an $(n, m, \min)$-sequence of this form is a *candidate* (for minimum square sum).

Viewing the positive and negative terms in order of increasing absolute value, we see that $\sum_{i=1}^{n} k_i = 0$ requires $s \ge n/2$. Also, $\sum_{i=1}^{n} k_i = 0$ implies

$$sm + s(s-1)/2 = (n-s+1)(m+u) + (n-s+1)(n-s)/2 - w \qquad (4)$$

for any candidate sequence.

For n even and $s = n/2$, (4) implies $w = (u+1)(n/2) + m + u$. Then the bound $w \le m+u+n-s$ implies $u = 0$ and $w = m + (n/2)$. Let K_0 denote the resulting $(n, m, \min)$-sequence, which consists of $\pm m, \pm(m+1), \ldots, \pm(m+(n/2)-1)$.

For n odd and $s = (n+1)/2$, (4) implies $w = u(n+1)/2$. Here the bound $m+u \le w$ implies $m \le u(n-1)/2$; since $1 \le m$, we see $0 < u$. The bound $w \le m+u+n-s$ implies $(u-1)(n-1) \le m$; from Proposition 1 we have $m \le n/4$, so $u \le 1$. Thus $u = 1$ and $w = (n+1)/2$. The resulting $(n, m, \min)$-sequence K_1 has terms $-m-((n-1)/2), \ldots, -m$ and $m+1, \ldots, m+((n+1)/2)$, except not $(n+1)/2$.

We claim that the minimum square sum among all $(n, m, \min)$-sequences is given by K_0 when n is even, and by K_1 when n is odd. The proof for even n is similar to but much simpler than the proof for odd n, so we give only the latter.

Let n be odd and let K', with parameters s', u', w', be a candidate sequence other than K_1. Since $s' \ge n/2$ and $K' \ne K_1$, we have $s' \ge (n+3)/2$. If also $u' \le 1$, then (4) gives $w' \le -(n+4m+1)/2 < 0$, which is not possible, so $u' > 1$.

Now, K' and K_1 have $(n+1)/2$ negative terms in common, and K' has $s'-((n+1)/2)$ more negative terms; in increasing order, they are $-m-s'+$

$1, \ldots, -m - ((n+1)/2)$. Comparing absolute values, it is easy to see that these $s' - ((n+1)/2)$ terms have greater square sum than the $s' - ((n+1)/2)$ largest positive terms of K_1. Thus we can conclude by showing that the $n - s'$ smallest positive terms of K_1, which are $m+1, \ldots, m+u+n-s'$ with one term $(n+1)/2$ missing, have smaller square sum than the $n - s'$ positive terms of K', which for some u are $m+u, \ldots, m+u+n-s'$ with one term w' omitted. This conclusion follows from $u' > 1$.

Then for even n, using K_0 with (3) gives $4\sum_{i=m}^{m+(n/2)-1} i^2 \leq (n^3 - n)/3$, which simplifies to $(4m + n - 2)^2 \leq (7n^2 + 28)/3$. This reduces to the desired bound for $f(n)$ when n is even.

For odd n, using K_1 with (3) similarly implies $(4m + n + 1)^2 \leq (7n^2 + 2n + 3)/3$, and then the final bound of the theorem. ∎

Notation. For each $n \geq 4$, let $B(n)$ denote the lower bound for $F(n)$ derived in Theorem 2, and let $b(n)$ denote the upper bound for $f(n)$ derived there.

Table 1 shows that the bounds are quite good for $4 \leq n \leq 24$. Later we give solutions achieving the values in Table 1.

Definitions. Say that a solution S of the n-queens problem is an *upper champion* if $M(S) = F(n)$ and $m(S) = \max\{m(U) : U$ is a solution of the n-queens problem and $M(U) = F(n)\}$. Say S is a *lower champion* if $m(S) = f(n)$ and $M(S) = \min\{M(U) : U$ is a solution of the n-queens problem and $m(U) = f(n)\}$. If S is both an upper and a lower champion, then S is an *n-champion*, or simply a *champion*; here $M(S) = F(n)$ and $m(S) = f(n)$.

For large n, $b(n)/n \approx (\sqrt{7/3} - 1)/4 = 0.13188\ldots$ and $B(n)/n \approx (\sqrt{7/3} + 1)/4 = 0.63188\ldots$. Thus the sequences $-B(n), \ldots, -b(n)$ and $b(n), \ldots, B(n)$ together contain about n integers, and we may hope that an n-champion exists.

For some n, it is not difficult to see that no champion exists. For example, suppose there were a 16-champion S; then $m(S) = f(16) = 2$ and $M(S) = F(16) = 10$. Of the 18 integers $-10, \ldots, -2, 2, \ldots, 10$, all but two, say a_1 and a_2, occur as difference diagonal numbers for S. By (2), the difference diagonal numbers sum to zero, as do the integers $-10, \ldots, -2, 2, \ldots, 10$, so $a_1 + a_2 = 0$. Similarly, the sum diagonal numbers of S comprise all but two, say b_1, b_2, of those 18 integers, and $b_1 + b_2 = 0$. By (3) we have

$$(17 \cdot 16 \cdot 15)/3 = 4\sum_{i=2}^{10} i^2 - (a_1^2 + a_2^2 + b_1^2 + b_2^2).$$

This reduces to $a_1^2 + b_1^2 = 88$, but 88 is not the sum of two squares.

Surprisingly, it appears that at least for most small n, champions exist. As might be expected, the number $Ch(n)$ of n-champions appears to correlate with how free the choice of diagonals is; we next introduce a measure of this freedom.

Definition. A sum or difference diagonal is ***available*** (for a champion) if its number h satisfies $f(n) \leq |h| \leq F(n)$. Let $\Delta(n)$ denote the excess of the number of available diagonals of the $n \times n$ board over the number $2n$ of diagonals occupied by a solution of the n-queens problem.

For $4 \leq n \leq 24$, the values of $Ch(n)$ and $\Delta(n)$ are shown in Table 2.

We were curious about what effect a symmetry constraint on solutions S would have on the values of $M(S)$ and $m(S)$. For $n = 12, 16, \ldots, 48, 52$, a computer search was made to determine the maximum value of $m(S)$ subject to the requirement that S be a solution of the n-queens problem, symmetric by a quarter-turn about board center, and satisfying $M(S) = B(n)$. Let that maximum be $f^*(n)$. We found that $f^*(44) = 4 = b(44) - 2$, $f^*(48) = 3 = b(48) - 3$, and for the other values of n mentioned, $f^*(n) = b(n) - 1$. (Later we give an example for $n = 52$.) This shows for these n that $F(n) = B(n)$, and that the symmetry constraint does not have a strong effect.

3 Constructions

As mentioned earlier, Ahrens showed [1] by construction that solutions of the n-queens problem exist for all positive integers except 2 and 3. Other authors independently did the same (see the discussion and references of [5]). Most of these constructions are not useful here. For example, some methods take a previously found solution for $n - 1$ and add an edge row, edge column, and a queen square in the newly formed corner to obtain a solution S for n. But then $M(S) = n - 1$ and $m(S) = 0$ are poor values. More generally, methods that choose squares near any of the corners will not work well.

For odd n, another difficulty is that any solution S that is symmetric by a half-turn about board center necessarily includes the center square, and thus has $m(S) = 0$. So for odd $n \geq 9$, this type of symmetry cannot be used if we want minimum values of $m(S)$.

The first construction we present is a special case of one due to Erbas and Tanik [2], and can be used for all $n \equiv \pm 2 \pmod 6$. It gives n-champions for $n = 4, 8, 10$, but not for larger n, as the values of m are always one. Constructions that use a "repeated motion", as this one does, generally can give good values for M but not for m, as n increases.

Let $n \equiv \pm 2 \pmod 6$. Let q be the largest integer such that $18q+9 < n$, except that $q = 0$ if $n < 9$. Set $I = \lfloor \frac{n+6q}{4} \rfloor + 1$. Let $E_n = \{\pm(-3q - \frac{5}{2} + 2i, \frac{n+1}{2} - i) : 1 \le i \le I\} \cup \{\pm(-3q - n - \frac{5}{2} + 2i, \frac{n+1}{2} - i) : I < i \le \frac{n}{2}\}$. (A less formal description: E_n is symmetric under rotation about the board center by a half-turn, so we only need give the squares of E_n in the top half of the board. To generate these, begin with $(-3q - \frac{1}{2}, \frac{n-1}{2})$ and make successive knight moves "two right, one down" until a square has been chosen in each row of the top half; for $n > 4$, it will be necessary to pass once from the right edge of the board to the left edge, as though those edges were identified.)

Then E_n is a solution of the n-queens problem; this is easily verified for $n = 4, 8$, and for $n \ge 10$, it follows from [2, Construction 2]. (In the notation of [2], $d = 6q+3$ and $s = (n/2) - 3q$ here.) For each n, $m(E_n) = 1$. Straightforward calculation shows $M(E_4) = 2$, $M(E_8) = 5$, and otherwise

$$M(E_n) = \begin{cases} \lceil \frac{2n}{3} \rceil & \text{if } n \equiv 2, 4, 8, \text{ or } 16 \pmod{18}, \\ \lfloor \frac{2n}{3} \rfloor & \text{if } n \equiv 10 \text{ or } 14 \pmod{18}. \end{cases}$$

Thus for all $n \equiv \pm 2 \pmod 6$, $M(E_n)/n \approx 2/3$, which is not too far from $B(n)/n \approx 0.63188$.

It would be interesting to see, especially for $n \equiv 3 \pmod 6$, constructions giving good values of M and, if possible, of m.

The next construction, due to Pólya, uses any solution S_1 of the h-queens problem, $h \ge 4$, to generate solutions S for each $n = hk$, with $k \equiv \pm 1 \pmod 6$, such that $M(S)/n$ is only slightly larger than $M(S_1)/h$, and $m(S)/n$ is only slightly smaller than $m(S_1)/h$.

Definition. A solution S of the n-queens problem is *toroidal* if no two difference (respectively sum) diagonal numbers of S are congruent modulo n.

The name derives from the fact that if the left and right edges of the $n \times n$ board are identified, and the top and bottom edges also, so that the board occupies the surface of a torus (and each orthogonal and each diagonal becomes a loop of n squares), a toroidal solution retains the property that no two queens attack each other.

Pólya showed [4] that there are toroidal solutions for n if and only if $n \equiv \pm 1 \pmod 6$.

Proposition 3 (Pólya Construction) *Let h, k be integers with $h \ge 4$, $k \ge 1$, and $k \equiv \pm 1 \pmod 6$. Let $S_1 = \{(X_i, Y_i) : 1 \le i \le h\}$ be a solution of the h-queens problem. Let $S_2 = \{(x_i, y_i) : 1 \le i \le k\}$ be a toroidal solution of the k-queens problem. Then $S = \{(kX_i + x_j, kY_i + y_j) : 1 \le i \le h, 1 \le j \le k\}$ is a solution of the hk-queens problem. We have*

$$M(S) \le kM(S_1) + M(S_2) \text{ and } m(S) \ge km(S_1) - M(S_2).$$

Thus

$$F(hk) \le kF(h) + M(S_2) \text{ and } f(hk) \ge kf(h) - M(S_2). \quad (5)$$

Proof. Pólya [4] showed that S is a solution of the hk-queens problem.

We may choose i and j with the square $(kX_i + x_j, kY_i + y_j)$ on a diagonal d whose number has absolute value $M(S)$. Then (taking the plus from each $\pm$ below if d is a sum diagonal, and the minus if d is a difference diagonal)

$$M(S) = |k(Y_i \pm X_i) + y_j \pm x_j| \le k|Y_i \pm X_i| + |y_j \pm x_j| \le kM(S_1) + M(S_2),$$

Similarly, if $(kX_i + x_j, kY_i + y_j)$ is on a diagonal with absolute value $m(S)$, then

$$m(S) = |k(Y_i \pm X_i) + y_j \pm x_j| \ge k|Y_i \pm X_i| - |y_j \pm x_j| \ge km(S_1) - M(S_2).$$

The final claims of the proposition follow if we take an S_1 satisfying $M(S_1) = F(h)$, and then another satisfying $m(S_1) = f(h)$. ∎

By using a particular toroidal solution for each k, we get a more specific bound, which we state in quotient form.

Corollary 4 *Let h, k be integers with $h \ge 4$, $k \ge 1$, and $k \equiv \pm 1 \pmod 6$. Then*

$$\frac{F(hk)}{hk} < \frac{F(h)}{h} + \frac{3}{4h} \text{ and } \frac{f(hk)}{hk} > \frac{f(h)}{h} - \frac{3}{4h}.$$

Proof. Let $S_2 = \{(i, 2i) : -\frac{k-1}{2} \le i \le \frac{k-1}{2}\}$, where row coordinates are reduced modulo k so that their absolute values do not exceed $(k-1)/2$. Then S_2 is a toroidal solution (see [3] or [4] for discussion of such *regular* toroidal solutions) of the k- queens problem. Choose r in $\{1, -1\}$ such that $k \equiv r \pmod 4$. In [6, Lemma 7] it is shown that $M(S_2) = (3k - 2 - r)/4$. (This can be verified by checking that the square of S_2 nearest the lower right board corner is $((k + 2 + r)/4, (-k + 2 + r)/2)$, and that no square of S_2 has a diagonal number of larger absolute value than the difference diagonal of this square.) Then employing (5) with this S_2, dividing by hk, and using $|r| = 1$ gives the desired conclusion. ∎

We note in passing that for $k \equiv \pm 1 \pmod 6$, the S_2 used in the proof of Corollary 4 satisfies $m(S_2) = 0$, as does every toroidal solution, and $M(S_2)/k \approx 3/4$.

Our final construction requires some definitions.

Definitions. Let sgn : $R \to R$ be the signum function, defined by $\operatorname{sgn}(x) = 1$ if $x > 0$, $\operatorname{sgn}(x) = -1$ if $x < 0$, and $\operatorname{sgn}(0) = 0$. For $i = 1, 2$ define $f_i : R^2 \to R^2$ by $f_1(x, y) = (2x - (\operatorname{sgn}(y - x)/2), 2y + (\operatorname{sgn}(y - x)/2))$ and $f_2(x, y) = (2x - (\operatorname{sgn}(y + x)/2), 2y - (\operatorname{sgn}(y + x)/2))$.

Define $h : R^2 \to R^2$ by $h(x, y) = (x, -y)$.

For a set S of squares of the $n \times n$ board, let $M_d(S)$ and $m_d(S)$ (respectively $M_s(S)$ and $m_s(S)$) denote the maximum and minimum absolute values of difference diagonal (respectively sum diagonal) numbers of S.

The construction arose from our consideration of some solutions along with the usual alternating square coloring used on chessboards. For a square (x, y), this coloring can be thought of as the parity of $y + x$.

Examining solutions for $n = 4k$, we saw that for some k, there was a solution S' such that the even squares of S' formed a double sized, slightly distorted copy of a solution S for $n = 2k$ (the copy is $f_1(S)$), and the odd squares of S' formed a similar copy of S, but upside down (the copy is $f_2(h(S))$). An example is shown in Figure 1.

Proposition 5 (Doubling Construction) *Let n be an even positive integer and let S be a solution of the n-queens problem that is symmetric by a half-turn about the board center. Then $S' = f_1(S) \cup f_2(h(S))$ is a solution of the $2n$-queens problem that is symmetric by a half-turn about board center.*

If $M_d(S) < M_s(S)$ then $M_d(S') = M_s(S') = 2M_s(S)$, so $M(S') = 2M(S)$.

If $M_d(S) = M_s(S)$ then $M_d(S') = M_s(S') + 1 = 2M_d(S) + 1$, so $M(S') = 2M(S) + 1$.

If $M_d(S) > M_s(S)$ then $M_d(S') = M_s(S') + 2 = 2M_d(S) + 1$, so $M(S') = 2M(S) + 1$.

If $m_d(S) > m_s(S)$ then $m_d(S') = m_s(S') = 2m_s(S)$, so $m(S') = 2m(S)$.

If $m_d(S) = m_s(S)$ then $m_s(S') = m_d(S') - 1 = 2m_d(S) - 1$, so $m(S') = 2m(S) - 1$.

If $m_d(S) < m_s(S)$ then $m_s(S') = m_d(S') - 2 = 2m_d(S) - 1$, so $m(S') = 2m(S) - 1$.

Proof. Let n and S satisfy the hypotheses. Since n is even and S is symmetric by a half-turn about board center, neither sum nor difference diagonal 0 contains a square of S. Thus $m(S) > 0$, and the values of the signum function occurring in the row and column numbers of S' are not zero. We show that S' is a solution of the $2n$-queens problem.

For any (x, y) in S, the diagonal numbers of the corresponding squares in S' are below.

square	$f_1(x, y)$	$f_2(h(x, y))$
diff. diagonal no.	$2(y - x) + \text{sgn}(y - x)$	$-2(y + x)$
sum diagonal no.	$2(y + x)$	$-2(y - x) + \text{sgn}(y - x)$

No square of $f_1(S)$ can share a difference diagonal with a square of $f_2(h(S))$, as the numbers have different parity; similarly for sum diagonals.

Suppose that for distinct squares $s_a = (x_a, y_a)$ and $s_b = (x_b, y_b)$ of S, the squares $f_1(s_a)$ and $f_1(s_b)$ share a diagonal. If it is a sum diagonal, then $2(y_a + x_a) = 2(y_b + x_b)$ so s_a and s_b share a sum diagonal, which is a contradiction. If it is a difference diagonal, then $2(y_a - x_a) + \text{sgn}(y_a - x_a) = 2(y_b - x_b) + \text{sgn}(y_b - x_b)$. However, each side of this equation has the same sign as the first term of the side, so $\text{sgn}(y_a - x_a) = \text{sgn}(y_b - x_b)$, and then $y_a - x_a = y_b - x_b$. This contradicts the fact that s_a and s_b lie on distinct difference diagonals. A similar argument shows that $f_2(h(s_a))$ and $f_2(h(s_b))$ do not share a diagonal.

We then need to check that no two squares of S' share an orthogonal, or equivalently that S' occupies each row and column of the $2n$-board. The latter follows from the fact that that for each (x, y) in S, we also have $(-x, -y)$ in S, and then the four squares $f_1(\pm(x, y))$, $f_2(h(\pm(x, y)))$ jointly occupy columns $\pm 2x \pm (1/2)$ and rows $\pm 2y \pm (1/2)$.

Since applying any of f_1, f_2, h to a set that is symmetric by a half-turn about board center gives such a set, S' has this type of symmetry.

For the remaining claims, note that if $k > 0$ occurs as a difference diagonal number of S, then by symmetry so does $-k$, and then the corresponding diagonals of S' are difference diagonals numbered $\pm(2k + 1)$ and sum diagonals numbered $\pm(2k - 1)$. Similarly, if $k > 0$ occurs as a sum diagonal number of S, then the corresponding diagonals of S' are difference and sum diagonals numbered $\pm 2k$. We examine one of the six claims; the others are similar.

Suppose $M_d(S) > M_s(S)$. By the preceding paragraph, the diagonals of S' corresponding to sum diagonals of S all have numbers of absolute value at most $2M_s(S)$, so the difference diagonal number of S' of largest absolute value is $2M_d(S) + 1$, and the sum diagonal number of S' of largest absolute value is $2M_d(S) - 1$, which implies the claim. ∎

Notation. In the situation of Proposition 5, we write $d(S) = S'$.

Choose particular n, S satisfying the hypotheses of Proposition 5. Starting with S, there are many ways to use the functions d and h to inductively construct solutions of the $2^i n$-queens problem for each $i \geq 0$. We give two of these: first, the one that gives the largest values of the function m, then the one that gives the smallest values of M. Let $E = 1$ if $M_d(S) \geq M_s(S)$, and let $E = 0$ otherwise. Let $e = 1$ if $m_d(S) \leq m_s(S)$, and let $e = 0$ otherwise.

Let $L_0 = S$ and for each $i \geq 0$ define $L_{i+1} = d(L_i)$. Then L_i is a solution of the $2^i n$-queens problem for each i. If $e = 1$, the final statements of Proposition 5 imply that $m(L_{i+1}) = 2m(L_i) + 1$ for even i, and $m(L_{i+1}) = 2m(L_i)$ for odd i. Examining this and the case $e = 0$, it is easy to prove by induction that $m(L_i) = 2^i m(S) - (2^{i+1+e} + (-1)^{i+e} - 3)/6$ for $i \geq 0$.

Similarly, $M(L_i) = 2^i M(S) + 2^{i-1+E} - 1$ for $i \geq 1$.

These imply $m(L_i)/(2^i n) > (m(S) - (2^e/3))/n$ and $M(L_i)/(2^i n) < (M(S) + 2^{E-1})/n$ for $i \geq 1$.

Let $U_0 = S$ and for each $i \geq 0$ set $U_{i+1} = d(U_i)$ if i is even, and $U_{i+1} = d(h(U_i))$ if i is odd. (Note h interchanges sum and difference diagonals.) Then U_i is a solution of the $(2^i n)$-queens problem for each i, and an induction yields $m(U_i) = 2^i m(S) - 2^{i-1+e} + 1$ for $i \geq 1$, and $M(U_i) = 2^i M(S) + (2^{i+1+E} + (-1)^{i+E} - 3)/6$ for $i \geq 0$.

It follows that $m(U_i)/(2^i n) > (m(S) - 2^{e-1})/n$ and $M(U_i)/(2^i n) < (M(S) + (2^E/3))/n$ for $i \geq 1$.

For each n, $4 \leq n \leq 24$, we now give an n-champion, referred to herein as C_n, if one exists. If there is no champion, we give an upper champion followed by a lower champion. In the latter situation, for these n each upper champion S has $m(S) = f(n) - 1$, and with one exception each lower champion S has $M(S) = F(n) + 1$. The exception is $n = 22$, where $M(S) = F(n) + 2$.

We give the row numbers as the column number increases from $(1-n)/2$ to $(n-1)/2$. (Recall that for even n, the row and column numbers each are half an odd integer.) If the solution is symmetric by a half-turn about board center, then only the row numbers for the left half are given. Where possible, we describe the solution using d, h, E_n as defined previously.

$n = 4$. $\frac{1}{2}(1, -3)$. Symmetric by a quarter-turn.
$n = 5$. $(1, -2, 0)$. Symmetric by a quarter-turn.
$n = 6$. $\frac{1}{2}(1, -5, 3)$. Symmetric by a half-turn; see Figure 1.
$n = 7$. $(2, 0, 3, -3, -1, 1, 2)$.
$n = 8$. $\frac{1}{2}(3, -1, 5, -7)$. Symmetric by a half-turn; this is $d(C_4)$.
$n = 9$. $(2, 0, -3, 3, -4, -2, 4, 1, -1)$.
$n = 10$. $\frac{1}{2}(3, -5, 1, -7, 9)$. Symmetric by a half-turn; this is $h(E_{10})$.
$n = 11$. A. $(2, 0, -3, 3, 5, -5, -1, -4, 4, 1, -2)$.
B. $(3, 1, -1, -3, -5, 4, 2, 5, -4, -2, 0)$.
$n = 12$. $\frac{1}{2}(3, -1, 9, -11, 7, -5)$. Symmetric by a half-turn; this is $d(C_6)$. See Figure 1.
$n = 13$. $(2, 0, -3, -5, 4, 6, -6, -4, -1, 5, 3, 1, -2)$.
$n = 14$. $\frac{1}{2}(5, 1, -3, -11, 9, -13, 7)$. Symmetric by a half-turn.
$n = 15$. $(3, 1, 4, 2, -5, -7, -2, -6, 5, 7, 0, 6, -4, -1, -3)$.
$n = 16$. A. $\frac{1}{2}(-5, 7, 3, -1, 9, -11, 15, -13)$. Symmetric by a half-turn; this is $d(h(C_8))$.
B. $\frac{1}{2}(7, -5, -1, 3, 11, -9, 13, -15)$. Symmetric by a half-turn; this is $d(C_8)$.
$n = 17$. $(2, 4, 1, -3, -6, 6, -5, -8, 8, 5, -4, -7, 7, 0, -2, 3, -1)$.
$n = 18$. $\frac{1}{2}(5, -1, -7, -3, -13, 11, 15, 9, -17)$. Symmetric by a half-turn.
$n = 19$. $(3, 1, 4, -2, -7, 6, -6, -9, 4, -5, 9, -4, 8, -8, 7, 2, -1, -3, 0)$.

$n=20.$ $\frac{1}{2}(-5,7,11,-9,-1,3,15,-13,17,-19)$. Symmetric by a half-turn; this is $d(C_{10})$. Another champion is given by applying the Pólya construction to C_4 and C_5.

$n=21.$ A. $(3,1,4,2,-5,-8,-6,-9,5,10,8,-10,-3,9,7,-7,-2,6,-1,-4,0)$.

B. $(4,2,5,3,1,-7,-9,-6,-8,-10,9,6,10,7,0,-2,8,-5,-3,-1,-4)$.

$n=22.$ A. $\frac{1}{2}(7,3,9,5,1,-15,-19,-13,-17,-21,15,11,21,-9,19,13,17,-3,-7,-11,-1,-5)$.

B. $\frac{1}{2}(11,5,-1,-9,7,3,-21,-15,17,-13,-19)$. Symmetric by a half-turn.

$n=23.$ $(4,2,5,3,1,-3,-7,-9,-11,-8,-10,6,11,9,7,0,10,8,-4,-6,-1,-5,-2)$.

$n=24.$ $\frac{1}{2}(7,3,9,-11,-5,-1,-19,17,13,-21,15,-23)$. Symmetric by a half-turn.

For $n = 52$, we give a solution S that is symmetric by a quarter-turn, with $M(S) = F(52) = 33$ and $m(S) = f^*(52) = 6 = b(52) - 1$. By the symmetry, it suffices to give a set of 13 squares which generate S under rotation. The squares of this S in the leftmost 13 columns form such a set; we give their row numbers as their column numbers increase: $\frac{1}{2}(5,15,3,13,23,11,21,25,19,9,17,7,1)$.

We conclude with two conjectures.

Conjecture 1 *For all* $n \geq 4$, $f(n) \geq b(n) - 1$ *and* $F(n) \leq B(n) + 1$.

Conjecture 2 *There are infinitely many* n *for which* n*-champions exist.*

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n	4	5	6	7	8	9	10	11	12	13	14
$f(n)$	1	0	1	0	1	1	1	1	1	1	2
$F(n)$	2	3	4	**5**	5	6	6	7	8	8	9

n	15	16	17	18	19	20	21	22	23	24
$f(n)$	1	2	2	2	2	**2**	2	3	2	3
$F(n)$	**10**	10	11	11	12	13	13	14	15	15

Table 1: Values of $f(n)$ and $F(n)$ for $4 \leq n \leq 24$, found by computer search. Those in bold differ from the appropriate bound in Theorem 2 by 1; otherwise, the bound is attained.

n	4	5	6	7	8	9	10	11	12	13	14
$Ch(n)$	2	2	4	24	4	8	4	0	136	16	16
$\Delta(n)$	0	4	4	8	4	6	4	6	8	6	4

n	15	16	17	18	19	20	21	22	23	24
$Ch(n)$	2744	0	8	36	80	3860	0	0	296080	240
$\Delta(n)$	10	4	6	4	6	8	6	4	10	4

Table 2: Values of $Ch(n)$, the number of n-champions, found by computer search, and values of $\Delta(n)$, the excess of the number of available diagonals over the number of occupied diagonals, for $4 \leq n \leq 24$.

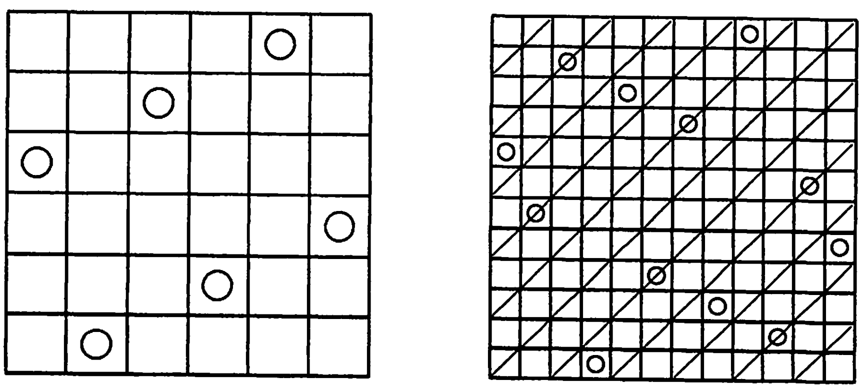

Figure 1: On the left a 6-champion S is shown. On the right is a 12-champion S', obtained from S by the doubling construction of Proposition 5. Diagonal lines mark the odd squares of the 12×12 board. The even squares in S' are the members of $f_1(S)$, and the odd squares in S' are the members of $f_2(h(S))$.